DUALITY FOR SOME FREE MODES*

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Abstract

The paper establishes a duality between a category of free sub-reducts of affine spaces and a corresponding category of generalized hypercubes with constants. This duality yields many others, in particular a duality between the category of (finitely generated) free barycentric algebras (simplices of real affine spaces) and a corresponding category of hypercubes with constants.

1. Introduction

It is well known that any finite-dimensional (i.e. finitely generated) vector space V over a field F is isomorphic to its second dual V^{**}. More generally, one may say that the category of finite-dimensional vector spaces with linear mappings as morphisms is dually equivalent (or just dual) to itself. However, this symmetry is lost when instead of vector spaces one considers affine spaces over a field: the first dual then has too many affine space homomorphisms to preserve the self-duality of vector spaces.

In the current paper we show how to overcome this difficulty. We provide a duality between the category of affine spaces over a fixed field on the one hand and a category of certain affine spaces over the same field with constants on the other. The duality is similar to the duality for vector spaces. In the case of finitely generated affine spaces, we actually prove a more general theorem concerning the category of free affine spaces over any commutative ring with unity. This duality does not involve any topology or additional relations. But as in the case of vector spaces, it cannot be directly extended to the infinite-dimensional case. Note, however, that in [14], a natural full duality (see e.g. [2] and [4]) was established between the category of all affine spaces over a finite field and the category of topological affine spaces over the same field with constant operations determined by the elements of the field. (For related results see also [5].) Since in general a variety of affine spaces over a commutative ring R does not satisfy the assumptions needed to apply the techniques of natural dualities, it was not possible to extend this result to general affine spaces using the same tools. (Even the extension of natural dualities to the case of an infinite schizophrenic object as proposed in [2, Exercise 2.9, p. 61] does not work in our context. For example the generator $\mathbb{R}$ of the variety of real affine spaces does not admit a compact Hausdorff topology.) Section 5 provides details of the corresponding duality theorems.

Affine spaces are considered here as certain abstract algebras belonging to a broad family of idempotent and entropic algebras or modes, as defined in [17], [21]. Another important class of modes is given by subreducts of affine spaces and certain sums of such algebras. Among them are barycentric

algebras. Particular models of barycentric algebras are provided by convex subsets of real affine spaces, and by semilattices. The basic facts about affine spaces and barycentric algebras are briefly outlined in Section 5 and Section 7.1. Recall the well known duality for semilattices, see [8], [6], and also [18], [19].

The main motivation for studying the topics of this paper was to initiate investigations that would eventually lead to a possible duality for the variety of barycentric algebras. While this is a rather long-term project, some progress has been made on an initial stage. In [15], a duality for quadrilaterals was established. In the current paper, using the above-mentioned duality for free affine spaces, we were able to find a similar type of duality for free subreducts of affine spaces of a given type. This duality is presented in Section 6. As a corollary, in Section 7.1 a duality for finitely generated free barycentric algebras, namely simplices, is developed. The dual objects are hypercubes with two extreme corners identified as constants, and a full duality between the category of simplices and the category of such hypercubes is established. Note that a quite different type of duality for finite-dimensional simplices was presented in [22]. Section 7 provides also two other applications of the above mentioned duality theorems, to free commutative binary modes and to free general binary modes.

With few exceptions, all the dualities considered in this paper are of the classical type, recalled briefly in Section 3, with the free algebra on two free generators as a "schizophrenic" object. (In the case of affine spaces over a commutative ring R, this object is the ring R considered as an affine space, and in the case of barycentric algebras it is the unit interval.)

For algebraic concepts and notations used in this paper, readers are referred to [23] and [21]. In particular, mappings are generally placed in the natural position on the right of their arguments, either in line or as an index. These conventions help to minimise the number of brackets, which otherwise proliferate in the study of non-associative systems such as affine spaces and their (sub)reducts.

2.   Modes

Most algebras considered in this paper are *modes* in the sense of [17] and [21], algebras in which each element forms a singleton subalgebra, and for which each operation is a homomorphism. For algebras (A, Ω) of a given type $\tau : \Omega \to \mathbb{N}$, these two properties are equivalent to satisfaction of the identity

$$(2.1) \qquad\qquad\qquad x \ldots x \omega = x$$

of *idempotence* for each operation ω in Ω, and the identity

$$(2.2) \qquad (x_{11} \ldots x_{1m} \omega) \ldots (x_{n1} \ldots x_{nm} \omega) \phi = (x_{11} \ldots x_{n1} \phi) \ldots (x_{1m} \ldots x_{nm} \phi) \omega$$

of *entropicity* for any two operations in Ω: (m-ary) ω and (n-ary) ϕ. Among the main models of modes are affine spaces, their reducts and subreducts (subalgebras of reducts), in particular convex sets, and normal bands. (For more information see [17] and [21].)

