COMBINATORIAL APPROACHES TO THE UNIT GROUPS OF CLIFFORD ALGEBRAS, II

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Abstract. This is the second part of a two-part paper intended to introduce the use of Latin square and quasigroup techniques in a combinatorial approach to the unit groups of Clifford algebras. The aim of this part is to apply the techniques to a detailed study of the 32-element unit group of the real Clifford algebra of Minkowski spacetime, which we describe as the Pauli hull. This important special case deserves attention, both because of its connections with special relativity, and also as a home for multiple copies of the Pauli group of quantum information theory. Particular attention focuses on the automorphism group of the Pauli hull, and on the configurations of the copies of the Pauli group within the Pauli hull.

Contents

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1. Introduction

As the second part of a two-part combinatorial approach to the unit groups of Clifford algebras, this paper is a direct continuation of the first part, [11], where the motivation and requisite background for the approach were presented. In particular, we recall the *geometric* and *multivalent* representations of the multiplication group of a group, as identified in [11, §1.3]. We also remind readers of the conventions for terminology and notation established in [11, §1.3.4] and elsewhere in that paper.

1.1. **The extraspecial Pauli hull.** The unit group of the real Clifford algebra $\mathrm{Cl}(3,1)$ of Minkowski space (with signature $+++-$) has order 32. It is the extraspecial 2-group E_2 in the notation of [4, p.405], or the group N_3 in the notation of [13, Table III]. In our discussions, it will mainly feature as a home for copies of the Pauli group. Thus, we refer to it as the *Pauli hull* (§2.2).

Section 2 begins with a reasonably comprehensive description of the dihedral group D_4 as the multiplication group of the quasigroup of fourth roots of unity in $\mathbb{C}$, with the scalar product $x\overline{y}$ as the quasigroup product (§2.1). Using the geometric representation for multiplication groups of groups, the Pauli hull is presented as the multiplication group of D_4 (§2.2). The Pauli group appears naturally inside the Pauli hull as the real Clifford algebra unit group $D_4 \odot C_4$, the unit group of the real Clifford algebras $\mathrm{Cl}(3,0)$ and $\mathrm{Cl}(1,2)$ [11, §3.7]. The quasigroup-theoretical description of the Pauli group matches nicely with other descriptions that are familiar from quantum information theory (§2.3). In addition, the Pauli hull contains a second real Clifford algebra unit group, namely $D_4 \times C_2$, the unit group of $\mathrm{Cl}(2,1)$.

1.2. **Quaternions.** Whereas Section 2 treated the Pauli hull as the multiplication group of D_4, Section 3 gives a parallel treatment of the Pauli hull as the multiplication group of the quaternion group Q_8. At first, the geometric representation of group multiplication groups is used for identification of the quaternion group within the Pauli group (Proposition 3.1), and to describe the Pauli hull as the multiplication group of Q_8 (§3.2). Table 1 in §3.3 provides a dictionary to correlate the two geometric representations of the Pauli group inside the Pauli hull (in terms of D_4 and Q_8) with its abstract presentation, with its

Pauli matrix description, and with its occurence as the unit group of the real Clifford algebra of three-dimensional Euclidean space.

Subsequently, the multivalent representation assumes the primary role. This is particularly seen in the account of the automorphism group of the Pauli hull presented in §3.4, based on the identification of the automorphism group of Q_8 with the group of symmetries of the Bloch octahedron (§3.4.1). As a toy qubit model, the Bloch octahedron is spanned by 1-qubit stabiliser states (cf. [1, Fig. 4], [7, Fig. 2.1]), geometrically dual to the *quantum cube* of [1].

The six vertices of the Bloch octahedron are usually defined in linear-algebraic terms as eigenspaces of maximal abelian subgroups of the Pauli group. An alternative view emerges from Theorem 3.19 in §3.5, which identifies the six copies of the Pauli group that appear in the Pauli hull as a single orbit under its automorphism group. Table 2 presents their respective central quotients (phase spaces). Table 3 records the combinatorial structure underlying the distribution of the 12 elements of order 4 in the Pauli hull among its six Pauli subgroups. Finally, Proposition 3.20 identifies the setwise stabiliser of a Pauli subgroup under the action of the automorphism group of the Pauli hull as the group $D_4 \times S_4$ of order 192.

1.3. **Future development.** The most immediate issue for subsequent development of this part of our work is to give an account of the Clifford group $\mathcal{X}_1$ (of quantum information theory) [7, §4.1.2], [12, Ex. 6.4.1(1)] in combinatorial terms. The stabiliser described in Proposition 3.20 has the same order as the (complex) Clifford group $\mathcal{X}_1$ (compare [12, Ex. 6.4.1(1)], for example), but the two groups are not isomorphic. In particular, unlike $\mathcal{X}_1$, the group $D_4 \times S_4$ has no element of order 8.

2. The extraspecial Pauli hull

2.1. **Fourth roots of unity.** On the set $C_4 = \{\pm 1, \pm i\}$ of fourth roots of unity, a quasigroup structure $(C_4, \cdot)$ is given by (the restriction of) the complex inner product $x \cdot y = x\,\overline{y}$ on $\mathbb{C}$. (In this context, we use juxtaposition for the usual associative multiplication of complex numbers). Note that $C_2 = \{\pm 1\}$ forms a subgroup of the quasigroup $(C_4, \cdot)$. Also, throughout, note that $\overline{p}^2 = p^2$ for $p \in C_4$. The following theorem essentially represents the special case $d = 4$ of [10, Prop. 2.1.4], which identified a general finite dihedral group D_d of degree d (and thus of order $2d$) as the multiplication group of the quasigroup of integer residues modulo d under subtraction. However, in the present context, an exponential map has been applied to give a multiplicative version of the subtraction quasigroup on $\mathbb{Z}/_4$, and for subsequent reference it will be useful to have a full record of the resulting multiplicative version of [10, Prop. 2.1.4].

Theorem 2.1. *Consider the group* $D_4 = \mathrm{Mlt}(C_4, \cdot)$.

(a) *The composition relations (in diagrammatic notation reading from left to right)*

(2.1)
$$\begin{array}{c|cc} & R(q) & L(q) \\ \hline R(p) & R(pq) & L(pq) \\ L(p) & L(p\,\overline{q}) & R(p\,\overline{q}) \end{array}$$

hold for p, q *in* C_4.

(b) *In* $\mathrm{Mlt}(C_4, \cdot)$, *the identity element is* $R(1)$, *while*

(2.2)
$$R(q)^{-1} = R(\overline{q}) \quad \text{and} \quad L(q)^{-1} = L(q)$$

for each element q *of* C_4.

(c) *The multiplication group of* $(C_4, \cdot)$ *is the union* $L(C_4) \cup R(C_4)$.

(d) *The subsets* $L(C_4)$ *and* $R(C_4)$ *of the multiplication group* $\mathrm{Mlt}(C_4, \cdot)$ *are disjoint.*

(e) *There is an exact sequence*

$$\{\,1\,\} \longrightarrow C_4 \xrightarrow{R} \mathrm{Mlt}(C_4, \cdot) \longrightarrow C_2 \longrightarrow \{\,1\,\}$$

$$L(q) \longmapsto -1$$

of groups, split by $C_2 \to \mathrm{Mlt}(C_4, \cdot); -1 \mapsto L(1)$.

(f) *Conjugation relations* $g \uparrow h := g^h = h^{-1}gh$ *in* $\mathrm{Mlt}(C_4, \cdot)$ *are given by*

(2.3)
$$\begin{array}{c|cc} \uparrow & R(q) & L(q) \\ \hline R(p) & R(p) & R(\overline{p}) \\ L(p) & L(p\,q^2) & L(q^2\,\overline{p}) \end{array}$$

for p, q *in* C_4.

(g) *Commutators* $[g, h] = g^{-1+h} := g^{-1}g^h = g^{-1}h^{-1}gh$ *in* $\mathrm{Mlt}(C_4, \cdot)$ *are given by*

(2.4)
$$\begin{array}{c|cc} [\ ,\ ] & R(q) & L(q) \\ \hline R(p) & R(1) & R(p^2) \\ L(p) & R(q^2) & R(p^2q^2) \end{array}$$

for p, q *in* C_4.

(h) *The conjugacy classes of* $\mathrm{Mlt}(C_4, \cdot)$ *are given by*

$$R(p)^{\mathrm{Mlt}(C_4,\cdot)} = \{\ R(p), R(\overline{p})\ \} \tag{2.5}$$

and

$$L(p)^{\mathrm{Mlt}(C_4,\cdot)} = \{\ L(\pm p), L\,(\pm\,\overline{p}\,)\ \} \tag{2.6}$$

for each element p *of* C_4.

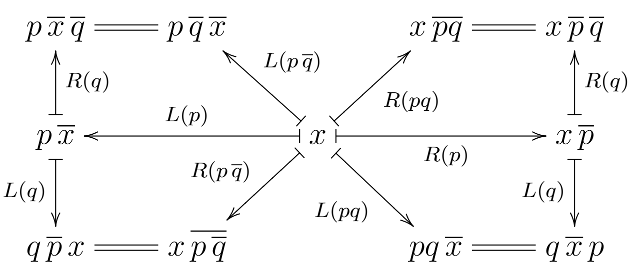
(i) *The set* $\{\ R(\pm 1)\ \}$ *forms the center* $Z\mathrm{Mlt}(C_4, \cdot)$ *and the commutator or derived subgroup* $\mathrm{Mlt}(C_4, \cdot)'$ *of* $\mathrm{Mlt}(C_4, \cdot)$.

(j) *Complex conjugation induces an outer automorphism*

$$\mathrm{Mlt}(C_4, \cdot) \to \mathrm{Mlt}(C_4, \cdot); R(p) \mapsto R(\overline{p})\,,\ \ L(p) \mapsto L(\overline{p})$$

of $\mathrm{Mlt}(C_4, \cdot)$.

Proof. (a): The relations expressed in (2.1) are verified by



The statements (b)–(g) then follow directly, in sequence. Consider, say,

$$L(p)^{R(q)} \overset{(2.2)}{=} R(\overline{q})L(p)R(q) \overset{(2.1)}{=} L(\overline{q}\,p)R(q)$$
$$\overset{(2.1)}{=} L(\overline{q}\,p\,\overline{q}) = L(p\,\overline{q}^2) = L(p\,q^2)$$

for $p, q \in C_4$, confirming the bottom left entry of the body of the table (2.3).

(h): The equation (2.5) is immediately apparent from the top row of the body of the table (2.3). Now consider a left multiplication $L(p)$. The containment

$$L(p)^{\mathrm{Mlt}(C_4,\cdot)} \subseteq \{\ L(\pm p), L\,(\pm\,\overline{p}\,)\ \}$$

is immediately apparent from the bottom row of the body of the table (2.3). The converse follows by considering the bottom row of the body of the table (2.3), with $q = 1, i$.

(i): By (2.6), no left multiplication is central. Thus in specifying the center, it suffices to restrict attention to the right multiplications $R(p)$.

By (2.5), the conjugacy class of a right multiplication $R(p)$ is a singleton iff p is real. The remaining statement follows by (2.4).

(j) is a direct consequence of (2.1), since $\overline{p}\,\overline{q} = \overline{pq}$ and $\overline{p}\,\overline{\overline{q}} = \overline{p\overline{q}}$. □

Remark 2.2. In [11, Table 4], the elements of D_4 were displayed in its $\mathrm{Mlt}(C_4, \cdot)$ guise as units within the negative type real Clifford algebra $\mathrm{Cl}(2, 0)$ and the positive type real Clifford algebra $\mathrm{Cl}(1, 1)$.

