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DUALITY FOR QUADRILATERALS

K. J. PSZCZOŁA, A. B. ROMANOWSKA, AND J. D. H. SMITH

ABSTRACT. Based on the duality for finite-dimensional simplices presented in [PRS02], we construct a duality for quadrilaterals (two-dimensional affine images of the three-dimensional simplex Δ_3).

One of the major research programmes in the theory of modes is the search for a duality theory of barycentric algebras. Since a complete theory seems quite elusive at the moment, the programme is proceeding by the gradual assembly of various pieces. Of course, the starting point is the now classical duality theory for semilattices (see [HMS74]). Recently, progress has been made on duality for certain convex subsets of simplices (the so-called *feasible sets* of ordered sets [RS01]) and, in a different direction, for finite-dimensional simplices themselves (see [PRS02]), the latter being finitely generated free barycentric algebras. The next stage is to find an algebraic duality for polytopes in real affine spaces, as homomorphic images of finite-dimensional simplices. In this paper we show that there exists a duality between the class of all quadrilaterals (two-dimensional affine images of the three-dimensional simplex Δ_3) and a certain class of topological convex sets.

Duality for finite-dimensional simplices can be formulated as follows (see [PRS02]):

Theorem 1 ([PRS02]). *There exists a duality between the category $\underline{\underline{\mathrm{S}}}$ of finite-dimensional simplices and the category $\underline{\underline{\breve{\mathrm{B}}}}$ of hypercubes with constants. The duality is realised by functors*

$$\mathrm{D} : \underline{\underline{\mathrm{S}}} \to \underline{\underline{\breve{\mathrm{B}}}};\ \Delta_n \mapsto \underline{\underline{\mathrm{S}}}(\Delta_n, \mathrm{I}) \cong \breve{\mathrm{I}}^{n+1} \quad \textit{and} \quad \mathrm{E} : \underline{\underline{\breve{\mathrm{B}}}} \to \underline{\underline{\mathrm{S}}};\ \breve{\mathrm{I}}^{n+1} \mapsto \underline{\underline{\breve{\mathrm{B}}}}(\breve{\mathrm{I}}^{n+1}, \breve{\mathrm{I}}) \cong \Delta_n,$$

acting on morphisms in a standard fashion:

$$D : (f : \Delta_n \to \Delta_m) \mapsto (fD : \Delta_m D \to \Delta_n D);\ afD = fa,$$

where $a \in \Delta_m D = \underline{\underline{\mathrm{S}}}(\Delta_m, \mathrm{I})$, and

$$E : (g : \breve{\mathrm{I}}^k \to \breve{\mathrm{I}}^l) \mapsto (gE : \breve{\mathrm{I}}^l E \to \breve{\mathrm{I}}^k E);\ xgE = gx,$$

where $x \in \breve{\mathrm{I}}^l E = \underline{\underline{\breve{\mathrm{B}}}}(\breve{\mathrm{I}}^l, \breve{\mathrm{I}})$.

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One of the major research programmes in the theory of modes is the search for a duality theory of barycentric algebras. Since a complete theory seems quite elusive at the moment, the programme is proceeding by the gradual assembly of various pieces. Of course, the starting point is the now classical duality theory for semilattices (see [HMS74]). Recently, progress has been made on duality for certain convex subsets of simplices (the so-called *feasible sets* of ordered sets [RS01]) and, in a different direction, for finite-dimensional simplices themselves (see [PRS02]), the latter being finitely generated free barycentric algebras. The next stage is to find an algebraic duality for polytopes in real affine spaces, as homomorphic images of finite-dimensional simplices. In this paper we show that there exists a duality between the class of all quadrilaterals (two-dimensional affine images of the three-dimensional simplex Δ_3) and a certain class of topological convex sets.

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where $x \in \breve{\mathrm{I}}^l E = \underline{\underline{\breve{\mathrm{B}}}}(\breve{\mathrm{I}}^l, \breve{\mathrm{I}})$.

The structure dual to the simplex Δ_3 is the hypercube $\check{\mathrm{I}}^4$. We expect the structure dual to a homomorphic image of Δ_3 to be a subalgebra of the hypercube $\check{\mathrm{I}}^4$.