The following theorem providing a characterization of entropicity will play an essential role in this paper. Let $\underline{\tau}$ be the variety of all τ-algebras.

Theorem 2.1 ([17], [21]). *A τ-algebra A is entropic iff for each τ-algebra X, the morphism set $\underline{\tau}(X, A)$ is a subalgebra of the power τ-algebra A^X.*

Corollary 2.2. *If $\underline{K}$ is a prevariety of entropic algebras, then for each pair A, B of $\underline{K}$-algebras, the morphism set $\underline{K}(B, A)$ is again a $\underline{K}$-algebra.*

Recall that the power algebra on the set A^B is defined by

$$\omega : (A^B)^m \to A^B; (f_1, \ldots, f_m) \mapsto f_1 \ldots f_m \omega = (f : B \to A),$$

where

$$f : B \to A; x \mapsto xf = x(f_1 \ldots f_m \omega) = (xf_1 \ldots xf_m)\omega$$

for each (m-ary) ω in Ω.

In particular, Theorem 2.1 and Corollary 2.2 hold for each variety of modes.

3. Duality

In general, a *duality* (or a *dual equivalence of categories*) is a concept of category theory. One says that there is a (full) duality between categories $\mathfrak{A}$ and $\mathfrak{X}$, if there are contravariant functors $D : \mathfrak{A} \to \mathfrak{X}$ and $E : \mathfrak{X} \to \mathfrak{A}$ such that DE is naturally isomorphic with the identity functor $1 : \mathfrak{A} \to \mathfrak{A}$, and similarly ED is naturally isomorphic with the identity functor $1 : \mathfrak{X} \to \mathfrak{X}$

(see, e.g., [11], p. 91). We are interested in the case where $\mathfrak{A}$ is a category of algebras and $\mathfrak{X}$ is a category of relational topological spaces. (Note however that $\mathfrak{X}$ may have only relational or only topological structure.) We speak about a duality for the category $\mathfrak{A}$ or a duality for $\mathfrak{A}$-algebras if such a duality exists.

In many cases, the functors D and E of the duality are represented by a *schizophrenic object* [6], [4], [2]. The schizophrenic object T appears simultaneously as an object $\underline{T}$ of $\mathfrak{A}$ and as an object $\underline{\underline{T}}$ of $\mathfrak{X}$, in such a way that both D and E are hom-functors, $D = \mathfrak{A}(\ ,\underline{T})$ and $E = \mathfrak{X}(\ ,\underline{\underline{T}})$. Moreover, the underlying sets of $\underline{T}$ and $\underline{\underline{T}}$ coincide (with T). The functors D and E are defined on objects and morphisms by

$$D : (f : A \to B) \mapsto (fD : \mathfrak{A}(B, \underline{T}) \to \mathfrak{A}(A, \underline{T}); x \mapsto fx),$$

$$E : (\varphi : X \to Y) \mapsto (\varphi E : \mathfrak{X}(Y, \underline{\underline{T}}) \to \mathfrak{X}(X, \underline{\underline{T}}); \alpha \mapsto \varphi\alpha).$$

The natural isomorphisms e of DE with the identity functor and ε of ED with the identity functor are given by the following evaluations:

$$e_A : A \to ADE; a \mapsto (ae_A : x \mapsto ax),$$

$$\varepsilon_X : X \to XED; x \mapsto (x\varepsilon_X : \alpha \mapsto x\alpha).$$

Most dualities considered in the paper are of this type.

4.   GENERAL CONTEXT OF DUALITIES FOR MODES

In this section we provide a method of representing free algebras in a prevariety of modes. Recall that prevarieties are abstract classes of similar algebras closed under subalgebras and direct products, i.e. classes of the form $\underline{\underline{P}} = \mathsf{SP}(\underline{P})$. Equivalently, prevarieties can be described as classes axiomatized by implications. Prevarieties, when considered as categories with algebras as objects and homomorphisms as morphisms, are complete and cocomplete categories. In particular, each prevariety $\underline{P}$ has all products $\prod_{i\in I} A_i$ and all coproducts $\sum_{i\in I} A_i$ of families of algebras A_i in the prevariety. (See [23], Theorems IV2.1.3 and IV2.2.3.) All algebras considered in this section will be Ω-algebras (A, Ω) of a fixed type $\tau : \Omega \to \mathbb{N}$.