2.2. **The Pauli hull as a multiplication group.** The *Pauli hull* is defined to be the central square $D_4 \odot D_4$ of the dihedral group of degree 4. It is the extraspecial 2-group E_2 in the notation of [4, p.405], or the group N_3 in the notation of [13, Table III]. Recall that D_4 is the multiplication group of the complex inner product quasigroup $(C_4, \cdot)$ on the set of fourth roots of unity, according to Theorem 2.1. The Pauli hull is then recognized by [11, Th. 2.14] as the multiplication group of $\mathrm{Mlt}(C_4, \cdot)$. In turn, [11, Prop. 2.13] suggests a quasigroup-theoretical normal form for elements of the Pauli hull, given in §2.2.2 after an initial abstract summary of the groups under discussion. The quasigroup-theoretical normal forms fit nicely with many subgroups of the Pauli hull, most notably the Pauli group itself (§2.2.4).

2.2.1. *The abstract groups.* In the following proposition, relations on the groups are specified by words which equate to the identity element.

Proposition 2.3. *The groups $D_4, D_4 \odot C_4$, and $D_4 \odot D_4$ have abstract presentations as follows.*

(a) *The group D_4 is generated by elements x_0, x_1 subject to the set*

$$R_0 = \{\, x_0^2\,,\ x_1^4\,,\ (x_0 x_1)^2 \,\}$$

of relations.

(b) *The group $D_4 \odot C_4$ is generated by elements x_1, x_2, x_3 subject to the set*

$$R_1 = \{\, x_1^4\,,\ x_2^2\,,\ x_3^2 x_1^2\,,\ [x_1, x_2]\,,\ [x_1, x_3]\,,\ x_3^{-1} x_2 x_1^2 x_3 x_2 \,\}$$

of relations.

(c) *The group $D_4 \odot D_4$ is generated by elements x_0, x_1, x_2, x_3 subject to the union of $R_0 \cup R_1$ with the set*

$$R_2 = \{\, [x_0, x_3] \,\}$$

of relations.

2.2.2. *The Pauli hull.*

Definition 2.4. For an element p of C_4, consider the respective left and right multiplications $L(p), R(p)$ by p in the complex inner product quasigroup $(C_4, \cdot)$. Then define

$$\begin{aligned} TL(p) &:= T\left(L(p)\right), \ TR(p) := T\left(R(p)\right) \\ RL(p) &:= R\left(L(p)\right), \ RR(p) := R\left(R(p)\right), \end{aligned} \tag{2.7}$$

as abbreviated forms for the corresponding elements of the Pauli hull.

Lemma 2.5. *The relations*

$$TR(p) = TR(-p) \quad \textit{and} \quad TL(p) = TL(-p) \tag{2.8}$$

hold for $p \in C_4$.

Proof. The statement follows from Theorem 2.1(f). □

In view of Lemma 2.5, we often write $TR(\pm p)$ and $TL(\pm p)$ for the respective common values of the expressions in (2.8).

Theorem 2.6. *Consider the Pauli hull as the multiplication group of the multiplication group of the complex inner product quasigroup on the set of fourth roots of unity.*

(a) *It forms the set*

$$\{\, TR(\pm p)RR(q), TR(\pm p)RL(q), TL(\pm p)RR(q), TL(\pm p)RL(q)\,\}$$

where p and q range over all the elements of C_4. Since there are 2 choices for the pairs $\pm p$ and 4 choices for the elements q, the given description of the Pauli hull exhibits its cardinality of $32 = 4 \times 2 \times 4$.

(b) *Its multiplication table is given by*

	$TR(p_2)RR(q_2)$	$TR(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(p_1p_2)RR(q_1q_2)$	$TR(p_1p_2)RL(q_1q_2)$
$TR(p_1)RL(q_1)$	$TR(p_1p_2)RL(p_2^2\,q_1\,\overline{q_2})$	$TR(p_1p_2)RR(p_2^2\,q_1\,\overline{q_2})$
$TL(p_1)RR(q_1)$	$TL(p_1\,\overline{p_2})RR(q_1q_2)$	$TL(p_1\,\overline{p_2})RL(q_1q_2)$
$TL(p_1)RL(q_1)$	$TL(p_1\,\overline{p}_2)RL(p_2^2\,q_1\,\overline{q_2})$	$TL(p_1\,\overline{p}_2)RR(p_2^2\,q_1\,\overline{q_2})$

	$TL(p_2)RR(q_2)$	$TL(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TL(p_1p_2)RR(\overline{q_1}\,q_2)$	$TL(p_1p_2)RL(\overline{q_1}\,q_2)$
$TR(p_1)RL(q_1)$	$TL(p_1p_2)RL(p_2^2\,\overline{q_1q_2})$	$TL(p_1p_2)RR(p_2^2\,\overline{q_1q_2})$
$TL(p_1)RR(q_1)$	$TR(p_1\,\overline{p_2})RR(\overline{q_1}\,q_2)$	$TR(p_1\,\overline{p_2})RL(\overline{q_1}\,q_2)$
$TL(p_1)RL(q_1)$	$TR(p_1\,\overline{p}_2)RL(p_2^2\,\overline{q_1q_2})$	$TR(p_1\,\overline{p}_2)RR(p_2^2\,\overline{q_1q_2})$

for $p_i, q_i \in C_4$*, with identity element* $TR(1)RR(1)$*.*

(c) *Inversion in the group is given by*

$$
\begin{aligned}
TR(p)RR(q)^{-1} &= TR(\overline{p})RR(\overline{q})\,,\\
TR(p)RL(q)^{-1} &= TR(\overline{p})RL(p^2q)\,,\\
TL(p)RR(q)^{-1} &= TL(p)RR(q)\,,\\
TL(p)RL(q)^{-1} &= TL(p)RL(p^2\,\overline{q})\,.
\end{aligned}
$$

(d) *The group has a faithful transitive permutation representation of degree* 8*.*

(e) *Conjugation relations $g \uparrow h := g^h = h^{-1}gh$ within the Pauli hull are given by*

$\uparrow$	$TR(p_2)RR(q_2)$	$TR(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(p_1)RR(q_1)$	$TR(p_1)RR(p_1^2\,\overline{q_1})$
$TR(p_1)RL(q_1)$	$TR(p_1)RL(p_2^2 q_1 q_2^2)$	$TR(p_1)RL(p_1^2\,p_2^2\,\overline{q_1}q_2^2)$
$TL(p_1)RR(q_1)$	$TL(p_1p_2^2)RR(q_1q_2^2)$	$TL(p_1p_2^2)RR(p_1^2\,\overline{q_1}\,q_2^2)$
$TL(p_1)RL(q_1)$	$TL(p_1p_2^2)RL(p_2^2\,q_1)$	$TL(p_1p_2^2)RL(p_1^2\,p_2^2\,\overline{q_1})$

$\uparrow$	$TL(p_2)RR(q_2)$	$TL(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(\overline{p_1})RR(\overline{q_1})$	$TR(\overline{p_1})RR(p_1^2\,q_1)$
$TR(p_1)RL(q_1)$	$TR(\overline{p_1})RL(p_2^2\,\overline{q_1}\;q_2^2)$	$TR(\overline{p_1})RL(p_1^2\,p_2^2\,q_1q_2^2)$
$TL(p_1)RR(q_1)$	$TL(\overline{p_1}\,p_2^2)RR(\overline{q_1}\,q_2^2)$	$TL(\overline{p_1}\,p_2^2)RR(p_1^2\,q_1q_2^2)$
$TL(p_1)RL(q_1)$	$TL(\overline{p_1}p_2^2)RL(p_2^2\,\overline{q_1})$	$TL(\overline{p_1}\,p_2^2)RL(p_1^2\,p_2^2\,q_1)$

In particular, the center is $\{\, TR(\pm 1)RR(\pm 1)\,\}$.

(f) *Commutators $[g,h] = g^{-1+h} := g^{-1}g^h = g^{-1}h^{-1}gh$ within the Pauli hull are given by*

$[\ ,\ ]$	$TR(p_2)RR(q_2)$	$TR(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(1)RR(1)$	$TR(1)RR(p_1^2q_1^2)$
$TR(p_1)RL(q_1)$	$TR(1)RR(p_2^2q_2^2)$	$TR(1)RR(p_1^2p_2^2q_1^2q_2^2)$
$TL(p_1)RR(q_1)$	$TR(p_2^2)RR(q_2^2)$	$TR(p_2^2)RR(p_1^2\,q_1^2\,q_2^2)$
$TL(p_1)RL(q_1)$	$TR(p_2^2)RR(p_2^2)$	$TR(p_2^2)RR(p_1^2\,p_2^2\,q_1^2)$

$[\ ,\ ]$	$TL(p_2)RR(q_2)$	$TL(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(p_1^2)RR(q_1^2)$	$TR(p_1^2)RR(p_1^2)$
$TR(p_1)RL(q_1)$	$TR(p_1^2)RR(p_2^2q_1^2q_2^2)$	$TR(p_1^2)RR(p_1^2p_2^2q_2^2)$
$TL(p_1)RR(q_1)$	$TR(p_1^2p_2^2)RR(q_1^2q_2^2)$	$TR(p_1^2p_2^2)RR(p_1^2q_2^2)$
$TL(p_1)RL(q_1)$	$TR(p_1^2p_2^2)RR(p_2^2q_1^2)$	$TR(p_1^2p_2^2)RR(p_1^2p_2^2)$

In particular, the derived subgroup is $\{\, TR(\pm 1)RR(\pm 1)\,\}$.

Proof. (a) This part follows by [11, Prop 2.13(a)]

(b) Using [11, (2.6)], we have

$$T(h_1)R(g_1)T(h_2)R(g_2) = T(h_1h_2)R\big(g_1^{h_2}g_2\big)$$

in the notation of Theorem 2.1(f). Theorem 2.1(a),(f) then supplies the computations required to complete the multiplication table.

(c) The statements follow from the multiplication tables in (b).

(d) The statement follows immediately by [11, Prop. 2.9].

(e), (f): The tables are obtained using (c) and the multiplication table from (b). □

2.2.3. *The subgroup $D_4 \times C_2$.* In this section, recall that the group $D_4 \times C_2$ is the unit group of the real Clifford algebra $\mathrm{Cl}(2,1)$ [11, Table 2].

Proposition 2.7. *Within the Pauli hull, the subset*

$$\{\, TR(\pm p)RR(q), TL(\pm p)RR(q) \mid p,q \in C_4 \,\} \tag{2.9}$$

of order 16 *forms a subgroup.*

Proof. In the tables of Theorem 2.6(b), consider the first columns, with $TR(p_2)RR(q_2)$ and $TL(p_2)RR(q_2)$ as their respective labels. In each table, restrict consideration to the respective first and third rows. Then the corresponding entries of the bodies of the tables are seen to lie in the subset (2.9). Indeed, the multiplication table

(2.10)

	$TR(p_2)RR(q_2)$	$TL(p_2)RR(q_2)$
$TR(p_1)RR(q_1)$	$TR(p_1p_2)RR(q_1q_2)$	$TL(p_1p_2)RR(\overline{q_1}\,q_2)$
$TL(p_1)RR(q_1)$	$TL(p_1\,\overline{p_2})RR(q_1q_2)$	$TR(p_1\,\overline{p_2})RR(\overline{q_1}\,q_2)$

is obtained. □

Corollary 2.8. *Consider the subgroup described in Proposition 2.7.*

(a) *Powers in the subgroup are given by*

$$TL(p)RR(q)^2 = TR(1)RR(1)$$

and

$$TR(p)RR(q)^n = TR(p^n)RR(q^n) \tag{2.11}$$

for all integers n.

