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DIFFERENTIAL GROUPOIDS

1. Introduction

A *differential groupoid* $(G,\cdot)$ is a set G equipped with a binary operaton $\cdot$ called *multiplication* satisfying the following identities:

$$(1.1) \qquad \begin{cases} x\cdot x = x & (\textit{idempotence}) \\ (x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t) & (\textit{entropicity}) \\ x\cdot(y\cdot z) = x\cdot y & (\textit{reduction}). \end{cases}$$

Apart from *left-zero semigroups* (where the multiplication $(x,y)\mapsto x\cdot y = x$ is just projection onto the left factor), the first examples of differential groupoids appearing in the literature were Płonka's "cyclic groupoids" [P1] [P2]. Differential groupoids in general were studied in [Ro] [RR1] [RR2]. In [RR1, Prop. 1.1] it was shown that differential groupoids could be characterised by satisfaction of the identity

$$(1.2) \qquad (x\cdot y)\cdot z = (x\cdot z)\cdot y$$

of *left normality* (familiar from semigroup theory) along with idempotence and reduction. For this reason differential groupoids were described acronymically in [Ro] [RR1] [RR2] as "LIR-groupoids". The purpose of the current paper is to point out some connections between differential groupoids, differentials, and differentiation (whence the terminology). Some of the theory of differential groupoids developed in [Ro] is recapitulated in new form in the second section. Each differential groupoid has projections onto left zero semigroups such that the fibres are also left zero semigroups. The kernel congruence of such a projection is said to be a *left zero decomposition* of the differential groupoid. The two most important left zero decompositions of a differential groupoid $(G,\cdot)$ are the kernel γ of the right multiplication mapping (2.1) and the kernel β of the projection of $(G,\cdot)$ onto its left zero semigroup replica, the largest left zero semigroup that is a quotient of $(G,\cdot)$. These decompositions are described in the second section.

The third section examines the linearisation of differential groupoids, the variety of differential groupoids that are reducts of affine spaces. The main result, Theorem 3.6, identifies these differential groupoids as affine spaces over the dual numbers. They are "differential groups" in the terminology used by Mac Lane [ML, II.1] in his direct approach to homology groups. The general left zero decompositions β and γ reduce to difference by boundaries and cycles respectively in this context (Proposition 3.10).

The last two sections are concerned with differentiation. Theorem 4.4 shows how tangent line approximations to differentiable real functions may be described by a simple functional equation in the differential groupoid on the real dual numbers. This functional equation is then abstracted in the fifth section to give a general theory of differentiation in differential groupoids. The basic idea is that derivations of a differentiable function "repair the failure of the function to be a groupoid homomorphism". The left zero replica congruence β gives a corresponding notion of continuity. The final result, Theorem 5.8, describes how continuity is necessary for differentiability.

As idempotent and entropic algebras, differential groupoids are *modes* in the sense of the book [RS1], which may serve as a reference for the universal algebraic concepts used here, such as replicas and tensor product varieties. The subtitle of [RS1] described modal theory as a [universal] "algebraic approach to order, geometry, and convexity." "Geometry" there predominantly meant affine geometry. Later [RS2], a modal theoretic approach to projective geometry was given. The present paper is intended to initiate a modal theoretic approach to differential geometry.

2. Cocyclic and cobordic elements

Left normality (1.2) is a consequence of the defining identities (1.1) via $(x\cdot y)\cdot z = (x\cdot y)\cdot(z\cdot z) = (x\cdot z)\cdot(y\cdot z) = (x\cdot z)\cdot y$. For each element y of a differential groupoid $(G,\cdot)$, consider the *right multiplication*

(2.1) $$R(y):\ G\to G;\ x\mapsto x\cdot y$$

by y. The set $GR = \{R(y) \mid y\in G\}$ of right multiplications generates a submonoid $R(G)$ of the endomorphism monoid $\mathrm{End}(G,\cdot)$ of the differential groupoid. This submonoid $R(G)$ is called the *right mapping monoid* of the differential groupoid. By left normality, it is a commutative monoid. Its elements are referred to as *right mappings* of $(G,\cdot)$. The endomorphism monoid $(\mathrm{End}(G,\cdot),\cdot)$ with multiplication defined by map composition also supports an *addition* operation $+$ with

(2.2) $$x(\vartheta+\varphi) = (x\vartheta)\cdot(x\varphi)$$

for x in G and endomorphisms ϑ, φ. The map

(2.3) $$R:\ (G,\cdot)\to(\mathrm{End}(G,\cdot),+);\ x\mapsto R(x)$$

is a groupoid homomorphism. Its image is a left zero semigroup $(GR,+)$, since $x(R(y)+R(z)) = (xy)\cdot(xz) = (x\cdot x)\cdot(y\cdot z) = x\cdot(y\cdot z) = x\cdot y = xR(y)$ by (1.1). The kernel of R is a congruence γ on $(G,\cdot)$. Elements of G related by γ are said to be *cocyclic*. If x and y are cocyclic, then $x\cdot y = xR(y) = xR(x) = x\cdot x = x$. Thus the fibres of R are left zero semigroups, and altogether γ furnishes a left zero decomposition of $(G,\cdot)$.