Lemma 4.1 . *Let $\underline{\underline{P}}$ be a prevariety of Ω-modes. Let B and A_i, for $i \in I$, be members of $\underline{P}$. Then there is an Ω-mode isomorphism*

$$(4.1) \qquad \underline{\underline{P}}\left(\sum_{i\in I} A_i, B\right) \cong \prod_{i\in I}\underline{\underline{P}}(A_i, B).$$

Proof. Since $\underline{\underline{P}}$ is a prevariety, all products and all coproducts in $\underline{\underline{P}}$ exist. Since $\underline{\underline{P}}$ is entropic, Corollary 2.2 implies that the sets on both sides of (4.1) also have the structure of $\underline{\underline{P}}$-algebras. Now let $\iota_i : A_i \to \sum_{i\in I} A_i$ be insertions of the coproduct $\sum_{i\in I} A_i$. Define the mapping

$$(4.2) \quad \Phi : \underline{\underline{P}}\left(\sum_{i\in I} A_i, B\right) \to \prod_{i\in I}\underline{\underline{P}}(A_i, B)\,;\ \left(h : \sum_{i\in I} A_i \to B\right) \mapsto (\iota_i h)_{i\in I}.$$

We will show that Φ is an Ω-isomorphism. It is well known that the co-product of algebras in an idempotent prevariety is in fact the so-called *free product*, a coproduct with injective insertions ι_i. In this case the coproduct $\sum_{i\in I} A_i$ is generated by the union $\bigcup_{i\in I}(A_i\iota_i)$. This is an easy consequence of e.g. Corollary 2 of Section 2 in [7]. In particular, each element of $\sum_{i\in I} A_i$ has the form $a_{i_1}\iota_{i_1}\dots a_{i_n}\iota_{i_n}w$ for some Ω-word w. Given homomorphisms $f_i : A_i \to B$ for each i in I, a homomorphism

$$f := \sum_{i\in I} f_i : \sum_{i\in I} A_i \to B :\ a_{i_1}\iota_{i_1}\dots a_{i_n}\iota_{i_n}w \mapsto a_{i_1}f_{i_1}\dots a_{i_n}f_{i_n}w$$

is specified uniquely by the equations

$$\iota_j\left(\sum_{i\in I} f_i\right) = f_j$$

for j in I. Hence the mapping Φ is surjective. It is also injective. To see this, assume that given homomorphisms $g_i : A_i \to B$ for each i in I, one has $f_i = g_i$. Then $a_i\iota_i(\sum_{i\in I} f_i) = a_i f_i = a_i g_i = a_i\iota_i(\sum_{i\in I} g_i)$ for any a_i in A_i. Since all $a_i\iota_i$ for i in I generate $\sum_{i\in I} A_i$, it follows that $\sum_{i\in I} f_i = \sum_{i\in I} g_i$.

To show that Φ is a homomorphism, let (n-ary) ω be in Ω, and let us consider homomorphisms $h_k : \sum_{i\in I} A_i \to B$ for $k = 1, \dots, n$. Note that

$$(h_1\dots h_n\omega)\Phi = (\iota_i(h_1\dots h_n\omega))_{i\in I}.$$

We have to show that for each i in I,

$$\iota_i h_1\dots \iota_i h_n\omega = \iota_i(h_1\dots h_n\omega).$$

Indeed, for each a_i in A_i one has the following:

$$a_i(\iota_i h_1 \ldots \iota_i h_n \omega)$$

$$= (a_i \iota_i h_1) \ldots (a_i \iota_i h_n)\omega$$

$$= ((a_i \iota_i)h_1) \ldots ((a_i \iota_i)h_n)\omega$$

$$= (a_i \iota_i)(h_1 \ldots h_n \omega).$$

$\blacksquare$

For a prevariety $\underline{\underline{P}}$ of Ω-modes, denote by XP the free $\underline{\underline{P}}$-algebra over X and by nP the free $\underline{\underline{P}}$-algebra on n free generators. Note that $0P$ is the empty algebra.