(b) *There are* 4 *elements of order* 4*. It follows that the subgroup is isomorphic to $D_4 \times C_2$* [6, pp.39,150], [13, Table II].

Proof. (a) The first statement is obtained as an immediate consequence of the multiplication table (2.10). The second statement follows by induction on n.

(b) Note

$$\begin{cases} TR(\pm i)RR(\pm i)^2 & = TR(\pm 1)RR(-1) \quad \text{and} \\ TR(\pm 1)RR(\pm i)^2 & = TR(\pm 1)RR(-1) \end{cases} \tag{2.12}$$

by (a). Other elements square to the identity. □

2.2.4. *The Pauli subgroup.* In this section, recall that the Pauli group $D_4 \odot C_4$ is the unit group of the real Clifford algebras $\mathrm{Cl}(3,0)$ and $\mathrm{Cl}(1,2)$ — [11, Table 5].

Proposition 2.9. *Within the Pauli hull, the subset*

$$\{\, TR(\pm p)RR(q), TR(\pm p)RL(q) \mid p, q \in C_4 \,\} \tag{2.13}$$

of order 16 *forms a subgroup.*

Proof. In (2.13), there are 2 choices for $TR(\pm p)$, and 4 choices for q, yielding $(2 \times 4) + (2 \times 4) = 16$ elements altogether. The (closed) multiplication table for these elements is given by the top fragment

	$TR(p_2)RR(q_2)$	$TR(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(p_1p_2)RR(q_1q_2)$	$TR(p_1p_2)RL(q_1q_2)$
$TR(p_1)RL(q_1)$	$TR(p_1p_2)RL(p_2^2\, q_1\, \overline{q_2})$	$TR(p_1p_2)RR(p_2^2\, q_1\, \overline{q_2})$

of the first table of Theorem 2.6(b). □

Corollary 2.10. *Consider the subgroup described in Proposition 2.9.*

(a) *Powers in the subgroup are given by*

$$TR(p)RL(q)^2 = TR(p^2)RR(p^2)\,,\ TR(p)RL(q)^4 = TR(1)RR(1)$$

together with (2.14) *below, and*

$$TR(p)RR(q)^n = TR(p^n)RR(q^n)$$

for all integers n.

(b) *Inversion in the subgroup is given by*

$$TR(p)RL(q)^{-1} = TR(\overline{p})RL(p^2 q) \tag{2.14}$$

and

$$TR(p)RR(q)^{-1} = TR(\overline{p})RR(\overline{q})$$

for all integers n.

(c) *Conjugation relations* $g \uparrow h := g^h = h^{-1}gh$ *within the subgroup are given by*

$\uparrow$	$TR(p_2)RR(q_2)$	$TR(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(p_1)RR(q_1)$	$TR(p_1)RR(p_1^2\,\overline{q_1})$
$TR(p_1)RL(q_1)$	$TR(p_1)RL(p_2^2q_1q_2^2)$	$TR(p_1)RL(p_1^2\,p_2^2\,\overline{q_1}q_2^2)$

In particular, the center is $\{\, TR(p)RR(p) \mid p \in C_4 \,\}$.

(d) *Commutators* $[g,h] = g^{-1+h} := g^{-1}g^h = g^{-1}h^{-1}gh$ *within the subgroup are given by*

$[\ ,\ ]$	$TR(p_2)RR(q_2)$	$TR(p_2)RL(q_2)$
$TR(p_1)RR(q_1)$	$TR(1)RR(1)$	$TR(1)RR(p_1^2q_1^2)$
$TR(p_1)RL(q_1)$	$TR(1)RR(p_2^2q_2^2)$	$TR(1)RR(p_1^2p_2^2q_1^2q_2^2)$

In particular, the derived subgroup is $\{\, TR(1)RR(\pm 1) \,\}$.

(e) *There are* 8 *elements of order* 4.

(f) *The subgroup* (2.13) *exhibited by Proposition 2.9 is the Pauli group* $C_4 \odot D_4$.

Proof. (a), (b): In each case, the first statement is obtained as an immediate consequence of the multiplication table from the proof of Proposition 2.9, while the second statements recall (2.11).

(c), (d): The tables are obtained using (b) and the multiplication table from the proof of Proposition 2.9. In connection with the specification of the center, note that $p_2^2q_2^2 = 1$ if $p_2 = q_2$.

(e): Four elements with non-trivial squares are exhibited by (2.12). The computations

$$\begin{cases} TR(\pm i)RL(\pm i)^2 & = TR(\pm 1)RR(-1) \quad \text{and} \\ TR(\pm i)RL(\pm 1)^2 & = TR(\pm 1)RR(-1) \end{cases}$$

exhibit the remaining four.

(f): By (c), the center is C_4. By (e), there are eight elements of order four. The only group of order 16 with 8 elements of order four and with C_4 as its center, appearing as a subgroup of the Pauli hull, is the Pauli group [6, pp.39,150], [13, Table II]. □

2.3. **Fourth roots of unity and the Pauli group.** According to Corollary 2.10(f), the subgroup

$$\{\, TR(\pm p)RR(q), TR(\pm p)RL(q) \mid p, q \in C_4 \,\}$$

of the Pauli hull exhibited in quasigroup-theoretical terms by (2.13) is the Pauli group. The goal of this section is to correlate the quasigroup-theoretical description of the Pauli group with other descriptions that are more familiar in quantum information theory.

2.3.1. *Recognizing the Pauli group.* The center of the Pauli group G_1 is $\{ pI \mid p \in C_4 \}$, so in order to make a connection with the multiplication group notation, the first identification to be made is

$$pI = TR(p)RR(p) \tag{2.15}$$

for $p \in C_4$ using Corollary 2.10(c).

In both the negative and the positive parts of [11, Table 4], the Pauli matrix X, appearing there as e_1, was identified with $L(1)$. This element of D_4 embeds into the Pauli hull as $RL(1) = TR(\pm 1)RL(1)$, so we may identify

$$x_1 = i = TR(i)RR(i) \text{ and } x_2 = X = TR(1)RL(1) \tag{2.16}$$

invoking the abstract presentation of Proposition 2.3(b). An element x_3 of order 4 must then be found to yield $x_2x_3x_2 = x_1^2x_3$. In the positive part of [11, Table 4], we have f_2 of order 4 with $e_1f_2e_1 = -f_2$, identified with $R(\pm i)$, so we may take

$$x_3 = iY = TR(1)RR(i) \tag{2.17}$$

and

$$Y = -i(iY) = TR(-i)RR(-i)TR(1)RR(i) = TR(-i)RR(1)\,.$$

Finally, $Z = -i(XY)$ or

$$Z = X(-iY) = TR(1)RL(1)TR(1)RR(-i) = TR(1)RL(i)\,.$$

We may summarize as follows.

Proposition 2.11. *Setting*

$$X = TR(1)RL(1)\,, \quad Y = TR(i)RR(1)\,, \quad Z = TR(1)RL(i) \tag{2.18}$$

along with (2.15) *interprets the quantum information-theoretic Pauli group* [11, (3.26)] *within the quasigroup-theoretical version* (2.13).

Remark 2.12. The specifications (2.18) may be recorded as

$$X = RL(1)\,, \quad Y = TR(i)\,, \quad Z = RL(i)$$

in abbreviated form. Thus, the bit flip X is the image of the dihedral reflexion $L(1)$ in the multiplication group, while the phase flip Z is the image of the dihedral reflexion $L(i)$.

2.3.2. *The abstract presentation.* The previous section called upon the abstract presentation of Proposition 2.3(b) to identify the quantum information-theoretic and quasigroup-theoretic versions of the Pauli group. In this section, we directly verify that the multiplication group elements chosen there do satisfy the required relations. In particular, this reconfirms the assertion of Corollary 2.10(f), at that point justified on the basis of the known subgroup structure of the Pauli hull (i.e., the extraspecial group of order 32) [6, p.150], that the Pauli subgroup (2.13) exhibited by Proposition 2.9 is indeed the Pauli group $C_4 \odot D_4$.

Proposition 2.13. *The relations*

$$x_1^4 = x_2^2 = 1\,, \quad x_1^2 = x_3^2\,, \tag{2.19}$$

$$[x_1, x_2] = [x_1, x_3] = 1\,, \tag{2.20}$$

$$x_2 x_3 x_2 = x_1^2 x_3 \tag{2.21}$$

are satisfied by

$$x_1 = TR(i)RR(i)\,, \quad x_2 = TR(1)RL(1)\,, \quad x_3 = TR(1)RR(i)$$

in the Pauli subgroup.

Proof. The power relations (2.19) are obtained as direct consequences of Corollary 2.10(a). For example,

$$x_1^2 = TR(-1)RR(-1) = TR(1)RR(-1) = x_3^2\,.$$

The commutation relations (2.20) follow by Corollary 2.10(c). For (2.21), we have

$$x_2 x_3 x_2 = TR(1)RR(i) \uparrow TR(1)RL(1) = TR(1)RR(-i)$$

from the conjugation table in Corollary 2.10(c), and again

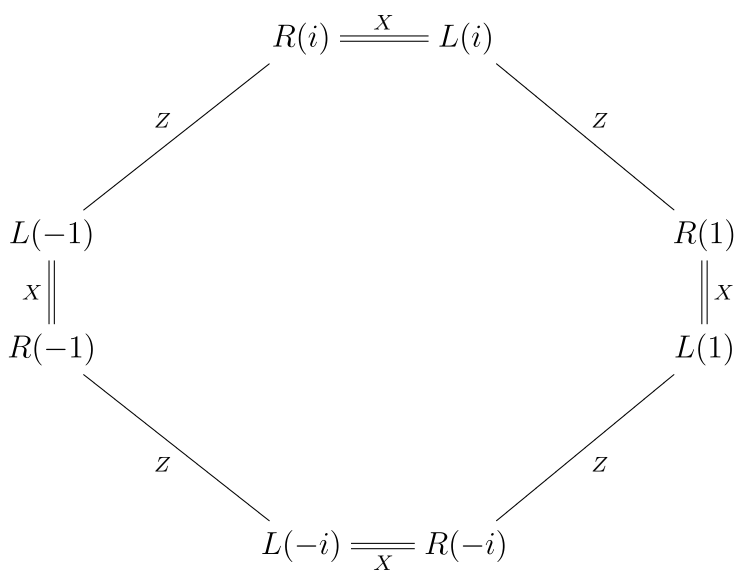
$$\begin{aligned} x_1^2 x_3 &= TR(i)RR(i)^2 TR(1)RR(i) \\ &= TR(-1)RR(-1)TR(1)RR(i) = TR(-1)RR(-i)\,. \end{aligned}$$

The second equation uses Corollary 2.10(a), while the third uses the multiplication table of Corollary 2.10(b). □

2.3.3. *The Cayley diagram.* By Theorem 2.6(d), the extraspecial Pauli hull Mlt D_4 has a faithful natural transitive permutation representation of degree 8 on the multiplication group $D_4 = \mathrm{Mlt}(C_4, \cdot)$ of the complex inner-product quasigroup $(C_4, \cdot)$. The restriction of this permutation representation to the Pauli subgroup of the Pauli hull provides a faithful transitive permutation representation of the Pauli group, which will be presented here using the quantum information-theoretic generators X, Z and iI identified in Proposition 2.11.