For an element x of G, the set

(2.4) $$xR(G) = \{x\vartheta \mid \vartheta \in R(G)\}$$

is called the *orbit* of x in G. Two elements x and y of G are said to be *cobordic*, or in the relation β, if their orbits intersect:

(2.5) $$x \beta y \Leftrightarrow \exists z \in xR(G) \cap yR(G).$$

The significance of cobordism is given by the following.

Theorem 2.6 [Ro, 2.5-6].

(i) *The relation* β *of* (2.5) *is a congruence relation on* $(G,\cdot)$.

(ii) *The quotient* $(G^{\beta},\cdot)$ *is the left zero replica of* $(G,\cdot)$.

Proof. By (2.5), β is reflexive and symmetric. Suppose $x\beta y \beta z$, say $\exists \vartheta, \varphi, \chi, \psi \in R(G)$. $x\vartheta = y\varphi$ and $y\chi = z\psi$. Then $x\vartheta\chi = y\varphi\chi = y\chi\varphi = z\psi\varphi$, so that $x\beta z$ and β is transitive. Now for t in G, $(x\cdot t)\vartheta = xR(t)\vartheta = x\vartheta R(t) = y\varphi R(t)$, so that $(x\cdot t)\beta y$. Thus for any u in G, and in particular for $t\beta u$, one has $(x\cdot t)\beta y \beta x \beta (y\cdot u)$, whence $(x\cdot t)\beta(y\cdot u)$ by the transitivity, so that β is a congruence on $(G,\cdot)$. Moreover $x^{\beta}\cdot t^{\beta} = (x\cdot t)^{\beta} = y^{\beta} = x^{\beta}$, so that $(G^{\beta},\cdot)$ is a left zero semigroup. Finally, suppose that $(G^{\alpha},\cdot)$ is a left zero semigroup. Set $\vartheta = R(g_1)\ldots R(g_m)$ and $\varphi = R(h_1)\ldots R(h_n)$. Then $x^{\alpha} = x^{\alpha}R(g_1^{\alpha})\ldots R(g_m^{\alpha}) = (x\vartheta)^{\alpha} = (y\varphi)^{\alpha} = y^{\alpha}R(h_1^{\alpha})\ldots R(h_n^{\alpha}) = y^{\alpha}$, whence $\beta \leq \alpha$. □

Since $(G^{\gamma},\cdot)$ is a left zero semigroup, an immediate corollary of Theorem 2.6 (ii) is $\beta \leq \gamma$. In other words, cobordic elements are cocyclic. For each element x of G, the set

(2.7) $$x^{\gamma}\mathrm{nat}\,\beta = \{y^{\beta} \mid y\gamma x\}$$

is called the *homology set* of $(G,\cdot)$ at x. It is a left zero semigroup under the multiplication it inherits from $(G,\cdot)$. The differential groupoid $(G,\cdot)$ is said to have *trivial homology* if its homology sets are all singletons, i.e. if each pair of cocyclic elements is cobordic. The opposite extreme is represented by a left zero semigroup, which consists of a single homology set.

3. Linearisation of differential groupoids

For a commutative ring R (with a unit element), *affine spaces* over R are defined algebraically as idempotent reducts of unital R-modules. Affine spaces over R form a variety $\underline{\underline{R}}$ [RS1, 255]. A *Mal'cev operation* is a ternary operation P on an algebra A satisfying the identities

(3.1) $$(x,y,y)P = x = (y,y,x)P.$$

Then the variety $\underline{\underline{\mathbb{Z}}}$ of affine spaces over the integers may be realised as the variety of algebras of type $\{(P,3)\}$ for which P is an entropic Mal'cev operation, i.e. for which $P:(A^3,P)\rightarrow(A,P)$ is a homomorphism. The variety $\underline{\underline{R}}$ may be realised as a variety of algebras of type $(R\times\{2\})\cup\{(P,3)\}$. The action of P on an affine space A is as the entropic Mal'cev operation