Proposition 4.2. *Let $\underline{\underline{P}}$ be a prevariety of Ω-modes. Then for each free $\underline{\underline{P}}$-algebra XP, the $\underline{\underline{P}}$-algebra $\underline{\underline{P}}(XP, 2P)$ of homomorphisms from XP into $2P$ is isomorphic to the power $\underline{\underline{P}}$-algebra $(2P)^X$,*

$$\underline{\underline{P}}(XP, 2P) \cong (2P)^X.$$

Proof. First note that in the case X is the empty set, the statement above obviously holds. So assume now that X is non-empty. It is well-known that the free algebra XP is isomorphic to the coproduct of the family of free $\underline{\underline{P}}$-algebras $(1P)_x$, for $x \in X$, each on one free generator. (The proof of this fact goes like the case of varieties, see e.g. [1], Theorem I.20.11.) Then Lemma 4.1 implies the following:

$$\underline{\underline{P}}(XP, 2P)$$

$$\cong \underline{\underline{P}}\left(\sum_{x \in X}(1P)_x, 2P\right)$$

$$\cong \prod_{x \in X} \underline{\underline{P}}(\{x\}, 2P)$$

$$\cong (2P)^X,$$

since the free $\underline{\underline{P}}$-algebra on one generator x is isomorphic to the one-element algebra $\{x\}$, and the $\underline{\underline{P}}$-algebra $\underline{\underline{P}}(\{x\}, 2P)$ is isomorphic to $2P$. $\blacksquare$

In particular, Proposition 4.2 shows that the structures dual to the free $\underline{\underline{P}}$-algebras should be the power $\underline{\underline{P}}$-algebras $(2P)^X$, possibly extended by some additional relations or topology. Since other members of the prevariety $\underline{\underline{P}}$ are homomorphic images of the free $\underline{\underline{P}}$-algebras, one may expect that the structures in the category dual to $\underline{\underline{P}}$ will be substructures of the powers $(2P)^X$.

5.   Duality for free affine spaces

Let R be a commutative ring with 1. An *affine space* over the ring R (or an *affine R-space*) can be defined as the reduct $(A, P, \underline{R})$ of a module $(A, +, R)$, where P is the Mal'cev operation

$$P : A^3 \to A; (x_1, x_2, x_3) \mapsto x_1 x_2 x_3 P = x_1 - x_2 + x_3,$$

and $\underline{R}$ is the family of binary operations

$$\underline{r} : A^2 \to A; (x_1, x_2) \mapsto x_1 x_2 \underline{r} = x_1(1 - r) + x_2 r$$

for each $r \in R$. (See, e.g., [17] and [21].) Equivalently, each algebra $(A, P, \underline{R})$ can be described as the (full) idempotent reduct of the corresponding R-module $(A, +, R)$. The class $\underline{\underline{R}}$ of all affine R-spaces $(A, P, \underline{R})$ is a variety.

For a commutative ring R and any natural number n, the free affine R-space $(n + 1)R$ on $n + 1$ free generators, say $x_0, \dots x_n$, is isomorphic to the affine R-space R^n, and consists of all linear combinations

$$x_0 a_0 + \dots + x_n a_n$$

such that $\sum_{i=0}^n a_i = 1$. These facts together with Proposition 4.2 immediately give the following.

Proposition 5.1. *Let R be a commutative ring with 1. Then the affine space $\underline{\underline{R}}(\sum_{x \in X}\{x\}, R)$ of homomorphisms from the free affine R-space over X into the free affine R-space R on two free generators is isomorphic to the affine space R^X. If X is finite, non-empty, consisting of $n + 1$ elements, then*

$$\underline{\underline{R}}\left(\sum_{x \in X}\{x\}, R\right) \cong \underline{\underline{R}}(R^n, R) \cong R^{n+1}.$$

■

Note as well that the constant functions $\breve{0} : R^n \to R; (a_1, \ldots, a_n) \mapsto 0$ and $\breve{1} : R^n \to R; (a_1, \ldots, a_n) \mapsto 1$ correspond to the elements $\bar{0} = (0, \ldots, 0)$ and $\bar{1} = (1, \ldots, 1)$ of R^{n+1}. These two constants generate other constants $\bar{p} = (p, \ldots, p)$ of R^{n+1}, for all $p \in R$. Indeed $\bar{0}\bar{1}\underline{p} = (01\underline{p}, \ldots, 01\underline{p}) = (p, \ldots, p) = \bar{p}$.

Now let $\underline{\breve{R}}$ be the category of affine spaces $\breve{R}^n = (R^n, P, \underline{R}, \bar{0}, \bar{1})$, for non-negative integers n, with two constants $\bar{0}$ and $\bar{1}$. Morphisms of $\underline{\breve{R}}$ are affine space homomorphisms preserving both constants. Note that each homomorphism h preserves also the constants $\bar{p}$ for all $p \in R$, since $\bar{p}h = \bar{0}\bar{1}\underline{p}h = \bar{0}h\bar{1}h\underline{p} = 01\underline{p} = p$.