Theorem 2.14. *The Cayley diagram of the Pauli group with respect to right actions of the generating set* $\{X, Z, iI\}$*, including the involutions* X, Z*, is given as*

(2.22)



together with the rotation *actions*

$$iI : R(p) \mapsto R(ip)\,,\ \ L(p) \mapsto L(ip) \tag{2.23}$$

for $p \in C_4$*.*

Proof. Towards (2.23), note that

$$\begin{aligned} R(p)TR(i)RR(i) &= R(p)LR(i)^{-1}RR(i)RR(i) \\ &= R(-i)R(p)R(-1) = R(ip) \end{aligned}$$

and

$$\begin{aligned} L(p)TR(i)RR(i) &= L(p)LR(i)^{-1}RR(i)RR(i) \\ &= R(-i)L(p)R(-1) = L(ip) \end{aligned}$$

for $p \in C_4$. The other computations are straightfoward. They use the identifications of Proposition 2.11 and the multiplication table of Theorem 2.1(a). □

Remark 2.15. The Cayley diagram (2.22) may be envisaged as being located in the complex plane, where each single complex number p from C_4 appears with two distinct chiral representations $L(p)$ and $R(p)$. The rotation actions (2.23) then correspond to multiplication by i, the counterclockwise rotation of the complex plane by $\pi/2$.

2.3.4. *Central quotients.* By Theorem 2.1(a), the set

$$\{ R(\pm 1), L(\pm 1) \} \tag{2.24}$$

forms a Klein 4-subgroup of $D_4 = \mathrm{Mlt}(C_4, \cdot)$.

Proposition 2.16. (a) *There is a surjective group homomorphism*

$$\mathrm{Mlt}(C_4, \cdot) \to \{ R(\pm 1), L(\pm 1) \}\,; R(p) \mapsto R(p^2)\,,\ L(p) \mapsto L(p^2) \tag{2.25}$$

whose kernel is $\{ R(\pm 1) \}$.

(b) *The Klein* 4*-group* (2.24) *is the central quotient of* $D_4 = \mathrm{Mlt}(C_4, \cdot)$.

Proof. (a) Theorem 2.1(a) yields a direct verification that the mapping (2.25) is a group homomorphism. Taking $p = 1$ and $p = i$ shows that the mapping is surjective. Finally, note that $\{ R(\pm 1) \}$ is the preimage of the identity element $R(1)$ of $D_4 = \mathrm{Mlt}(C_4, \cdot)$.

(b) By Theorem 2.1(i), we have $Z\mathrm{Mlt}(C_4, \cdot) = \{ R(\pm 1) \}$. Thus, by the First Isomorphism Theorem, (2.24) represents the central quotient $\mathrm{Mlt}(C_4, \cdot)/Z\mathrm{Mlt}(C_4, \cdot)$ of $\mathrm{Mlt}(C_4, \cdot)$. □

Theorem 2.17. (a) *There is a surjective group homomorphism*

$$\theta\colon \{ TR(\pm p)RR(q), TR(\pm p)RL(q) \mid p, q \in C_4 \} \to \{ R(\pm 1), L(\pm 1) \}\,;$$

$$TR(\pm p)RR(q) \mapsto R(p^2q^2)\,,\ TR(\pm p)RL(q) \mapsto L(p^2q^2) \tag{2.26}$$

whose kernel is $\{ TR(p)RR(p) \mid p \in C_4 \}$.

(b) *The Klein* 4*-group* (2.24) *is the central quotient of the Pauli subgroup.*

Proof. (a) Note that

$$TR(\pm p_1p_2)RR(q_1q_2)^\theta = TR(\pm p_1p_2)RR(p_2^2\, q_1\, \overline{q_2})^\theta = R(p_1^2p_2^2q_1^2q_1^2)$$

and

$$TR(\pm p_1p_2)RL(q_1q_2)^\theta = TR(\pm p_1p_2)RL(p_2^2\, q_1\, \overline{q_2})^\theta = R(p_1^2p_2^2q_1^2q_1^2)\,.$$

By the top fragment of the multiplication table of Theorem 2.6(b), together with Theorem 2.1(a), it is then apparent that θ is a group homomorphism. Taking $p = 1$, and $q = 1$ or $q = i$ in (2.26), shows that θ is surjective. Furthermore, the containment

$$\{ TR(p)RR(p) \mid p \in C_4 \} \subseteq \operatorname{Ker}\theta \tag{2.27}$$

is immediate from (2.26). By the First Isomorphism Theorem, we have $16/|\operatorname{Ker}\theta| = 4$ or $|\operatorname{Ker}\theta| = 4$. The containment (2.27) is thus seen to be improper, as required.

(b) By Corollary 2.10(c), the set $\{ TR(p)RR(p) \mid p \in C_4 \}$ is the center of the Pauli subgroup. Thus, by the First Isomorphism Theorem, (2.24) represents the central quotient of the Pauli subgroup. □

2.4. **The Heisenberg group interpretation.** Section 2.3.4 presented direct syntactical verifications, in terms of multiplication groups, that the central quotients of both D_4 and the Pauli group are isomorphic to the Klein 4-group, modeled by the subgroup (2.24) of D_4. We now take a geometric approach to these common central quotients: as a classical phase space, leading to the identification of the Pauli group as a Heisenberg-Weyl group [5, §0.2], [15, §IV.D].

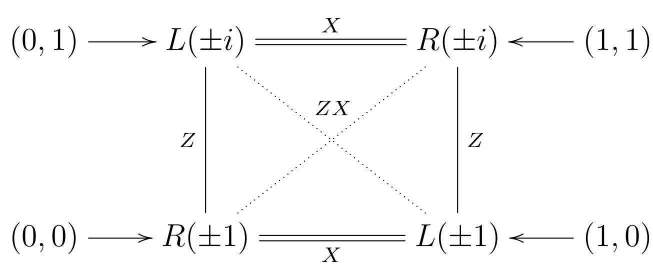
Here, we will recognize the central quotient of a group Q as the space Q^ζ of equivalence classes of the center congruence $\zeta(Q)$ (or just ζ) of Q, recalling that the *center congruence* $\zeta(Q)$ is the largest subgroup of Q^2 normalizing the diagonal subgroup $\widehat{Q} = \{ (q,q) \mid q \in Q \}$ [14, §3.2]. The normality of $\widehat{Q}$ in ζ means that $\widehat{Q}$ is an equivalence class of a (unique) *centering congruence* W on ζ [14, Prop. 3.4]. While the center congruence makes sense in any quasigroup, it takes a special form in a group Q. In this situation, the equivalence classes $q^{\zeta(Q)}$ are the cosets $qZ(Q) = Z(Q)q$ of the center $Z(Q)$, for $q \in Q$.

Recalling from Theorem 2.1(i) that the center of $D_4 = \mathrm{Mlt}(C_4, \cdot)$ is $R(\pm 1)$, the ζ-classes of D_4 are given as the antipodal pairs

$$\{ R(\pm 1) \}\,, \quad \{ R(\pm i) \}\,, \quad \{ L(\pm 1) \}\,, \quad \{ L(\pm i) \} \tag{2.28}$$

in the Cayley diagram (2.22) of the Pauli group. Thus, even though D_4 is not a quotient group of the Pauli group, our quasigroup-theoretical approach provides a quotient relationship at the graph-theoretical level. Involutive actions of Pauli generators by right multiplication on the set of antipodal pairs in (2.22) are given by the central part of

(2.29)



with an abbreviated notation omitting the braces from the elements (2.28).

The edge-labeled graph in the middle of (2.29) is obtained as the quotient of the edge-labeled graph (2.22) resulting from identification of antipodal vertices. It may be interpreted as the Cayley diagram of the Klein 4-group D_4/D_4'. In this way, (2.29) introduces coordinates for the set D_4^ζ of four central classes taken from the set $\{0,1\}^2 = (\mathbb{Z}/_2)^2$, recognizing the configuration as the classical phase space for a single qubit [16, §3]. The first component u_1 of a binary vector $u = (u_1, u_2)$ corresponds to a (generalized) position, while the second component u_2 corresponds to a (generalized) momentum in terms of Hamiltonian mechanics. The 3 edge classes inside the Cayley diagram (2.29), drawn respectively as doubled, single, dotted, and labelled respectively by X, Z, ZX, are the three *striations* of [16, Fig. 4].

It should be noted from Theorem 2.1(i) that the commutator map

$$D_4 \times D_4 \to D_4' = Z(D_4); (x,y) \mapsto [x,y]$$

descends to a scalar product

$$D_4/D_4' \times D_4/D_4' \to Z(D_4); (x^\zeta, y^\zeta) \mapsto [x,y] \tag{2.30}$$

tabulated by (2.4) (compare [4]). Identifying the $\zeta(D_4)$-classes by their phase space coordinates in $(\mathbb{Z}/_2)^2$, and recognizing $Z(D_4)$ as $\mathbb{Z}/_2$, the form

$$\omega\colon (\mathbb{Z}/_2)^2 \times (\mathbb{Z}/_2)^2 \to \mathbb{Z}/_2; \big((u_1,u_2),(v_1,v_2)\big) \mapsto u_1v_2 + u_2v_1 \tag{2.31}$$

may be used to rewrite the scalar product (2.30).

We now turn to the Pauli group G_1. In our multiplication group interpretation, the center $Z(G_1)$ is given as $\{ TR(\lambda)RR(\lambda) \mid \lambda \in C_4 \}$ according to Corollary 2.10(c). Using the top fragment of the first table of Theorem 2.6(b), as in the proof of Proposition2.9, the four center congruence classes are then obtained as

$$TR(\pm p)RR(q)^{\zeta(G_1)} = \{ TR(\pm\lambda p)RR(\lambda q) \mid \lambda \in C_4 \} \text{ and}$$
$$TR(\pm p)RL(q)^{\zeta(G_1)} = \{ TR(\pm\lambda p)RL(\lambda q) \mid \lambda \in C_4 \}$$

with $p, q \in C_4$. Choosing $\lambda = p^{-1}$, they may be represented as

$$TR(\pm1)RR(1)^{\zeta(G_1)}, \quad TR(\pm1)RR(i)^{\zeta(G_1)},$$
$$TR(\pm1)RL(1)^{\zeta(G_1)}, \quad TR(\pm1)RL(i)^{\zeta(G_1)},$$

respectively corresponding to the classes

$$R(\pm1)\,, R(\pm i)\,, L(\pm1)\,, L(\pm i)$$

of D_4 with their phase space labels from $(\mathbb{Z}/_2)^2$ as displayed in (2.29).

By virtue of its provenance as a quotient of the Cayley diagram (2.22) for the Pauli group, the middle part of (2.29) correctly reflects the action of the Pauli generators on the phase space. Setting

$$X(0) = Z(0) = 1\,, \quad X(1) = X\,, \quad \text{and} \quad Z(1) = Z\,,$$

and recalling

$$[Z, X] = [TR(\pm 1)RL(i), TR(\pm 1)RL(1)] = TR(\pm 1)RR(-1)$$

from Corollary 2.10(d), we recover the description of the Pauli group as a Heisenberg-Weyl group.