(3.2) $$(x,y,z)P = x-y+z,$$

while the action of an element r of R is as

(3.3) $$xy\underline{r} = (1-r)x+ry.$$

Given varieties $\mathfrak{V}_i$ of type $\tau_i : \Omega_i \to \mathbb{N}$, for $i=1,2$, the *tensor product* $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ is the variety of algebras A of type $\bigcup_{n=1}^{\infty}(\tau_1^{-1}(n) \cup \tau_2^{-1}(n)) \times \{n\}$ satisfying the identities that (A,Ω_1) be a $\mathfrak{V}_1$-algebra, that (A,Ω_2) be a $\mathfrak{V}_2$-algebra, and that each operation ω of Ω_1 is a homomorphism $(A^{\omega\tau_1},\Omega_2) \to (A,\Omega_2)$ [RS1, 232]. There are reductions $\pi_i : \mathfrak{V}_1 \otimes \mathfrak{V}_2 \to \mathfrak{V}_i$ by means of which $\mathfrak{V}_1 \otimes \mathfrak{V}_2$-algebras may be considered just as $\mathfrak{V}_i$-algebras. For a variety $\mathfrak{V}$, the *linearisation* is defined to be the variety $\underline{\underline{\mathbb{Z}}} \otimes \mathfrak{V}$. Note that $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{R}} = \underline{\underline{R}}$ for a commutative ring R, so that varieties of affine spaces are their own linearisations.

For a commutative ring R, the ring $R[d]$ of *dual numbers* over R is the abelian group $R \oplus dR = \{r+ds \mid r,s \in R\}$ with multiplication

(3.4) $$(r+ds)(t+du) = rt+d(st+ru).$$

The element d with $d^2=0$ is called the *differential*. Elements of the ideal dR are sometimes described as *infinitesimal*, while the projection

(3.5) $$\pi : R[d] \to R;\ r+ds \mapsto r$$

is described as *taking the finite part* r of $r+ds$. The linearisation of the variety $\mathfrak{D}$ of differential groupoids may then be identified as the variety of affine spaces over the dual numbers $\mathbb{Z}[d]$:

Theorem 3.6. $\underline{\underline{\mathbb{Z}}} \otimes \mathfrak{D} = \underline{\underline{\mathbb{Z}[d]}}$.

Proof. Let A be an affine space over $\mathbb{Z}[d]$. Then the multiplication

(3.7) $$x \cdot y = x - dx + dy$$

given by (3.3) with $r=d$ makes A a differential groupoid. Conversely, consider the variety $\underline{\underline{\mathbb{Z}}} \otimes \mathfrak{D}$. Since it is a variety of Mal'cev modes, it is the variety of affine spaces over some ring [RS1, 254]. Further, this ring is a quotient of the polynomial ring $\mathbb{Z}[X]$, where the indeterminate X furnishes the differential groupoid multiplication as

(3.8) $$x \cdot y = (1-X)x+Xy.$$

By the reduction law (1.1), $x \cdot (y \cdot z) = (1-X)x+X(1-X)y+X^2z = (1-X)x+Xy = x \cdot y$. Equating coefficients of z, one sees that X^2 annihilates affine space elements, so that the affine spaces are over a quotient ring $\mathbb{Z}[X]/\langle X^2 \rangle = \mathbb{Z}[d]$. This quotient cannot be proper, since affine spaces over the dual numbers are already in $\underline{\underline{\mathbb{Z}}} \otimes \mathfrak{D}$. □

In Mac Lane's terminology [ML, II.1], $\mathbb{Z}[d]$-modules are differential groups. Under the operation (3.7), differential groups are differential groupoids. Let $(K,+)$ be a differential group, with the abelian group endomorphism or "boundary operator"

(3.9) $$d : (K,+) \to (K,+);\ x \mapsto dx$$

satisfying $d^2=0$. Elements of $\operatorname{Ker} d$ are called *cycles*, and elements of $\operatorname{Im} d$ are called *boundaries*. The *homology group* $H(K)$ of $(K,+,d)$ is defined as the quotient $\mathrm{Ker}\, d/\mathrm{Im}\, d$. These constructions for differential groups illuminate the general constructions for differential groupoids given in the previous section as follows:

Proposition 3.10. *Let* $(K,+,d)$ *be a differential group, with corresponding differential groupoid* $(K,\cdot)$ *defined by* (3.7).