Corollary 5.2. *For a commutative ring R*

$$\underline{R}(R^n, R) \in \underline{\breve{R}} \text{ and } \underline{R}(R^n, R) \cong \breve{R}^{n+1}.$$

■

Proposition 5.3. *The set $\underline{\breve{R}}(\breve{R}^{n+1}, \breve{R})$ has the structure of an affine R-space. Moreover, this affine space is isomorphic to the affine R-space R^n, i.e.*

$$\underline{\breve{R}}(\breve{R}^{n+1}, \breve{R}) \cong R^n.$$

Proof. First note that by Proposition 5.1, one has the affine space isomorphism

$$\underline{R}(R^{n+1}, R) \cong R^{n+2}.$$

We want to select those homomorphisms of $\underline{R}(R^{n+1}, R)$ that preserve the constants $\bar{0}$ and $\bar{1}$. Let $x_0, x_1, \ldots, x_{n+1}$ be free generators of the affine space R^{n+1}. We may assume that $x_0 = \bar{0}$, and that x_i has 1 in the i-th position and 0 otherwise. By choosing x_0 to be the zero of the module, R^{n+1} becomes a free module on $n+1$ free generators $x_1, \ldots, x_{n+1}$. The affine space homomorphisms preserving $\bar{0}$ are precisely the module homomorphisms. Now for the category $\underline{\underline{Mod}}_R$ of R-modules one has that

$$\underline{\underline{Mod}}_R(R^{n+1}, R) = \underline{\underline{Mod}}_R\left(\sum_{i=1}^{n+1} R, R\right) = \prod_{i=1}^{n+1} \underline{\underline{Mod}}_R(R, R) = R^{n+1}.$$

The affine space homomorphisms preserving $\bar{0}$ form a subspace $\widetilde{R}$ isomorphic to the affine R-space R^{n+1}. The space $\widetilde{R}$ is also equipped with the structure

of a module. In fact this module is also free with free generators $\breve{x}_i$, where $i = 1, \ldots, n + 1$. The generators $\breve{x}_i : R^{n+1} \to R$ are defined on the free generators $x_1, \ldots, x_{n+1}$ of R^{n+1} by $x_i \mapsto 1$ and $x_j \mapsto 0$ for $j \neq i$. Hence each element of $\widetilde{R}$ is a linear combination

$$\breve{x}_1 a_1 + \cdots + \breve{x}_{n+1} a_{n+1},$$

where the a_i are in R. We want our homomorphisms to preserve the constant $\bar{1}$. Note that $\bar{1}$ can also be written as $x_1 + \cdots + x_{n+1}$ and

$$(x_1 + \cdots + x_{n+1})(\breve{x}_1 a_1 + \cdots + \breve{x}_{n+1} a_{n+1})$$

$$= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} x_i \breve{x}_j a_j$$

$$= \sum_{i=1}^{n+1} x_i \breve{x}_i a_i = \sum_{i=1}^{n+1} a_i.$$

Hence for $\bar{1} = x_1 + \cdots + x_{n+1}$ to be mapped to 1, one needs that $\sum_{i=1}^{n+1} a_i = 1$. This gives the set

$$\bar{R} = \left\{ \breve{x}_1 a_1 + \cdots + \breve{x}_{n+1} a_{n+1} \;\middle|\; a_i \in R, \; \sum_{i=1}^{n+1} a_i = 1 \right\}.$$

On the other hand, the affine R-space R^n was described as defined on the set

$$\left\{ x_0 a_0 + \cdots + x_n a_n \;\middle|\; a_i \in R, \; \sum_{i=0}^{n} a_i = 1 \right\}.$$

It is easy to see that by assigning to each generator x_i the generator $\breve{x}_{i+1}$, one obtains the uniquely defined affine R-space isomorphism from R^n to $\bar{R}$. It follows that the affine R-spaces $\bar{R}$ and R^n are isomorphic. ∎

Let $\underline{FR}$ be the category of free affine R-spaces on $n = 0, 1, 2 \ldots$ free generators, with affine space homomorphisms as morphisms. Define two functors $D : \underline{FR} \to \underline{\breve{R}}$ and $E : \underline{\breve{R}} \to \underline{FR}$ as follows:

$$
\begin{array}{ccccc}
R^n & & \underline{R}(R^n, R) & & fh \\
f \downarrow & \longmapsto & \uparrow fD & & \updownarrow \\
R^m & & \underline{R}(R^m, R) & & h
\end{array}
$$

$$
\begin{array}{ccccc}
\breve{R}^{n+1} & & \underline{\breve{\underline{R}}}(\breve{R}^{n+1}, \breve{R}) & & \phi\alpha \\
\phi \downarrow & \longmapsto & \uparrow \phi E & & \uparrow \\
\breve{R}^{m+1} & & \underline{\breve{\underline{R}}}(\breve{R}^{m+1}, \breve{R}) & & \alpha
\end{array}
$$