Theorem 2.18. *Consider the Pauli group G_1 and the phase space D_4^ζ.*

(a) *As a set,* $G_1 = \{ i^j X(u_1) Z(u_2) \mid j \in \mathbf{Z}/_4 ,\ u \in D_4^\zeta \}$.

(b) *The product in G_1 is given as*

$$i^j X(u_1)Z(u_2) \cdot i^k X(v_1)Z(v_2) = i^{j+k+2u_2v_1} X(u_1 + u_2)Z(u_2 + v_2)\,.$$

(c) *The commutator in G_1 is given as*

$$[i^j X(u_1)Z(u_2), i^k X(v_1)Z(v_2)] = (-1)^{\omega(u,v)}$$

with the form (2.31).

(d) *Consider the Heisenberg-Weyl element $i^j X(u_1)Z(u_2)$ for $u = (u_1, u_2)$ in the phase space D_4^ζ.*
   (i) *If $u = (0, 0)$, it corresponds to $TR(\pm i^j)RR(i^j)$ in the Pauli subgroup* (2.13).
   (ii) *If $u = (0, 1)$, it corresponds to $TR(\pm i^j)RL(i^{j+1})$ in the Pauli subgroup* (2.13).
   (iii) *If $u = (1, 0)$, it corresponds to $TR(\pm i^j)RL(i^j)$ in the Pauli subgroup* (2.13).
   (iv) *If $u = (1, 1)$, it corresponds to $TR(\pm i^j)RR(i^{j-1})$ in the Pauli subgroup* (2.13).

(e) (i) *Consider the Pauli subgroup element $TR(\pm p)RR(q)$. Then*

$$TR(\pm p)RR(q) = \begin{cases} qX(0)Z(0) & \text{if } q \in \{ \pm p \}\ ; \\ iqX(1)Z(1) & \text{if } q \notin \{ \pm p \}\ . \end{cases}$$

   (ii) *Consider the Pauli subgroup element $TR(\pm p)RL(q)$. Then*

$$TR(\pm p)RL(q) = \begin{cases} qX(1)Z(0) & \text{if } q \in \{ \pm p \}\ ; \\ -iqX(0)Z(1) & \text{if } q \notin \{ \pm p \}\ . \end{cases}$$

Proof. Parts (a)–(c) are standard computations in the matrix group [11, (3.26)]. Part (d) follows by computation of

$$TR(\pm i^j)RR(i^j)\,[TR(\pm 1)RL(1)]^{u_1}\,[TR(\pm 1)RL(i)]^{u_2}$$

in the Pauli subgroup (2.13) using the first table of Theorem 2.6(b). Part (e) follows by inversion of part (d). ☐

Corollary 2.19. *Define*

$$\iota\colon C_4 \to G_1; i^j \mapsto TR(\pm i^j)RR(i^j)$$

as the insertion of the center into G_1*, and*

$$\nu\colon G_1 \to G_1^\zeta; g \mapsto g^\zeta$$

as the natural projection onto the phase space. Then the sequence

$$1 \longrightarrow C_4 \xrightarrow{\iota} G_1 \xrightarrow{\nu} G_1^\zeta \longrightarrow 1$$

is exact.

3. Quaternions

3.1. **The quaternion group within the Pauli group.** Although the quaternion group does not appear as the multiplication group of a quasigroup (according to [11, Cor. 2.12]), it does appear as a subgroup of the Pauli group, and may thus be represented in terms of $\mathrm{Mlt}\, D_4$ as follows.

Proposition 3.1. *The subset* Q *of the Pauli subgroup consisting of the elements*

$$TR(\pm i)RL(\pm 1)\,, \quad TR(\pm i)RL(\pm i)\,, \quad TR(\pm 1)RR(\pm i)\,, \tag{3.1}$$

$$TR(\pm 1)RR(\pm 1) \tag{3.2}$$

forms a quaternion group. Here, the elements (3.1) *have order* 4*, while the elements* (3.2) *are central*

Proof. The quaternion group has 6 elements of order 4, none of which is central. On the other hand, the Pauli group has 8 elements of order 4, as described by Corollary 2.10(e). Of the 4 listed in (2.12), the first two are central in the Pauli group, by Corollary 2.10(c), and cannot appear in the quaternion group. The remainder are listed in (3.1), while (3.2) presents their even powers. □

Corollary 3.2. *Since the quaternion subgroup*

$$\{\, \pm I, \pm iX, \pm iY, \pm iZ \,\}$$

of the Pauli group G_1 *consists of all its elements of determinant* 1*, the index two subgroup* Q *of the Pauli subgroup is the kernel of the determinant homomorphism to* C_2*.*

3.2. **The Pauli hull in quaternion terms.** In Proposition 3.1, the quaternion group was expressed as a subgroup of the multiplication group of the dihedral group D_4, the central square $D_4 \odot D_4$ of the dihedral group. In order to complete our understanding of the Pauli group from the standpoint of quasigroup theory, we now take a converse approach. Since $\mathrm{Mlt}\, D_4 \cong D_4 \odot D_4 \cong Q_8 \odot Q_8 \cong \mathrm{Mlt}\, Q_8$, we may investigate the extraspecial Pauli hull as the multiplication group of the quaternion group.

The following theorem, expressing the extraspecial Pauli hull as the multiplication group of the quaternion group, forms a counterpart of Theorem 2.6, which expressed the Pauli hull as the multiplication group of the dihedral group.

Theorem 3.3. *Consider the Pauli hull as the multiplication group of the quaternion group Q_8.*

(a) *It forms the set*

$$\{\, T(\pm h)R(g) \mid h, g \in Q_8 \,\} \,. \tag{3.3}$$

Since there are 4 choices for the pairs $\pm h$, and 8 choices for the elements g, the given description of the Pauli hull exhibits its cardinality of $32 = 4 \times 8$.

(b) *Its multiplication is given by*

$$T(\pm h_1)R(g_1)T(\pm h_2)R(g_2) = T(\pm h_1 h_2)R\left(g_1^{h_2} g_2\right)$$

for $h_1, g_1, h_2, g_2 \in Q_8$.

(c) *Inversion in the group is given by $T(\pm h)R(g)^{-1} = T(\pm h)R\left(g^{-h}\right)$ with $g^{-h} = (g^h)^{-1}$.*

(d) *The group has a faithful permutation representation of degree 8.*

(e) *Conjugation relations within the Pauli hull are given by*

$$T(\pm h_1)R(g_1) \uparrow T(\pm h_2)R(g_2) = T(h_1)R\left(g_2^{-h_1} g_1^{h_2} g_2\right)$$

for $h_1, g_1, h_2, g_2 \in Q_8$. Thus, the center is $\{\, T(\pm 1)R(\pm 1) \,\}$.

(f) *Commutators $[g, h] = g^{-1+h} := g^{-1}g^h = g^{-1}h^{-1}gh$ within the Pauli hull are given by*

$$[T(\pm h_1)R(g_1), T(\pm h_2)R(g_2)] = T(\pm 1)R\left(g_1^{-1} g_2^{-h_1} g_1^{h_2} g_2\right)$$

for $h_1, g_1, h_2, g_2 \in Q_8$. In particular, the derived subgroup is $\{\, T(\pm 1)R(\pm 1) \,\}$.

(g) *An element $T(\pm h)R(g)$ is of order 4 if and only if $g^h \neq g^{-1}$. In particular, there are 12 elements of order 4 in the Pauli hull, namely*

$$T(\pm 1)R(g) \quad \textit{and} \quad T(\pm g)R(g)$$

for $g \notin Z(Q_8)$.

(h) *There is an injective group homomorphism*

$$g \mapsto T(\pm 1)R(g) \tag{3.4}$$

from Q_8 *to the Pauli hull.*

Proof. (a) This part follows by [11, Prop 2.13(a)].

(b) This part follows directly from [11, (2.6)]

(c) From (b), we have

$$T(\pm h)R(g^{-h})T(\pm h)R(g) = T(\pm h^2)R(g^{-h^2}g) = T(\pm 1)R(1)$$

for $h, g \in Q_8$, since $T(h^2) = 1$ for $h \in Q_8$.

(d) The statement follows immediately by [11, Prop. 2.9].

(e) By (c) and (b), we have $T(\pm h_1)R(g_1) \uparrow T(\pm h_2)R(g_2)$

$$\begin{aligned}
&= \big(T(\pm h_2)R(g_2)\big)^{-1}T(\pm h_1)R(g_1)T(\pm h_2)R(g_2) \\
&= T(\pm h_2)R(g_2^{-h_2})T(\pm h_1)R(g_1)T(\pm h_2)R(g_2) \\
&= T(\pm h_2)R(g_2^{-h_2})T(\pm h_1 h_2)R(g_1^{h_2}g_2) \\
&= T(h_1)R(g_2^{-h_2h_1h_2}g_1^{h_2}g_2) = T(h_1)R\left(g_2^{-h_1}g_1^{h_2}g_2\right)
\end{aligned}$$

for $h_1, g_1, h_2, g_2 \in Q_8$.

(f) By (c), (e) and (b), we have $[T(\pm h_1)R(g_1), T(\pm h_2)R(g_2)]$

$$\begin{aligned}
&= T(\pm h_1)R(g_1)^{-1} \cdot T(\pm h_1)R(g_1) \uparrow T(\pm h_2)R(g_2) \\
&= T\left(\pm h_1^{-1}\right)R\left(g_1^{-h_1}\right)T(\pm h_1)R\left(g_2^{-h_1}g_1^{-h_2}g_2\right) \\
&= T(\pm 1)R\left(g_1^{-1}g_2^{-h_1}g_1^{h_2}g_2\right)
\end{aligned}$$

for $h_1, g_1, h_2, g_2 \in Q_8$.

(g) The first statement follows from (c), since the possible orders for elements of the extraspecial Pauli hull are $1, 2$ and 4.

(h) This part follows directly from (b). □

Theorem 3.3 (and its dihedral group analogue Theorem 2.1) use the representation of group multiplication group elements that is provided by [11, Prop. 2.13]. When dealing with the multiplication group of the quaternion group, however, it is almost always better to employ the multivalent notation. This notation works well with the quaternion group $Q_8 = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$, since its center is $\{\pm 1\}$. We begin with an analysis of the elements of order 4 in the Pauli hull $\operatorname{Mlt} Q_8$, and their commutation properties.

Definition 3.4. In the following, $g, h \in Q_8 \smallsetminus Z(Q_8)$.

(a) An element $\langle L'(\pm 1), R(g)\rangle$ of $\mathrm{Mlt}\, Q_8$ is of *right type.*
(b) An element $\langle L'(\pm h), R(\pm 1)\rangle$ of $\mathrm{Mlt}\, Q_8$ is of *left type.*
(c) An element $\langle L'(h), R(g)\rangle$ of $\mathrm{Mlt}\, Q_8$ is of *mixed type.*
(d) The elements $\langle L'(\pm 1), R(\pm 1)\rangle$ of $\mathrm{Mlt}\, Q_8$ are of *central type.*

Remark 3.5. The types appearing in Definition 3.4 appear as follows in the notation of [11, Prop. 2.13]:

R: $\langle L'(\pm 1), R(g)\rangle = T(\pm 1)R(g)$;
L: $\langle L'(\pm h), R(\pm 1)\rangle = T(\pm h)R(h)$;
M: $\langle L'(h), R(g)\rangle = T(\pm h)R(hg)$

(compare [11, Lemma 2.18]).