(i) *Elements of* K *are cocyclic iff they differ by a cycle.*

(ii) *Elements of* K *are cobordic iff they differ by a boundary.*

Proof. Let x,y,u be elements of K.

(i) $x\gamma y \Leftrightarrow R(x) = R(y)$

$$\Leftrightarrow \forall z \in K,\ z - dz + dx = z - dz + dy$$

$$\Leftrightarrow dx = dy \Leftrightarrow x - y \in \mathrm{Ker}\, d.$$

(ii) $x\cdot(x+u) = x + du$. Thus $xR(K) = x + \mathrm{Im}\, d$.

Then $x\beta y \Leftrightarrow \exists z \in xR(K) \cap yR(K)$

$$\Leftrightarrow \exists z \in (x + \mathrm{Im}\, d) \cap (y + \mathrm{Im}\, d)$$

$$\Leftrightarrow x - y \in \mathrm{Im}\, d.$$ □

The homology sets of elements of K may then all be identified with the underlying set of the homology group $H(K)$.

4. Differential calculus

The dual numbers $\mathbb{R}[d]$ over the reals form a differential groupoid under the multiplication

(4.1) $$x \cdot y = x - dx + dy$$

(cf. (3.7)). This differential groupoid provides a convenient framework for certain aspects of real differential calculus. Consider an everywhere-differentiable function $f : \mathbb{R} \to \mathbb{R}$. At each point a of $\mathbb{R}$, f has the tangent line approximation $f_a : \mathbb{R} \to \mathbb{R}$ with

(4.2) $$f_a(a+x) = f(a) + xf'(a).$$

These tangent line approximations may be used to extend the function $f : \mathbb{R} \to \mathbb{R}$ to a function $f : \mathbb{R}[d] \to \mathbb{R}[d]$ by means of the formula

(4.3) $$f(a+dx) = f(a) + f'(a)dx$$

for real x, which may be interpreted to mean that the tangent line approximation (4.2) is exact for points $a+dx$ infinitesimally close to a (cf. (3.5)). In general, the extended functions $f : \mathbb{R}[d] \to \mathbb{R}[d]$ are not homomorphisms of the differential groupoid structure on $\mathbb{R}[d]$, although affine functions such as extensions of (4.2) to $\mathbb{R}[d]$ are. The following theorem expresses the relationship of a differentiable function to its tangent line approximation by saying that these approximations "repair the failure of the function to be a differential groupoid homomorphism". The "finite part" map $x \mapsto x\pi$ is as in (3.5).

Theorem 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be an everywhere-differentiable function. Extend it to $f : \mathbb{R}[d] \to \mathbb{R}[d]$; $x \mapsto f(x\pi)+f'(x\pi)(x-x\pi)$. Then for all dual numbers x and y,*

(4.5) $$f(x \cdot y) = f(x) \cdot f_{x\pi}(y).$$

Furthermore, for each real a, any function $g : \mathbb{R}[d] \to \mathbb{R}[d]$ satisfying the functional equation

(4.6) $$f(a \cdot x) = f(a) \cdot g(x)$$

for all dual numbers x is infinitesimally close to the tangent line approximation f_a.

Proof. Let $x,y \in \mathbb{R}[d]$. By (4.2),

$$\begin{aligned} f_{x\pi}(y) &= f_{x\pi}(y\pi) + (f_{x\pi})'(y\pi)(y - y\pi) \\ &= f(x\pi) + (y\pi - x\pi)f'(x\pi) + f'(x\pi)(y - y\pi) \\ &= f(x\pi) + f'(x\pi)(y - y\pi), \end{aligned}$$

where $f_{x\pi} : \mathbb{R} \to \mathbb{R}$ is extended to $f_{x\pi} : \mathbb{R}[d] \to \mathbb{R}[d]$ by (4.2). Then

$$\begin{aligned} f(x) \cdot f_{x\pi}(y) &= [f(x\pi) + f'(x\pi)(x - x\pi)] \cdot [f(x\pi) + f'(x\pi)(y - x\pi)] \\ &= f(x\pi) + f'(x\pi)(x - x\pi) - df(x\pi) - f'(x\pi)d(x - x\pi) \\ &\quad + df(x\pi) + f'(x\pi)(y - x\pi) \\ &= f(x\pi) + f'(x\pi)(x - x\pi - dx + dy) \\ &= f(x - dx + dy) = f(x \cdot y), \end{aligned}$$

verifying (4.5). Finally, if (4.6) holds, then for all z in $\mathbb{R}[d]$ one has $f'(a)dz = f(a+dz) - f(a) = f(a \cdot (a+z)) - f(a) = f(a) \cdot g(a+z) - f(a) = dg(a+z) - df(a)$, whence $g(a+z)$ is cocyclic with $f(a) + xf'(a) = f_a(a+z)$. In other words, g and f_a are infinitesimally close. □

5. Abstract differentiation

The connections between differentiation of real functions and functional equations in the differential groupoid of real dual numbers given by Theorem 4.4 motivate a general theory of differentiation in differential groupoids. It is convenient to revert to postfix notation for functions. The basic definition is as follows.