Theorem 5.4. *The functors D and E establish a full duality between the categories $\underline{FR}$ and $\underline{\breve{R}}$.*

Proof. First note that the empty affine R-space and one-element algebra in $\underline{\breve{R}}$ are dual to each other. Then the fact that both evaluations e and ε are isomorphisms follows easily by Corollary 5.2 and Proposition 5.3, and is proved as for vector spaces. For the evaluation

$$
e_{R^n} : R^n \to \underline{\breve{\underline{R}}}(\underline{R}(R^n, R), \breve{R}) = \underline{\breve{\underline{R}}}(\breve{R}^{n+1}, \breve{R}) \cong R^n,
$$

the morphisms $e_{R^n}(x_i)$ are defined on generators $\breve{x}_0, \ldots, \breve{x}_n$ by $\breve{x}_i \mapsto 1$ and for $j \neq i$ by $\breve{x}_j \mapsto 0$. And for the evaluation

$$
\varepsilon_{\breve{R}^{n+1}} : \breve{R}^{n+1} \to \underline{R}(\underline{\breve{\underline{R}}}(\breve{R}^{n+1}, \breve{R}), R) = \underline{R}(R^n, R) \cong \breve{R}^{n+1},
$$

the morphisms $\varepsilon_{\breve{R}^{n+1}}(\breve{x}_i)$ are defined on generators $x_0, \ldots, x_n$ by $x_i \mapsto 1$ and for $j \neq i$ by $x_j \mapsto 0$. ■

6. Duality for free subreducts of affine spaces

It is well known that subreducts of algebras in a given quasivariety again form a quasivariety. (See [12].) In particular, all subreducts of a given type $\tau : \Omega \to \mathbb{N}$ of affine R-spaces form a quasivariety. (See [21].) Free algebras in such a quasivariety $\underline{\Omega R}$ are also free in the variety generated by $\underline{\Omega R}$. We start with a characterization of free $\underline{\Omega R}$-algebras.

Lemma 6.1. *Let R be a commutative ring with 1. Let $\underline{\Omega R}$ be the quasivariety of Ω-subreducts of affine R-spaces. Then the free $\underline{\underline{\Omega R}}$-algebra $X\Omega R$ over X is isomorphic to the Ω-subreduct $\langle X \rangle_\Omega$, generated by X, of the free affine R-space $(XR, P, \underline{R})$.*

Proof. First note that the universality property for free affine R-spaces assures commutativity of the following diagram for each affine R-space A and a mapping $h : X \to A$:

$$X \overset{e}{\hookrightarrow} \quad \langle X \rangle_\Omega \overset{i}{\hookrightarrow} \quad (XR, \Omega)$$
$$\searrow h \qquad \qquad \nearrow \bar{h}_\Omega$$
$$(A, \Omega)$$

Here $\bar{h}_\Omega$ is the restriction of the (uniquely defined) affine space homomorphism $\bar{h} : XR \to A$ to the Ω-reduct.

Now let (B, Ω) be a subalgebra of (A, Ω). We want to show that for any mapping $h' : X \to B$, there is a (uniquely defined) Ω-homomorphism $\bar{h}' : \langle X \rangle_\Omega \to (B, \Omega)$ such that $e\bar{h}' = h'$. Let $j : B \to A; x \mapsto x$ be an embedding of (B, Ω) into (A, Ω). Take h equal to $h'j$. Then $\bar{h}_\Omega$ maps XR into $j(B)$, and similarly $i\bar{h}_\Omega$ maps $\langle X \rangle_\Omega$ into $j(B)$. Define $\bar{h}' := i\bar{h}_\Omega j^{-1}$. It is easy to see that $\bar{h}'$ satisfies the required properties. Hence $\langle X \rangle_\Omega$ is the free $\underline{\underline{\Omega R}}$-algebra $X\Omega R$ over R. ∎

Note that as in the case of affine spaces, the free $\underline{\underline{\Omega R}}$-algebra over the empty set is the empty algebra. Let J be the set of elements of $2\Omega R$.