Lemma 3.6. *An element* $\langle L'(h), R(g)\rangle$ *of* $\mathrm{Mlt}\, Q_8$ *has order* 4 *if and only if* $h^2 \neq g^2$.

Proof. By [11, Prop. 2.17], we have $\langle L'(h), R(g)\rangle^2 = \langle L'(h^2), R(g^2)\rangle$. The latter element is the identity element of $\mathrm{Mlt}\, Q_8$ if and only if $h^2 = g^2$. □

Lemma 3.7. *There are* 12 *elements of order* 4 *in* $\mathrm{Mlt}\, Q_8$*:* 6 *of right type and* 6 *of left type.*

Proof. By Lemma 3.6, the 6 elements $\langle L'(1), R(g)\rangle = \langle L'(-1), R(-g)\rangle$ of right type have order 4. Again by Lemma 3.6, the 6 elements $\langle L'(h), R(1)\rangle = \langle L'(-h), R(-1)\rangle$ of left type have order 4. On the other hand, an element $\langle L'(h), R(g)\rangle$ of mixed type, having $h^2 = -1 = g^2$, is not of order 4. The elements of central type have orders 1 and 2. □

In contrast with the preceding results, the following lemma is based on the group multiplication group notation of [11, Prop. 2.13(a)].

Lemma 3.8. *Suppose* $g, h \in Q_8 \smallsetminus Z(Q_8)$*, with* $\langle g\rangle \neq \langle h\rangle$*.*

(a) $[T(\pm 1)R(g), T(\pm h)R(h)] = 1$.
(b) $[T(\pm 1)R(g), T(\pm 1)R(h)] \neq 1$.
(c) $[T(\pm g)R(g), T(\pm h)R(h)] \neq 1$.

Proof. (a) By Theorem 3.3(f), we have $[T(\pm 1)R(g), T(\pm h)R(h)] =$

$$T(\pm 1)R\left(g^{-1}h^{-1}g^h h\right) = T(\pm 1)R\left(g^{-1}h^{-1}hgh^{-1}h\right) = 1\,.$$

Alternatively, one may invoke Remark 3.5 and note the commuting of right type elements with left type elements, by [11, Prop. 2.17],

(b) By Theorem 3.3(h), we have

$$[T(\pm 1)R(g), T(\pm 1)R(h)] = T(\pm 1)R\left([g,h]\right) \neq 1\,.$$

(c) By Theorem 3.3(f), we have

$$\begin{aligned}[T(\pm g)R(g),T(\pm h)R(h)] &= T(\pm 1)R\left(g^{-1}(h^{-1})^g g^h h\right)\\ &= T(\pm 1)R\left((g^{-1}gh^{-1}g^{-1}hgh^{-1}h\right) = T(\pm 1)R\left([h,g]\right) \neq 1\,.\end{aligned}$$

In (a) and (c), we freely use results such as $h^{-1}gh = hgh^{-1}$. □

3.3. **The Pauli group dictionary.** This section presents Table 1, which correlates the various descriptions of the Pauli group that have appeared up to this point in the paper. For the quaternion group multiplication group descriptions from §3.2 in the third column of the table, the non-central elements of the quaternion group are written as $\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$ to distinguish them, say, from phase terms like $\pm i$. The first column in Table 1 identifies elements with the quantum information theory notation taken from [11, (3.26)]. The second column records the dihedral multiplication group descriptions. The fourth column refers to the abstract presentation from §2.3.2, completed using (2.16) and (2.17). The critical issue at this stage is to identify a specific abstract Pauli group within the extraspecial Pauli hull concretely represented as $\mathrm{Mlt}\, Q_8$. Later, in §3.5, we provide a more general overview of Pauli subgroups in $\mathrm{Mlt}\, Q_8$.

The $1 + 3 + 3 + 1$ breakup of the rows in the body of the table generally corresponds to the column breakup in [11, Table 5], while the fifth column of Table 1 gives expressions corresponding to the unit group of the Clifford algebra Cl(3.0). The 8 rows of Table 1 only list half of the 16 elements of the Pauli group: a *transversal* in the sense of [11, Def'n. 3.18]. Representations of the remaining elements, the negatives of the elements listed, are obtained upon multiplication by the square of the element appearing in the last row of the table.

The Pauli group elements in the third set of body rows satisfy

$$\begin{cases} (-iX)(iY) &= (iZ)\,, \\ (iY)(iZ) &= (-iX)\,, \quad \text{and} \\ (iZ)(-iX) &= (iY) \end{cases} \tag{3.5}$$

by [11, (3.27)], and thus generate a quaternion subgroup of the Pauli group. In our assignment of specific $\mathrm{Mlt}\, Q_8$ elements to the Pauli group, we aim to align this quaternion subgroup with the image of the homomorphism (3.4) of Theorem 3.3(h). By Lemma 3.8(a), the elements $T(\pm h)R(h)$ of $\mathrm{Mlt}\, Q_8$ centralize the elements of the image. We break the symmetry of the quaternion generators by choosing $T(\pm\mathbf{i})R(\mathbf{i})$ as our specific centralizing element, matching $h = \mathbf{i}$ to the phase element iI in the Pauli group G_1. This symmetry-breaking motivates our choice of $-iX$ as the single "negated" quaternion generator in (3.5).

Pauli	Dihedral	Quaternion	Abstract	Cl(3, 0)
I	$TR(\pm 1)RR(-1)$	$T(1)R(1)$	1	e_0
X	$TR(\pm 1)RL(1)$	$T(\pm\mathbf{i})R(1)$	x_2	e_1
Y	$TR(\pm i)RR(1)$	$T(\pm 1)R(-\mathbf{k})$	$x_1^3x_3$	e_2
Z	$TR(\pm 1)RL(i)$	$T(\pm\mathbf{i})R(\mathbf{j})$	$x_1^2x_2x_3$	e_3
$-iX$	$TR(\pm i)RL(-i)$	$T(\pm 1)R(\mathbf{i})$	x_1x_2	e_3e_2
iY	$TR(\pm 1)RR(i)$	$T(\pm 1)R(\mathbf{j})$	x_3	e_3e_1
iZ	$TR(\pm i)RL(-1)$	$T(\pm 1)R(\mathbf{k})$	$x_1^3x_2x_3$	e_1e_2
iI	$TR(\pm i)RR(i)$	$T(\pm\mathbf{i})R(\mathbf{i})$	x_1	$e_1e_2e_3 = \omega$

Table 1. The descriptions of the Pauli group.

Remark 3.9. As noted in connection with the specification of $\operatorname{Aut} Q_8$ given in §3.4.1 below, the three possible symmetry-breakings involved in our assignment process correspond nicely with the three possible choices of a single octahedron from the three that are displayed in [8, Fig. 9]. Our choice, singling out the quaternion pair $\pm\mathbf{i}$, corresponds to the middle octahedron from that figure.

It is interesting to observe that the same symmetry-breaking occurs when superalgebra structure is placed on the quaternion algebra, as discussed in [11, Ex. 3.16].

To verify the correctness of our chosen assignments of quaternion elements, we have the following analogue of Proposition 2.13.

Proposition 3.10. *The relations*

$$x_1^4 = x_2^2 = 1\,, \quad x_1^2 = x_3^2\,, \tag{3.6}$$

$$[x_1, x_2] = [x_1, x_3] = 1\,, \tag{3.7}$$

$$x_2x_3x_2 = x_1^2x_3 \tag{3.8}$$

are satisfied by

$$x_1 = T(\pm\mathbf{i})R(\mathbf{i})\,, \quad x_2 = T(\pm\mathbf{i})R(1)\,, \quad x_3 = T(\pm 1)R(\mathbf{j})$$

in the Pauli subgroup.

Proof. The power relations (3.6) follow by Theorem 3.3(c) and (g). The commutation relations (3.7) hold by Lemma 3.8(a). Note that the lemma explicitly gives the commuting of x_1 with x_1x_2, from which the first relation of (3.7) immediately follows. Finally, by (3.6), we have $x_2x_3x_2 = x_3^{x_2}$. Theorem 3.3(e) shows that

$$T(\pm 1)R(\mathbf{j}) \uparrow T(\pm \mathbf{i})R(1) = T(\pm 1)R(-\mathbf{j})\,.$$

Theorem 3.3(c) now gives $T(\pm 1)R(\mathbf{j})^{-1} = T(\pm 1)R(-\mathbf{j})$, confirming that $x_2x_3x_2 = x_3^{-1} = x_1^2x_3$ as required for (3.8). □

3.4. **Automorphisms of the Pauli hull.** A comparison of the two respective quasigroup-theoretical descriptions of the extraspecial Pauli hull given by Theorems 2.6 and 3.3 is instructive. The former is more basic: it reduces computations down to multiplication and conjugation of complex numbers. On the other hand, while the latter is more elegant and conceptual, its computations do involve multiplication and conjugation (this time, in the group-theoretical sense!) of quaternions. The value of the quaternion group approach is best seen in its efficient description of the 1152 automorphisms of the extraspecial Pauli hull in multiplication group terms (Theorem 3.16 below). Remark 3.17 notes that D_4 could not serve as the basis for a similar description.

3.4.1. *Automorphisms of the quaternion group.* The following result is well known (cf. [8, §3.4]). The proof presented here uses the geometric idea from [8] of encoding group products as cyclic orderings of vertices of the faces of a triangulation of an oriented 2-manifold. However, while the general method of [8] uses three (boundaries of) octahedra for Q_8, a single octahedron suffices for our current purpose. It will nevertheless transpire that selection of a single octahedron from the triple appearing in [8, Fig. 9] exactly tracks the symmetry-breaking discussed in §3.3 below (see Remark 3.9).

Proposition 3.11. *Consider the quaternion group Q_8.*

(a) *Each automorphism of Q_8 fixes -1.*
(b) *The automorphism group of Q_8 is the symmetric group S_4.*

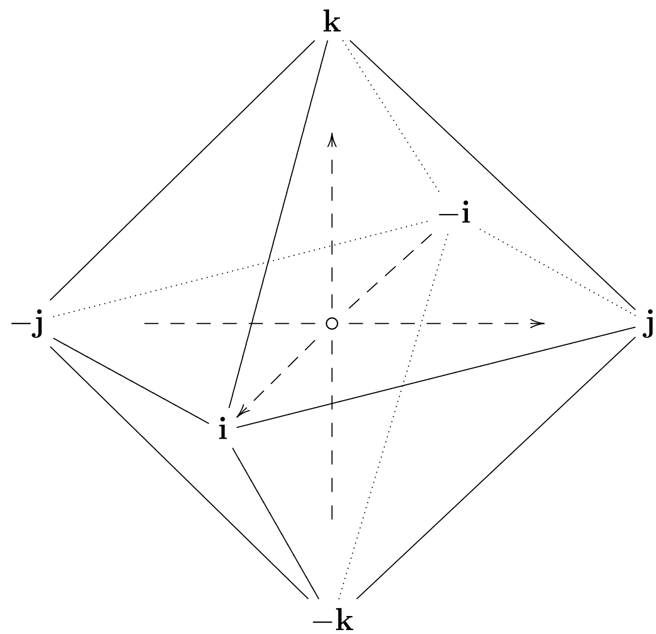
Proof. (a) The element -1 is fixed by automorphisms, since it is the unique element of order 2.