For each $n \in \mathbb{N}$ there is, up to isomorphism, exactly one simplex Δ_n. For quadrilaterals, the situation is different: there are infinitely many non-isomorphic quadrilaterals. In fact, two proper quadrilaterals are isomorphic as barycentric algebras if and only if their diagonals intersect in the same proportions. For a quadrilateral with vertices a, b, c, d, the condition that the diagonals intersect in the proportions p, q for some $p, q \in \mathrm{I}$, becomes $ac\underline{p} = bd\underline{q}$. When speaking of a "concrete" quadrilateral, we will thus mean a quadrilateral with given parameters p and q.

We introduce the following notation. Let $\underline{\underline{\mathrm{B}}}$ be the category of all barycentric algebras (and homomorphisms between them). Then $\check{\underline{\underline{\mathrm{B}}}}$ will denote the category of barycentric algebras with a unit interval of constants (cf. [PRS02]). Let $\underline{\underline{\mathrm{Q}}}$ denote the full subcategory of $\underline{\underline{\mathrm{B}}}$ whose objects are all quadrilaterals. Then $\widehat{\underline{\underline{\mathrm{Q}}}}$ will denote the category of barycentric algebras with a unit interval of constants, each of whose objects is of the form $\underline{\underline{\mathrm{B}}}(\mathrm{Q}, \mathrm{I})$ for a certain object Q of $\underline{\underline{\mathrm{Q}}}$. We adopt the convention that $\widehat{\ }$ will denote dual spaces and objects, while $\widehat{\widehat{\ }}$ will refer to the second dual.

For a given quadrilateral Q specified by the parameters p, q we will describe the subalgebra $\widehat{\mathrm{Q}} = \underline{\underline{\mathrm{B}}}(\mathrm{Q}, \mathrm{I}) \leq \underline{\underline{\mathrm{B}}}(\Delta_3, \mathrm{I}) \cong \check{\mathrm{I}}^4$.

The elements of the hypercube $\check{\mathrm{I}}^4 \cong \underline{\underline{\mathrm{B}}}(\Delta_3, \mathrm{I})$ correspond to the affine homomorphisms $\Delta_3 \to \mathrm{I}$ from the free barycentric algebra on 4 generators x_0, x_1, x_2, x_3. These homomorphisms are uniquely determined by their values on the free generators; the i-th coordinate of a point of the hypercube represents the image of the i-th free generator under the homomorphism.

The elements of the subalgebra $\widehat{\mathrm{Q}} = \underline{\underline{\mathrm{B}}}(\mathrm{Q}, \mathrm{I})$ correspond to the homomorphisms $\mathrm{Q} \to \mathrm{I}$. The i-th coordinate of such an element represents the image under the homomorphism of the i-th vertex of the quadrilateral Q. Denote the vertices of the quadrilateral by x_0h, x_1h, x_2h, x_3h. Because the diagonals of the quadrilateral intersect in the proportions p and q, the vertices satisfy the equation $x_0hx_2h\underline{p} = x_1hx_3h\underline{q}$. It follows that the coefficients (x, y, z, t) of each element of the subalgebra satisfy the corresponding equation

$$xz\underline{p} = yt\underline{q}. \tag{1}$$

Note that the condition (1) may be written in the form $x(1-p) - y(1-q) + zp - tq = 0$. This is the equation of a hyperplane in four-dimensional space, containing the point $\check{0}$. The desired figure $\widehat{\mathrm{Q}}$ is the intersection of the hypercube $\check{\mathrm{I}}^4$ with the hyperplane determined by equation (1). It is a polytope spanned by points lying on edges of the hypercube.

In order for a point with coordinates (x, y, z, t) to lie on an edge of the hypercube, at most one of the coordinates may differ from 0 or 1. If three of the coordinates are zero, then (1) implies that all four coordinates are zero. Consider the case where exactly two of the coordinates are zero. This may happen in any of $\binom{4}{2} = 6$ ways. In each such case, one of the two remaining coordinates will be 1, and the value of the remaining coordinate is completely determined by (1).