Lemma 6.2. *Let R and $\underline{\underline{\Omega R}}$ be defined as in Lemma* 6.1. *Then the set of elements of $X\Omega R$ for $X = \{x_0, \ldots, x_n\}$ coincides with the set*

$$\left\{ x_0 a_0 + \cdots + x_n a_n \,\middle|\, a_i \in J, \sum_{i=0}^{n} a_i = 1 \right\}.$$

Proof. The proof goes by induction with respect to the length of the elements of $X\Omega R$. ∎

Similarly as before, denote by $\underline{\underline{F\Omega R}}$ the category of free $\underline{\underline{\Omega R}}$-algebras on $n = 0, 1, 2, \ldots$ free generators with $\underline{\underline{\Omega R}}$-homomorphisms, and by $\underline{\underline{\breve{\Omega} R}}$ the category of direct products $(2\Omega R)^X$ with $|X| = n$ considered as Ω-algebras $(2\breve{\Omega} R)^X$ with constant operations $\bar{0}$ and $\bar{1}$, and Ω-homomorphisms preserving both constants. Then similarly as before one has the following.

Proposition 6.3. *Let $\underline{\underline{\Omega R}}$ be the quasivariety of Ω-subreducts of affine R-spaces. Then $\underline{\underline{\Omega R}}(X\Omega R, 2\Omega R)$ is a member of $\underline{\underline{\breve{\Omega} R}}$ and is isomorphic to $(2\breve{\Omega} R)^X$. In particular, for a finite non-empty set X with $|X| = n + 1$ one has*

$$\underline{\underline{\Omega R}}((n+1)\Omega R, 2\Omega R) \cong (2\breve{\Omega} R)^{n+1}.$$

∎

Proposition 5.3 is generalized in similar fashion.

Proposition 6.4. *The set $\underline{\breve{\Omega}R}((2\breve{\Omega}R)^{n+1}, 2\breve{\Omega}R)$ has the structure of an $\underline{\Omega R}$-algebra that is isomorphic to $(n+1)\Omega R$.*

Proof. First we embed the algebras $2\breve{\Omega}R$ and $2\breve{\Omega}R^{n+1}$ into corresponding affine spaces $\breve{R}$ and $\breve{R}^{n+1}$, repectively. Then we dualize the space $\breve{R}^{n+1}$ to obtain the affine R-space $\underline{\breve{R}}(\breve{R}^{n+1}, \breve{R})$ isomorphic to R^n, as in Proposition 5.3. Finally we restrict all the homomorphisms of the last space to the set J. In this way we obtain the set

$$\left\{ x_0 a_0 + \cdots + x_n a_n \ \middle| \ a_i \in J, \sum_{i=0}^{n} a_i = 1 \right\},$$

which by Lemma 6.2 is precisely the set of elements of $X\Omega R$. ∎

These two propositions yield a duality between the categories $\underline{F\Omega R}$ and $\underline{\breve{\Omega}R}$, generalizing the duality between the categories $\underline{FR}$ and $\underline{\breve{R}}$ given in Theorem 5.4 with functors D and E defined in a similar way.

Theorem 6.5. *The functors D and E establish a full duality between the categories $\underline{F\Omega R}$ and $\underline{\breve{\Omega}R}$.* ∎

7. APPLICATIONS AND ILLUSTRATIONS

This section will provide some applications of the above duality theorems for certain concrete classes of modes. The first one deals with barycentric algebras.

7.1. Free barycentric algebras.

We only briefly outline the basic facts about barycentric algebras that are needed in this section. For more details, readers may consult [16], [17] or [21]. Let I° denote the open real unit interval $]0, 1[$, and let I denote the closed unit interval $[0, 1]$. For p in I°, define $p' = 1 - p$.

Definition 7.1. A *barycentric algebra* A or $(A, \underline{I}^\circ)$ is an algebra of type $I^\circ \times \{2\}$, equipped with a binary operation

$$\underline{p} : A \times A \to A; (x, y) \mapsto xy\,\underline{p}$$

for each p in I°, satisfying the identities

$$(7.3) \qquad\qquad xx\,\underline{p} = x$$

of *idempotence* for each p in I°, the identities

$$(7.4) \qquad\qquad xy\,\underline{p} = yx\,\underline{p'}$$

of *skew-commutativity* for each p in I°, and the identities

$$(7.5) \qquad\qquad xy\,\underline{p}\ z\,\underline{q} = x\ yz\,\underline{q/(p'q')'}\ \underline{(p'q')'}$$

of *skew-associativity* for each p, q in I°. The variety of all barycentric algebras, construed as a category with the homomorphisms as morphisms, is denoted by $\underline{B}$.

A convex set C forms a barycentric algebra $(C, \underline{I}^\circ)$, with $xy\,\underline{p} = x(1-p) + yp$ for x, y in C and p in I°. A semilattice $(S, \cdot)$ becomes a barycentric algebra on setting $xy\,\underline{p} = x \cdot y$ for x, y in S and p in I°. Each barycentric algebra is a homomorphic image of a convex set. Moreover convex sets are precisely those barycentric algebras that are embeddable as subreducts into real affine spaces.