(b) By (a), the effect of automorphisms of Q_8 is determined on its subset

$$V = \{\, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k} \,\}$$

of 6 elements of order 4. Consider the octahedron that is the convex hull of these elements in the underlying three-dimensional real vector space consisting of the purely imaginary quaternions. Thus, V is the vertex set of an octahedron:

(3.9)



The octahedron's faces are the 8 triangles that are the convex hulls of linearly independent subsets of V. When the octahedron is oriented by outbound normals to these faces, the set of vertices of each face, such as $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is oriented in cyclic counterclockwise order $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$ around its outbound normal. An orientation of each edge of the face is induced, following this cyclic order.

This geometry encodes the non-trivial products in Q_8, namely those that are not witnessed by an abelian subgroup. Given an ordered linearly independent pair of elements of V, say $\mathbf{k} < \mathbf{i}$, the given ordering on that pair is induced from the cyclic counterclockwise ordering of the vertices of a unique face: in this case $\mathbf{k} \wedge \mathbf{i} \wedge \mathbf{j} = \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$. The product of the ordered linearly independent pair is then given by the third vertex of the face: in this case, $\mathbf{k}$.

It follows that the automorphism group of Q_8 is the symmetry group of the oriented octahedron, namely S_4 as the group of permutations of the 4 pairs of distinct closed faces of (3.9). □

Remark 3.12. (a) Our use of exterior algebra notation, such as $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$ for the oriented convex hull of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, follows [2, §IV.4], reprinted as [3, p.62].

(b) Consider a rotation by $2\pi/3$ of the octahedron (3.9) about the axis joining the barycenters of the opposite faces $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$ and $-\mathbf{k} \wedge -\mathbf{j} \wedge -\mathbf{i}$.

This octahedron symmetry realizes the outer automorphism

$$\theta_3 = (\mathbf{i}\ \mathbf{j}\ \mathbf{k})(-\mathbf{i}\ \ -\mathbf{j}\ \ -\mathbf{k}) \tag{3.10}$$

of Q_8.

(c) Consider a rotation by π of the octahedron (3.9) about the axis joining the barycenters of the opposite edges $\mathbf{i} \wedge \mathbf{j}$ and $-\mathbf{j} \wedge -\mathbf{i}$. This octahedron symmetry realizes the outer automorphism

$$\theta_2 = (\mathbf{k}\ \ -\mathbf{k})(\mathbf{i}\ \mathbf{j})(-\mathbf{i}\ \ -\mathbf{j}) \tag{3.11}$$

of Q_8.

(d) The conjugations

$$T(\mathbf{j}) = (\mathbf{k}\ \ -\mathbf{k})(\mathbf{i}\ \ -\mathbf{i}) \quad \text{and} \quad T(\mathbf{k}) = (\mathbf{i}\ \ -\mathbf{i})(\mathbf{j}\ \ -\mathbf{j}) \tag{3.12}$$

generate the inner automorphism group $C_2 \times C_2$ of Q_8. Note that $T(\mathbf{k})$ is the restriction of the involution automorphism of the Clifford algebra $\mathrm{Cl}(0, 2)$ to Q_8 as its unit group displayed in [11, Table 3].

(e) The 8-element subgroup of $\mathrm{Aut}\, Q_8$ that is generated by the outer automorphism $\theta_2^{\theta_3} = (\mathbf{i}\ -\mathbf{i})(\mathbf{j}\ \mathbf{k})(-\mathbf{j}\ -\mathbf{k})$ and the inner automorphisms (3.12) stabilizes the maximal abelian subgroup $\langle \mathbf{i} \rangle$ setwise. Since the outer automorphism conjugates $T(\mathbf{j})$ to $T(\mathbf{k})$ in $\mathrm{Aut}\, Q_8$, the stabilizer is isomorphic to $C_2 \wr C_2 \cong D_4$.

(f) The polytope (3.9) may be interpreted as an instance of the *Bloch octahedron* spanned by 1-qubit stabiliser states (cf. [1, Fig. 4], [7, Fig. 2.1]), geometrically dual to the *quantum cube* of [1]. While the vertices of the Bloch octahedron are usually defined in linear-algebraic terms as eigenspaces of maximal abelian subgroups of the Pauli group, our subsequent treatment of the Pauli subgroups of $\mathrm{Mlt}\, Q_8$ will offer more direct interpretations.

3.4.2. *Automorphisms of the quaternion multiplication group.* In this section, we will generally track elements of $\mathrm{Mlt}\, Q_8$ using the multivalent representation.

Proposition 3.13. *Let θ and ϕ be automorphisms of Q_8. Then*

$$(\theta, \phi)\colon\ \langle L'(h), R(g)\rangle \mapsto \langle L'(h^\theta), R(g^\phi)\rangle \tag{3.13}$$

gives a well-defined automorphism of $\mathrm{Mlt}\, Q_8$*.*

Proof. Consider a central element z of Q_8. By Proposition 3.11(a), we have $z^\theta = z = z^\phi$. Now

$$\begin{aligned}(\theta, \phi)\colon\ \langle L'(hz), R(gz)\rangle \mapsto \langle L'((hz)^\theta), R((gz)^\phi)\rangle &= \langle L'(h^\theta z^\theta), R(g^\phi z^\phi)\rangle \\ &= \langle L'(h^\theta z), R(g^\phi z)\rangle = \langle L'(h^\theta), R(g^\phi)\rangle\ ,\end{aligned}$$

so that (3.13) is well-defined. Note then that $(\theta,\phi)\colon \mathrm{Mlt}\,Q_8 \to \mathrm{Mlt}\,Q_8$ is invertible, with inverse (θ^{-1},ϕ^{-1}).

Finally, for elements h_1,h_2,g_1,g_2 and automorphisms θ,ϕ of Q_8, we have $\langle L'(h_1),R(g_1)\rangle^{(\theta,\phi)}\,\langle L'(h_2),R(g_2)\rangle^{(\theta,\phi)}$

$$= \langle L'(h_1^\theta),R(g_1^\phi)\rangle\,\langle L'(h_2^\theta),R(g_2^\phi)\rangle = \langle L'(h_1^\theta h_2^\theta),R(g_1^\phi g_2^\phi)\rangle$$
$$= \langle L'((h_1h_2)^\theta),R((g_1g_2)^\phi)\rangle = \langle L'(h_1h_2),R(g_1g_2)\rangle^{(\theta,\phi)}\ ,$$

so that (θ,ϕ) is an automorphism of $\mathrm{Mlt}\,Q_8$. □

Definition 3.14. For automorphisms θ,ϕ of Q_8, the automorphisms (θ,ϕ) of $\mathrm{Mlt}\,Q_8$ given by (3.13) are said to be *internal.*

Proposition 3.15. *The set*

$$\Pi = \{\,(\theta,\phi)\mid \theta,\phi\in \mathrm{Aut}\,Q_8\,\}$$

of internal automorphisms of $\mathrm{Mlt}\,Q_8$ *forms a subgroup of* $\mathrm{Aut}\,\mathrm{Mlt}\,Q_8$ *of order* 24^2*, isomorphic to the direct square of* $\mathrm{Aut}\,Q_8$*.*

Proof. For internal automorphisms (θ_1,ϕ_1) and (θ_2,ϕ_2), we have

$$\langle L'(h),R(g)\rangle\,(\theta_1,\phi_1)(\theta_2,\phi_2) = \langle L'(h^{\theta_1\theta_2}),R(g^{\phi_1\phi_2})\rangle$$
$$= \langle L'(h),R(g)\rangle\,(\theta_1\theta_2,\phi_1\phi_2)$$

for each pair h,g of elements of Q_8, establishing a group homomorphism

$$\pi\colon (\mathrm{Aut}\,Q_8)^2 \to \mathrm{Aut}\,\mathrm{Mlt}\,Q_8;\theta\otimes\phi \mapsto (\theta_1,\phi_1)\,.$$

Here, we use tensor product notation for the ordered pairs in $(\mathrm{Aut}\,Q_8)^2$.

The task is to show that the kernel of π is trivial. Suppose there are automorphisms θ,ϕ of Q_8 such that

$$\langle L'(h^\theta),R(g^\phi)\rangle = \langle L'(h),R(g)\rangle$$

for all $h,g\in Q_8$. In other words, $(h^{-1})^\theta x g^\phi = h^{-1}xg$, or equivalently, $xg^\phi g^{-1} = h^\theta h^{-1}x$, for all x in Q_8. Taking $x=1$ gives $g^\phi g^{-1} = h^\theta h^{-1}$, so that $g^\phi g^{-1} = z = h^\theta h^{-1}$ for some central element z.

We now have $g^\phi = gz$ and $h^\theta = hz$ for all $h,g\in Q_8$. In other words, $\phi = R(z) = \theta$. But θ and ϕ are automorphisms of Q_8, so $z=1$. Thus θ and ϕ are the identity automorphisms of Q_8, as required. □

Theorem 3.16. *The automorphism group of* $\mathrm{Mlt}\,Q_8$ *is isomorphic to the wreath product* $S_4\wr S_2$*, of order* $(24)^2\times 2 = 1152$*. Specifically, each automorphism has the form*

$$S^\varepsilon(\theta,\phi)S^\varepsilon = (\theta,\phi)\sigma^\varepsilon \tag{3.14}$$

with $S\colon Q_8\to Q_8; x\mapsto x^{-1}$ *or* σ *as in* [11, (2.8)]*, for automorphisms* θ,ϕ *of* Q_8 *and* $\varepsilon\in\{\,0,1\,\}$*.*

Proof. According to [10, Prop. 2.19], the conjugation σ by S is an automorphism of $\mathrm{Mlt}\, Q_8$. Since Q_8 is not Boolean, [11, Cor. 2.20] shows that σ is not internal. It follows from Proposition 3.15 that $\Pi \cup \Pi\sigma$ is a subgroup of $\mathrm{Aut}\,\mathrm{Mlt}\, Q_8$, of order $(24)^2 \times 2 = 1152$, isomorphic to the wreath product $S_4 \wr S_2$.

The group $\mathrm{Out}\,\mathrm{Mlt}\, Q_8$ of outer automorphisms of the extraspecial group $E_2 \cong \mathrm{Mlt}\, Q_8$ is isomorphic to the full orthogonal group $0^+(4,2)$ of the central quotient E_2^ζ of E_2 [9, Bem. III.13.9(b)]. Here, we are using the notation of [4] for the full orthogonal group. The orthogonal group $\Omega^+(4,2)$, isomorphic to $\mathsf{SL}_2(2) \times \mathsf{SL}_2(2)$, is a subgroup of index 2 in $0^+(4,2)$ [4, §2]. It follows that $|\mathrm{Out}\,\mathrm{Mlt}\, Q_8| = 6^2 \times 2 = 72$.

The inner automorphism group $\mathrm{Inn}\,\mathrm{Mlt}\, Q_8$ has order 16, as given, for example, by Theorem 2.6(e). Thus the order of the full automorphism group $\mathrm{Aut}\,\mathrm{Mlt}\, Q_8$ is $72 \times 16 = 1152$. It follows that every automorphism of $\mathrm{Mlt}\, Q_8$ has the form (3.14). $\square$

Remark 3.17. The dihedral group D_4 has inner automorphism group C_2^2 (cf. §2.4) and outer automorphism group C_2 — cf. Theorem 2.1(j). Thus, D_4 only has 8 automorphisms, an insufficient number for it to be able to provide an analogue of Theorem 3.16.