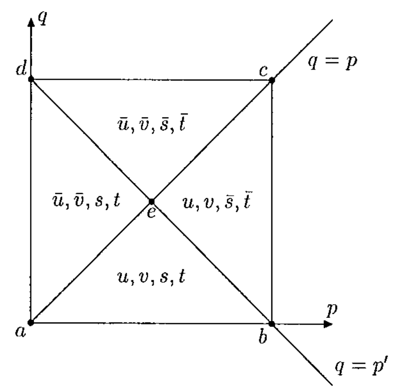
If the condition $xz\underline{p} = yt\underline{q}$ holds, then $x - xp + zp = y - yq + tq$, or equivalently $1 - x - p + xp + p - zp = 1 - y - q + yq + q - tq$. This in turn reduces to $(1-x)(1-z)\underline{p} = (1-y)(1-t)\underline{q}$. In other words, if (x, y, z, t) is a vertex of the figure, then so is $(1-x, 1-y, 1-z, 1-t)$. Certainly the points $(0,0,0,0)$ and $(1,1,1,1)$ are also vertices of the figure. Thus in total there are $\binom{4}{2} \cdot 2 \cdot 2 + 2 = 26$ potential vertices. They are described and named in Table 1, together with the conditions specifying the values of p and q for which each potential vertex does not fall outside the hypercube.

TABLE 1. Vertices of the image of the quadrilateral p, q lying on edges of the hypercube I^4

$\check{0}(0,0,0,0)$	$\check{1}(1,1,1,1)$	
$u(1, \frac{1-p}{1-q}, 0, 0)$	$u'(0, \frac{p-q}{1-q}, 1, 1)$	for $p \geq q$
$v(1, 1, \frac{p-q}{p}, 0)$	$v'(0, 0, \frac{q}{p}, 1)$	for $p \geq q$
$\bar{u}(\frac{1-q}{1-p}, 1, 0, 0)$	$\bar{u}'(\frac{q-p}{1-p}, 0, 1, 1)$	for $p \leq q$
$\bar{v}(1, 1, 0, \frac{q-p}{q})$	$\bar{v}'(0, 0, 1, \frac{p}{q})$	for $p \leq q$
$s(\frac{1-p-q}{1-p}, 1, 1, 0)$	$s'(\frac{q}{1-p}, 0, 0, 1)$	for $p \leq q'$
$t(0, \frac{p}{1-q}, 1, 0)$	$t'(1, \frac{1-q-p}{1-q}, 0, 1)$	for $p \leq q'$
$\bar{s}(0, 1, 1, \frac{q+p-1}{q})$	$\bar{s}'(1, 0, 0, \frac{1-p}{q})$	for $p \geq q'$
$\bar{t}(0, 1, \frac{1-q}{p}, 0)$	$\bar{t}'(1, 0, \frac{p+q-1}{p}, 1)$	for $p \geq q'$
$p_0(\frac{p}{p-1}, 0, 1, 0)$	$p_0'(\frac{1}{1-p}, 1, 0, 1)$	for $p = 0$
$p_1(1, 0, \frac{p-1}{p}, 0)$	$p_1'(0, 1, \frac{1}{p}, 1)$	for $p = 1$
$q_0(1, \frac{1}{1-q}, 1, 0)$	$q_0'(0, \frac{q}{q-1}, 0, 1)$	for $q = 0$
$q_1(0, 1, 0, \frac{q-1}{q})$	$q_1'(1, 0, 1, \frac{1}{q})$	for $q = 1$

The key task of the paper is to demonstrate the isomorphism $\breve{\underline{\underline{\mathrm{B}}}}(\underline{\underline{\mathrm{B}}}(\mathrm{Q}, \mathrm{I}), \check{\mathrm{I}}) \cong \mathrm{Q}$, enabling one to recover (the isomorphism class of) the quadrilateral barycentric algebra Q from its dual $\underline{\underline{\mathrm{B}}}(\mathrm{Q}, \mathrm{I})$. Depending on the values of p and q, there are 17 cases to consider: abe for $q < p$ and $q < p'$; bce for $q < p$ and $q > p'$; cde for $q > p$ and $q > p'$; ade for $q > p$ and $q < p'$; ae for $q = p$ and $q < p'$; be for $q < p$ and $q = p'$; ce for $q = p$ and $q > p'$; de for $q > p$ and $q = p'$, for $p, q \in \mathrm{I}^0$; ab for $q = 0, p \in \mathrm{I}^0$; bc for $p = 1, q \in \mathrm{I}^0$; cd for $q = 1, p \in \mathrm{I}^0$; da for $p = 0, q \in \mathrm{I}^0$; a for $q = 0$ and $p = 0$; b for $q = 0$ and $p = 1$; c for $q = 1$ and $p = 1$; d for $q = 1$ and $p = 0$; e for $p = q = \frac{1}{2}$. The names of the various cases refer to Figure 1. Note that the cases correspond to the first simplicial decomposition of the square. Note also that there are really only 5 different cases: e, x, xe, xy and xye, where $x, y \in \{a, b, c, d\}$ and $x \neq y$. Each of the 17 cases falls into one of these 5 types.