Definition 5.1. Let $(G,\cdot)$ be a differential groupoid. Then a function $f : G \to G$ is said to be *differentiable at an element* x of G if there is an endomorphism f_x of $(G,\cdot)$, called a *derivative of f at x*, such that

(5.2) $$(x \cdot y)f = xf \cdot yf_x$$

for all y in G. The function f is *differentiable* (*everywhere*) if it is differentiable at each element x of G.

Endomorphisms of $(G,\cdot)$ are of course differentiable, and may be taken as their own derivatives. One immediate consequence of the definition is:

Proposition 5.3 (Chain Rule). *If $f : G \to G$ is differentiable at $x \in G$, and $g : G \to G$ is differentiable at xf, then the composite fg is differentiable at x, with*

(5.4) $$(fg)_x = f_x g_{xf}.$$

Proof. For all y in G, one has $(x \cdot y)fg = (xf \cdot yf_x)g = xfg \cdot yf_x g_{xf}$. Since $f_x g_{xf}$ is an endomorphism of $(G,\cdot)$, the composite fg is differentiable at x, with this endomorphism as a derivative. □

Along with the concept of differentiability given by Definition 5.1, there is a corresponding concept of continuity for functions on differential groupoids.

Definition 5.5. Let $(G,\cdot)$ be a differential groupoid, with cobordism relation β. Then a function $f : G \to G$ is said to be *continuous at an element* x of G if

(5.6) $$x \beta y \Rightarrow xf \; \beta \; yf.$$

The function f is said to be *continuous* (*everywhere*) if it is continuous at each element of G.

Note that a continuous function $f : \mathbb{R}[d] \to \mathbb{R}[d]$ on the differential groupoid of real dual numbers yields a well-defined real function f^β such that the diagram

(5.7) $$\begin{array}{ccc} \mathbb{R}[d] & \xrightarrow{\;f\;} & \mathbb{R}[d] \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{R} & \xrightarrow[\;f^\beta\;]{} & \mathbb{R} \end{array}$$

commutes. Continuity of such a function f may be interpreted as saying that it maps infinitesimally close elements to infinitesimally close elements. Since cobordism on a left zero semigroup is equality, all functions of left zero semigroups are continuous.

The following theorem gives some connections between differentiability and continuity.

Theorem 5.8. *Let $(G,\cdot)$ be a differential groupoid, with a function $f : G \to G$.*

(i) *If f is differentiable at each element of the cobordism class of an element x of G, then f is continuous at x.*

(ii) *If f is differentiable, then it is continuous.*

Proof. Clearly, (ii) is a direct consequence of (i). Suppose that $x \in G$ and that f is differentiable at each element of x^β. Let $y \in x^\beta$. It will be shown that for all ϑ in $R(G)$,

(5.9) $$\exists \vartheta'_y \in R(G). \; y\vartheta f = yf\vartheta'_y.$$

Once this is proved, (i) follows: Suppose $x \beta y$, say $\exists \psi, \varphi \in R(G). \; x\psi = y\varphi$. Then by (5.9), $xf\psi'_x = x\psi f = y\varphi f = yf\varphi'_y$, whence $xf \; \beta \; yf$ as required. Now (5.9), as a property of an element ϑ of the monoid $R(G)$ generated by GR, may be proved by induction on the minimal length of ϑ as a word in the alphabet GR. For the empty word 1, (5.9) is true with $1'_y = 1$. Suppose (5.9) holds for some $\vartheta \in R(G)$. Note that f is differentiable at $y\vartheta$, since $y\vartheta \; \beta \; y \; \beta \; x$

implies $y\vartheta \; \beta \; x$ by the transitivity of β. Then for z in G one has $y\vartheta R(z)f = y\vartheta fR(zf_{y\vartheta}) = yf\vartheta'_y R(zf_{y\vartheta})$, which shows that one can take $\vartheta'_y R(zf_{y\vartheta})$ for $[\vartheta R(z)]'_y$. This verifies that (5.9) holds for $\vartheta R(z)$. □

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