For the following result, see [13], §2.1 of [17] and §5.8 of [21].

Theorem 7.2. *Let X be a finite set. The following structures are equivalent:*

  (a) *The free barycentric algebra XB on X;*

  (b) *The simplex spanned by X;*

  (c) *The set of all probability distributions on X.* ∎

Let $\underline{S}$ denote the category of finite-dimensional simplices Δ_n with barycentric algebra homomorphisms. Let $\underline{\underline{B}}$ denote the category of hypercubes $\breve{I}^n$ with barycentric algebra homomorphisms preserving both constants $\bar{0}$ and $\bar{1}$. Now Theorem 7.2 and Proposition 6.3 immediately imply the following propositions and theorem.

Proposition 7.3. *The set $\underline{S}(\Delta_n, I)$ has the structure of a $\underline{\underline{B}}$-algebra isomorphic to the hypercube $\breve{I}^{n+1}$.* ∎

Proposition 7.4. *The set $\underline{\underline{B}}(\breve{I}^{n+1}, \breve{I})$ has the structure of an $\underline{S}$-algebra isomorphic to the n-dimensional simplex Δ_n.* ∎

Theorem 7.5 . *The categories $\underline{S}$ of finite dimensional simplices and $\underline{\breve{B}}$ of hypercubes are dually equivalent.* ∎

7.2. Commutative binary and quasigroup modes.

A *quasigroup mode* is an idempotent and entropic quasigroup. It is *commutative* in the case the quasigroup multiplication is commutative. As a Mal'cev algebra each such quasigroup is equivalent to an affine space. In fact, the variety $\underline{CQM}$ of commutative quasigroup modes is known to be equivalent to the variety $\mathbf{D}$ of affine spaces over the ring $\mathbf{D}$ of rational dyadic numbers $m2^{-n}$ where m and n runs over all integers. (See e.g. [17].) In particular the quasigroup $(\mathbf{D}, \cdot, \diagup, \diagdown) = (\mathbf{D}, \underline{2^{-1}}, \underline{-1}, \underline{2})$ is a free $\underline{CQM}$-quasigroup on two free generators 0 and 1. The duality obtained for general free affine $\mathbf{D}$-spaces carry over to free $\underline{CQM}$-quasigroups.

A *commutative binary mode* is a mode with one commutative binary operation. The variety $\underline{CBM}$ of commutative binary modes has as a free object over a set X, the subgroupoid generated by X of the multiplicative reduct of the free $\underline{CQM}$-quasigroup over X (or equivalently of the corresponding affine $\mathbf{D}$-space.) See [9], [17], [21] for more details. In particular, the set $\mathbf{D}_1 = \mathbf{D} \cap I$ under the operation $\underline{2^{-1}}$ is a free $\underline{CBM}$-mode on two generators. Define a *dyadic hypercube* to be a power of the binary mode $\mathbf{D}_1$ with constants. As in the case of free barycentric algebras, the assumptions of Theorem 6.5 are satisfied, and one obtains a theorem analogous to 7.5.

Theorem 7.6 . *The category of free commutative binary modes is dually equivalent to the category of dyadic hypercubes.* ∎

7.3. Free binary modes.

The final application concerns free *binary modes*, modes with one binary multiplication. Such modes are cancellative, and hence embed as subreducts into affine spaces. (See [10], [20] and [21].) To be more specific, let $\mathbf{Z}[M]$ be the ring of integer polynomials with one indeterminate M. Consider the multiplicative reduct of the free affine $\mathbf{Z}[M]$-space $X\mathbf{Z}[M]$ over X defined by

$$(7.6) \qquad\qquad\qquad a \cdot b = ab\underline{M}.$$

Then the following holds for the variety $\underline{BM}$ of all binary modes.

Theorem 7.7 [21]. *The subgroupoid generated by X of the groupoid $(X\mathbf{Z}[M], \cdot)$ is the free binary mode XBM over the set X.* ∎

In particular, the free binary mode on $n + 1$ generators is (isomorphic to) the subgroupoid of $(\mathbf{Z}[M]^n, \cdot)$ generated by the (canonical) basis of the corresponding affine space. As in the previous examples, the assumptions of Theorem 6.5 are satisfied, and one obtains the following.

Theorem 7.8. *The category of free binary modes is dually equivalent to the category of powers of* $(\mathbf{Z}[M], \cdot, \bar{0}, \bar{1})$.   ∎

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