FIGURE 1



Recall the following (see e.g. [RS85], Proposition 159):

Theorem 2. *A variety* $\underline{\underline{\mathrm{V}}}$ *is a variety of entropic algebras if and only if, for each pair of* $\underline{\underline{\mathrm{V}}}$*-algebras* A *and* B*, the set* $\underline{\underline{\mathrm{V}}}(A, B)$ *is a subalgebra of the* $\underline{\underline{\mathrm{V}}}$*-algebra* B^A*.*

Because barycentric algebras are entropic, one has the following:

Corollary 3. *For each pair of barycentric algebras* A *and* B*, the set* $\underline{\underline{\mathrm{B}}}(A, B)$ *is a barycentric algebra.*

We now formulate the main result:

Theorem 4. *There is a duality between the category* $\underline{\underline{\mathrm{Q}}}$ *of quadrilaterals and the category* $\widehat{\underline{\underline{\mathrm{Q}}}}$ *of hypercube subalgebras with constants. The duality is realised by functors*

$$\mathrm{D} : \underline{\underline{\mathrm{Q}}} \to \widehat{\underline{\underline{\mathrm{Q}}}};\ \mathrm{Q} \mapsto \underline{\underline{\mathrm{B}}}(\mathrm{Q}, \mathrm{I})$$

and

$$\mathrm{E} : \widehat{\underline{\underline{\mathrm{Q}}}} \to \underline{\underline{\mathrm{Q}}};\ \widehat{\mathrm{Q}} \mapsto \underline{\underline{\breve{\mathrm{B}}}}(\widehat{\mathrm{Q}}, \breve{\mathrm{I}}),$$

with standard actions on morphisms

$$D : (f : \mathrm{Q} \to \mathrm{Q}') \mapsto (fD : \mathrm{Q}'D \to \mathrm{Q}D);\ afD = fa$$

for $a \in \mathrm{Q}'D = \underline{\underline{\mathrm{B}}}(\mathrm{Q}', \mathrm{I})$ *and*

$$E : (g : \widehat{\mathrm{Q}} \to \widehat{\mathrm{Q}}') \mapsto (gE : \widehat{\mathrm{Q}}'E \to \widehat{\mathrm{Q}}E);\ xgE = gx$$

for $x \in \widehat{\mathrm{Q}}'E = \breve{\underline{\underline{\mathrm{B}}}}(\widehat{\mathrm{Q}}', \breve{\mathrm{I}})$.

Proof. We will consider the cases abe, be, ab, b and e. We will show that in each of these cases, there is a barycentric algebra isomorphism $\breve{\underline{\underline{\mathrm{B}}}}(\underline{\underline{\mathrm{B}}}(\mathrm{Q}, \mathrm{I}), \breve{\mathrm{I}}) \cong \mathrm{Q}$.

Since we are fixing the constant $\breve{0}$, we may use the language of vectors, identifying each point α with the vector from $\breve{0}$ to α. We wish to determine a minimal set of vertices specifying the dual subalgebra $\widehat{\mathrm{Q}}$. For a given point α, set $\alpha' = \breve{1} - \alpha$. For each α in the potential vertex set $\{u, v, s, t, \bar{u}, \bar{v}, \bar{s}, \bar{t}, p_0, p_1, q_0, q_1\}$, one also has α' in the set. It thus suffices to choose just one of α and α'. Similarly, since $\breve{0}$ and $\breve{1}$ are always vertices, one does not need to mention them explicitly.

In dealing with each separate case we will assume, unless explicitly stated otherwise, that under $\widehat{\mathrm{Q}} \to \breve{\mathrm{I}}$ we have $u \mapsto x$ and $v \mapsto y$ for certain $x, y \in \mathrm{I}$. We will then examine the additional conditions on x and y that are needed to obtain a homomorphism. As the second dual we thus expect to obtain a certain subset of the square $\mathrm{I}^2 = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