3.5. **Pauli subgroups within the Pauli hull.** As an application of Table 1, and as a counterpart to Proposition 2.9, the following result identifies our chosen Pauli subgroup within the Pauli hull $\mathrm{Mlt}\, Q_8$, using the multivalent notation for the multiplication group.

Proposition 3.18. *Within the Pauli hull* $\mathrm{Mlt}\, Q_8$*, the subset*

$$L_{\mathbf{i}} = \{ \langle L'(h), R(g) \rangle \mid g \in Q_8 \,,\ h \in \langle \mathbf{i} \rangle \} \tag{3.15}$$

forms a subgroup of order 16 *modeling the Pauli group.*

Proof. The result is apparent on translating the elements of the third column of Table 1 into multivalent notation using [11, Lemma 2.18]. $\square$

Proposition 3.18 and Theorem 3.16 may now be used to give a fuller overview of the six Pauli subgroups in the Pauli hull (cf. [6, p.150]), manifesting the symmetry broken by the choice of $L_{\mathbf{i}}$ that underlies

Table 1. We extend the notation of (3.15) as follows:

$$\begin{aligned}
L_{\mathbf{i}} &= \{ \langle L'(h), R(g)\rangle \mid g \in Q_8 ,\ h \in \langle \mathbf{i}\rangle \} \ ; \\
L_{\mathbf{j}} &= \{ \langle L'(h), R(g)\rangle \mid g \in Q_8 ,\ h \in \langle \mathbf{j}\rangle \} \ ; \\
L_{\mathbf{k}} &= \{ \langle L'(h), R(g)\rangle \mid g \in Q_8 ,\ h \in \langle \mathbf{k}\rangle \} \ ; \\
R_{\mathbf{i}} &= \{ \langle L'(g), R(h)\rangle \mid g \in Q_8 ,\ h \in \langle \mathbf{i}\rangle \} \ ; \\
R_{\mathbf{j}} &= \{ \langle L'(g), R(h)\rangle \mid g \in Q_8 ,\ h \in \langle \mathbf{j}\rangle \} \ ; \\
R_{\mathbf{k}} &= \{ \langle L'(g), R(h)\rangle \mid g \in Q_8 ,\ h \in \langle \mathbf{k}\rangle \} \ .
\end{aligned} \tag{3.16}$$

Theorem 3.19. *In the extraspecial Pauli hull* $\operatorname{Mlt} Q_8$*, the six subgroups exhibited in* (3.16) *constitute the orbit of the chosen Pauli subgroup* (3.15) *under the automorphism group of* $\operatorname{Mlt} Q_8$*.*

Proof. By [10, Prop. 2.19], the automorphism σ of $\operatorname{Mlt} Q_8$ interchanges L_h with R_h, for each $h \in \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \}$.

Now, the outer automorphism θ_3 of Q_8 from (3.10) gives

$$(\theta_3, \phi)\colon L_{\mathbf{i}} \mapsto L_{\mathbf{j}} \mapsto L_{\mathbf{k}} \mapsto L_{\mathbf{i}}$$

in the notation of Theorem 3.16, for any automorphism ϕ of Q_8. On the other hand, for any automorphisms θ, ϕ of Q_8, we have

$$(\theta, \phi)\colon L_{\mathbf{i}} \mapsto \{ \langle L'(h^\theta), R(g^\phi)\rangle \mid g \in Q_8 ,\ h \in \langle \mathbf{i}\rangle \} \ . \tag{3.17}$$

Here, the set $\{ L'(h^\theta) \mid \ h \in \langle \mathbf{i}\rangle \}$ is one of the three maximal abelian subgroups $\langle \mathbf{i}\rangle , \langle \mathbf{j}\rangle , \langle \mathbf{k}\rangle$, of Q_8. Thus, the image of (3.17) is an element of $\{ L_{\mathbf{i}}, L_{\mathbf{j}}, L_{\mathbf{k}} \}$, as required to complete the proof of the theorem. □

Table 2 presents the 4 central classes (which constitute the phase space G_1^ζ) of the respective models (3.16) of the Pauli group inside $\operatorname{Mlt} Q_8$. The double lines in the table are designed to emphasize how

$$\text{the column vector } \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \text{ and matrix } \begin{bmatrix} \mathbf{ii} & \mathbf{ij} & \mathbf{ik} \\ \mathbf{ji} & \mathbf{jj} & \mathbf{jk} \\ \mathbf{ki} & \mathbf{kj} & \mathbf{kk} \end{bmatrix} ,$$

along with its transpose, underlie the structure of the table.

Table 2 may also be recast to show how the 12 elements of order 4 in $\operatorname{Mlt} Q_8$ (cf. Lemma 3.7) appear as the 8 elements $\pm iI, \pm iX, \pm iY, \pm iZ$ in each of the 6 models of the Pauli group G_1. See Table 3. This time, the double lines in the table are designed to emphasize how the

$$\text{column vector } \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \text{ and matrix } \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix}$$

underlie the structure of the table.

	$\langle i\rangle\, I$	$\langle i\rangle\, X$	$\langle i\rangle\, Y$	$\langle i\rangle\, Z$
$L_{\mathbf{i}}$	$\langle L'(\langle\mathbf{i}\rangle), R(1)\rangle$	$\langle L'(\langle\mathbf{i}\rangle), R(\mathbf{i})\rangle$	$\langle L'(\langle\mathbf{i}\rangle), R(\mathbf{j})\rangle$	$\langle L'(\langle\mathbf{i}\rangle), R(\mathbf{k})\rangle$
$L_{\mathbf{j}}$	$\langle L'(\langle\mathbf{j}\rangle), R(1)\rangle$	$\langle L'(\langle\mathbf{j}\rangle), R(\mathbf{i})\rangle$	$\langle L'(\langle\mathbf{j}\rangle), R(\mathbf{j})\rangle$	$\langle L'(\langle\mathbf{j}\rangle), R(\mathbf{k})\rangle$
$L_{\mathbf{k}}$	$\langle L'(\langle\mathbf{k}\rangle), R(1)\rangle$	$\langle L'(\langle\mathbf{k}\rangle), R(\mathbf{i})\rangle$	$\langle L'(\langle\mathbf{k}\rangle), R(\mathbf{j})\rangle$	$\langle L'(\langle\mathbf{k}\rangle), R(\mathbf{k})\rangle$
$R_{\mathbf{i}}$	$\langle L'(1), R(\langle\mathbf{i}\rangle)\rangle$	$\langle L'(\mathbf{i}), R(\langle\mathbf{i}\rangle)\rangle$	$\langle L'(\mathbf{j}), R(\langle\mathbf{i}\rangle)\rangle$	$\langle L'(\mathbf{k}), R(\langle\mathbf{i}\rangle)\rangle$
$R_{\mathbf{j}}$	$\langle L'(1), R(\langle\mathbf{j}\rangle)\rangle$	$\langle L'(\mathbf{i}), R(\langle\mathbf{j}\rangle)\rangle$	$\langle L'(\mathbf{j}), R(\langle\mathbf{j}\rangle)\rangle$	$\langle L'(\mathbf{k}), R(\langle\mathbf{j}\rangle)\rangle$
$R_{\mathbf{k}}$	$\langle L'(1), R(\langle\mathbf{k}\rangle)\rangle$	$\langle L'(\mathbf{i}), R(\langle\mathbf{k}\rangle)\rangle$	$\langle L'(\mathbf{j}), R(\langle\mathbf{k}\rangle)\rangle$	$\langle L'(\mathbf{k}), R(\langle\mathbf{k}\rangle)\rangle$

TABLE 2. The phase spaces of the Pauli subgroups (3.16).

	$\pm iI$	$\pm iX$	$\pm iY$	$\pm iZ$
$L_{\mathbf{i}}$	$\langle L'(\pm\mathbf{i}), R(1)\rangle$	$\langle L'(1), R(\pm\mathbf{i})\rangle$	$\langle L'(1), R(\pm\mathbf{j})\rangle$	$\langle L'(1), R(\pm\mathbf{k})\rangle$
$L_{\mathbf{j}}$	$\langle L'(\pm\mathbf{j}), R(1)\rangle$	$\langle L'(1), R(\pm\mathbf{i})\rangle$	$\langle L'(1), R(\pm\mathbf{j})\rangle$	$\langle L'(1), R(\pm\mathbf{k})\rangle$
$L_{\mathbf{k}}$	$\langle L'(\pm\mathbf{k}), R(1)\rangle$	$\langle L'(1), R(\pm\mathbf{i})\rangle$	$\langle L'(1), R(\pm\mathbf{j})\rangle$	$\langle L'(1), R(\pm\mathbf{k})\rangle$
$R_{\mathbf{i}}$	$\langle L'(1), R(\pm\mathbf{i})\rangle$	$\langle L'(\pm\mathbf{i}), R(1)\rangle$	$\langle L'(\pm\mathbf{j}), R(1)\rangle$	$\langle L'(\pm\mathbf{k}), R(1)\rangle$
$R_{\mathbf{j}}$	$\langle L'(1), R(\pm\mathbf{j})\rangle$	$\langle L'(\pm\mathbf{i}), R(1)\rangle$	$\langle L'(\pm\mathbf{j}), R(1)\rangle$	$\langle L'(\pm\mathbf{k}), R(1)\rangle$
$R_{\mathbf{k}}$	$\langle L'(1), R(\pm\mathbf{k})\rangle$	$\langle L'(\pm\mathbf{i}), R(1)\rangle$	$\langle L'(\pm\mathbf{j}), R(1)\rangle$	$\langle L'(\pm\mathbf{k}), R(1)\rangle$

TABLE 3. Elements of order 4 in the Pauli subgroups (3.16).

Proposition 3.20. *The setwise stabiliser of a Pauli subgroup under the action of* $\mathrm{Aut\,Mlt}\,Q_8$ *from Theorem 3.19 is the group* $D_4 \times S_4$ *. In particular, the subgroup*

$$\{\, (\theta, \phi) \in \mathrm{Aut\,Mlt}\,Q_8 \mid \langle\mathbf{i}\rangle\,\theta = \langle\mathbf{i}\rangle \,\} \tag{3.18}$$

is the setwise stabiliser of (3.15)*.*

Proof. Consider the Pauli subgroup $L_{\mathbf{i}}$ of (3.15). Suppose that the internal automorphism (θ, ϕ) stabilizes $L_{\mathbf{i}}$ setwise. Then ϕ may be any automorphism of Q_8, while θ must fix the maximal abelian subgroup $\langle\mathbf{i}\rangle$ of Q_8 setwise. Since there are 3 maximal abelian subgroups, there are 8 choices for θ within $\mathrm{Aut}\,Q_8$ — compare Remark 3.12(e). Thus (3.18) has order $24 \times 8 = 192$. There are 6 Pauli subgroups in the orbit (3.16) of $L_{\mathbf{i}}$, and $192 \times 6 = 1152 = |\mathrm{Aut\,Mlt}\,Q_8|$. It follows that (3.18) is the full setwise stabiliser of $L_{\mathbf{i}}$ in $\operatorname{Aut}\operatorname{Mlt} Q_8$. By Remark 3.12(e), the elements θ form a group isomorphic to D_4, while by Proposition 3.11, the independent elements ϕ form a group isomorphic to S_4. $\square$

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