CASE abe: Assume $p > q$ and $p < q'$. The dual subalgebra $\widehat{\mathrm{Q}}_1$ of the hypercube is then spanned by the points $\breve{0}, u, v, s, t, u', v', s', t', \breve{1}$. As observed earlier, it suffices to consider the points u, v, s, t, which already specify the subalgebra uniquely. Note that the matrix consisting of the vectors $\vec{u}, \vec{v}, \vec{s}, \vec{t}$ has rank 2, since the matrices

$$\begin{bmatrix} 1 & 1 & \frac{1-p-q}{1-p} & 0 \\ \frac{1-p}{1-q} & 1 & 1 & \frac{p}{1-q} \\ 0 & \frac{p-q}{p} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1-p}{1-q} & \frac{1-p}{1-q} & \frac{1-p-q}{1-q} & 0 \\ 0 & \frac{p-q}{p} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are row-equivalent. Express the vectors $\vec{s}$ and $\vec{t}$ as linear combinations of the vectors $\vec{u}$ and $\vec{v}$:

$$\vec{s} = \frac{q}{q-p}\frac{1-q}{1-p}\,\vec{u} + \frac{p}{p-q}\,\vec{v},$$

$$\vec{t} = \frac{p}{q-p}\,\vec{u} + \frac{p}{p-q}\,\vec{v}.$$

In order for the vertices s and t to lie in the interval I, the following conditions must be satisfied:

$$0 \le \frac{q}{q-p}\frac{1-q}{1-p}x + \frac{p}{p-q}y \le 1,$$

$$0 \leq \frac{p}{q-p}x + \frac{p}{p-q}y \leq 1.$$

These may be cast in the form

1. $y \geq \frac{q}{p}\frac{1-q}{1-p}x$,
2. $y \leq \frac{q}{p}\frac{1-q}{1-p}x + \frac{p-q}{p}$,
3. $y \geq x$,
4. $y \leq x + \frac{p-q}{p}$,

reducing in turn to the conditions 2 and 3. These conditions determine a certain subset $\widehat{\widehat{\mathrm{Q}}}_1$ of the square I^2. Note that $\widehat{\widehat{\mathrm{Q}}}_1$ is a quadrilateral whose corners lie at the points $a(0,0)$, $b(1,1)$, $c(\frac{1-p}{1-q},1)$ and $d(0,\frac{p-q}{p})$. The diagonal from a to c has the equation

$$y = \frac{1-q}{1-p}\, x.$$

The diagonal from d to b has the equation

$$y = \frac{q}{p}\, x + \frac{p-q}{p}.$$

The point of intersection of these diagonals is $s(1-p, 1-q)$. Thus

$$\vec{as} = [1-p, 1-q], \quad \vec{sc} = [\frac{q(1-p)}{1-q}, q] = \frac{q}{1-q}\,\vec{as} \quad \text{and}$$

$$\frac{|\vec{sc}|}{|\vec{ac}|} = \frac{\frac{q}{1-q}}{1+\frac{q}{1-q}} = q.$$

Similarly

$$\vec{ds} = [1-p, \frac{q}{p}(1-p)], \quad \vec{sb} = [p,q] = \frac{p}{1-p}\,\vec{ds} \quad \text{and}$$

$$\frac{|\vec{sb}|}{|\vec{db}|} = \frac{\frac{p}{1-p}}{1+\frac{p}{1-p}} = p.$$

It thus follows that the second dual $\widehat{\widehat{\mathrm{Q}}}_1$ is isomorphic to the original quadrilateral Q_1.

CASE *be*: Assume $p > q$ and $p = q'$. The dual subalgebra $\widehat{\mathrm{Q}}_2$ of the hypercube is then spanned by the points u, v, $s = t(0,1,1,0)$. Note once again that the matrix consisting of the vectors $\vec{u}, \vec{v}, \vec{s}, \vec{t}$ has rank 2. Express the vector $\vec{s}$ as a linear combination of the vectors $\vec{u}$ and $\vec{v}$:

$$\vec{s} = \vec{t} = -\frac{p}{p-q}\,\vec{u} + \frac{p}{p-q}\,\vec{v}.$$

We have the following condition:

$$0 \leq -\frac{p}{p-q}x + \frac{p}{p-q}y \leq 1,$$

which may be formulated as

$$x \leq y \leq x + \frac{p - q}{p}.$$

This condition determines a certain subset $\widehat{\widehat{Q}}_2$ of the square I^2. Note that $\widehat{\widehat{Q}}_2$ is the quadrilateral with the vertices $a(0,0)$, $b(1,1)$, $c(\frac{q}{p}, 1)$ and $d(0, \frac{p-q}{p})$. As in the previous case, we determine the point of intersection of its diagonals. The diagonal from a to c is

$$y = \frac{p}{q}\, x.$$

The diagonal from d to b is

$$y = \frac{q}{p}\, x + \frac{p - q}{p}.$$

Their point of intersection is $s(\frac{q}{p+q}, \frac{p}{p+q})$. Thus

$$\vec{as} = [\frac{q}{p+q}, \frac{p}{p+q}], \quad \vec{sc} = [\frac{q^2}{p(p+q)}, \frac{q}{p+q}] = \frac{q}{p}\, \vec{as} \quad \text{and}$$

$$\frac{|\vec{as}|}{|\vec{ac}|} = \frac{1}{1 + \frac{q}{p}} = p$$

(using the assumption $p = q'$). Similarly

$$\vec{ds} = [\frac{q}{p+q}, \frac{q^2}{p(p+q)}], \quad \vec{sb} = [\frac{p}{p+q}, \frac{q}{p+q}] = \frac{p}{q}\, \vec{ds} \quad \text{and}$$

$$\frac{|\vec{ds}|}{|\vec{db}|} = \frac{1}{1 + \frac{p}{q}} = q.$$

Thus again the second dual $\widehat{\widehat{Q}}_2$ is isomorphic to the original quadrilateral Q_2.

CASE *ab*: Assume $q = 0$. The dual subalgebra $\widehat{Q}_3$ of the hypercube is then determined by the points $u(1, 1-p, 0, 0)$, $v = s = q_0(1, 1, 1, 0)$, $t(0, p, 1, 0)$. The matrix composed of the vectors $\vec{u}, \vec{v}, \vec{s}, \vec{t}, \vec{q_0}$ has rank 2. Express the vector $\vec{t}$ as a linear combination of the vectors $\vec{u}$ and $\vec{v}$:

$$\vec{t} = \vec{v} - \vec{u}.$$

We have the following condition:

$$0 \leq y - x \leq 1,$$

which may be cast in the form

$$x \leq y \leq x + 1.$$

This condition determines a certain subset $\widehat{\widehat{Q}}_3$ of the square I^2. In this case $\widehat{\widehat{Q}}_3$ is a triangle with the vertices $a(0,0)$, $b(1,1)$ and $c(0,1)$. The triangle is a degenerate quadrilateral, in which the diagonals intersect on one of the sides. Thus the second dual $\widehat{\widehat{\mathrm{Q}}}_3$ is isomorphic to the original figure Q_3.

CASE *b*: Assume $q = 0$ and $p = 1$. The dual subalgebra $\widehat{\mathrm{Q}}_4$ of the hypercube is then determined by the points $u = p_1(1,0,0,0)$, $v = s = q_0(1,1,1,0)$, $t(0,1,1,0)$. The matrix consisting of the vectors $\vec{u}, \vec{v}, \vec{s}, \vec{t}, \vec{q_0}, \vec{p_1}$ is of rank 2. Proceeding as in the previous cases, one may express the vector $\vec{t}$ as a linear combination of the vectors $\vec{u}$ and $\vec{v}$, namely $\vec{t} = \vec{v} - \vec{u}$. We thus have the same situation as in the preceeding case.

CASE *e*: Assume $p = q$ and $p = q'$. The assumption implies $p = q = \frac{1}{2}$. In this case the dual subalgebra $\widehat{\mathrm{Q}}_5$ of the hypercube is determined by the points $u = v(1,1,0,0)$, $s = t(0,1,1,0)$. The matrix consisting of the vectors $\vec{u}, \vec{v}, \vec{s}, \vec{t}$ of course again has rank 2. Consider $u \mapsto x$, $s \mapsto y$. Since there are no additional conditions, we obtain the entire square $\mathrm{I}^2 = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ as the second dual $\widehat{\widehat{\mathrm{Q}}}_5$. It is isomorphic with the original parallelogram. Now from Corollary 3 we can deduce that $\underline{\underline{\mathrm{B}}}(\mathrm{Q},\mathrm{I})$ is an object of the category $\underline{\underline{\widehat{\mathrm{Q}}}}$ and $\underline{\underline{\breve{\mathrm{B}}}}(\widehat{\mathrm{Q}},\breve{\mathrm{I}})$ is an object of the category $\underline{\underline{\mathrm{Q}}}$. Furthermore, the functors E and D are well-defined.

The proof of Theorem 4 is now complete. □

Remark 5. In all the various cases, the second dual contains the diagonal of the square. In case *e* this diagonal is internal, but in the other cases it appears as one of the edges of the second dual.

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