COMBINATORIAL APPROACHES TO THE UNIT GROUPS OF CLIFFORD ALGEBRAS, I

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Abstract. This is the first part of a two-part paper introducing the application of Latin square and quasigroup techniques within a combinatorial approach to the unit groups of Clifford algebras. The two fundamental tools employed are a theorem showing that each unit group of a real Clifford algebra appears as a subgroup inside the multiplication group of a quasigroup, and the recently developed combinatorial theory of supersymmetry, which leads to a more uniform construction of the unit groups than is possible with the central product of groups.

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1. Introduction

1.1. Latin squares and quasigroups.

1.1.1. *Latin squares, quasigroups,* 3*-nets.* The primary combinatorial structures underlying this work are Latin squares, an example of which is exhibited on the left hand side of (1.1). Given a positive number n of distinct symbols, an $n \times n$ *Latin square* is defined as a square array that contains each of the n symbols in each row and in each column of the array. Algebraically, Latin squares are characterized in terms of *quasigroups*; the two structures are often conflated. Along with semigroups, quasigroups represent weakenings of group structure. Semigroups keep the associativity of groups, relaxing cancellativity. On the other hand, quasigroups retain the cancellativity (knowledge of any two of x, y, z in the equation $xy = z$ determines the third uniquely), but the product xy of x and y does not have to be associative. A Latin square becomes the body of the multiplication table of a quasigroup when its rows and columns are labelled by its symbols, as exhibited on the right hand side of (1.1). Conversely, the body of the multiplication table of any (finite, nonempty) quasigroup is a Latin square.

(1.1)

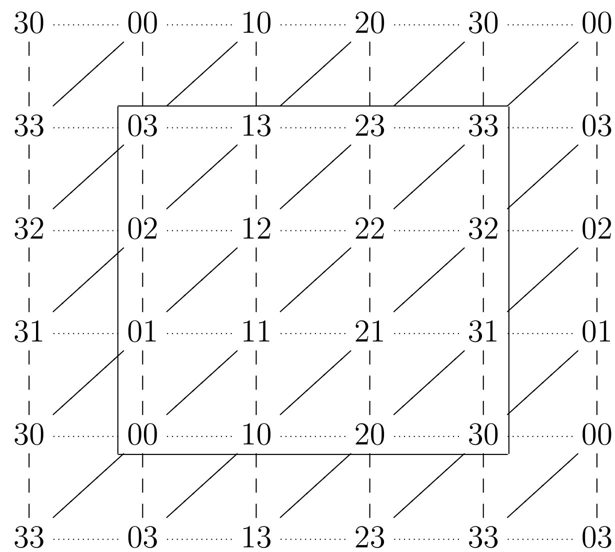
0	3	2	1
1	0	3	2
2	1	0	3
3	2	1	0

	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Latin squares and quasigroups have a geometric interpretation in terms of 3-*nets* [30, Defn. I.4.2], as illustrated in (1.2) on the basis of the Latin square and quasigroup of (1.1). Suppose that the set of symbols is Q, with multiplication xy. The underlying set $Q \times Q$ of the 3-net is partitioned into three pencils of parallel (i.e., mutually exclusive) classes (or *lines*):

- *Horizontal:* common second component y (dotted lines);
- *Vertical:* common first component x (dashed lines);
- *Diagonal:* common product xy (solid lines).

(1.2)



The actual 3-net in (1.2) is given by the box, which lives naturally on a torus. The additional points, provided for better illustration of the diagonal classes, are then identified with their namesake elements inside the box. Each point of the net is determined uniquely as the intersection of specific lines or classes from two different pencils. That point then intersects with a unique line or class from the remaining pencil. A 3-net such as (1.2) may be regarded as (a discrete version of) a *foliation*; the structure used, for example, by Brody and Hughston [7, Fig. 9] to describe expectations of the Hamiltonian, or by Hořava [18, §2.2] to capture the anisotropy in his approach to quantum gravity.

1.1.2. *Quasigroups and projective geometry.* Certain families of Latin squares or quasigroups model projective planes. To take an illustrative example, the projective plane over the 4-element field is modelled by the following Latin squares, which all differ from the four-element Latin square displayed in (1.1).

(1.3)

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	2	3	1
1	3	2	0
2	0	1	3
3	1	0	2

0	3	1	2
1	2	0	3
2	1	3	0
3	1	2	1

The classes in the 3-nets of these Latin squares, namely their common horizontal and vertical classes, along with their individual diagonal classes, correspond to the five *striations* displayed in the depiction of the phase space for a pair of qubits from [35, Fig. 6]. On the other hand, the 3-net (1.2) does not form part of a full projective plane. This ability to capture partial geometries forms a useful feature of Latin squares and quasigroup structure.

1.2. **Multiplication groups and their applications.**

1.2.1. *Multiplication groups.* Universal enveloping algebras provide an associative home for the non-associative products (commutators) in a Lie algebra. Similarly, certain groups, most notably the *multiplication groups*, provide associative products incorporating the non-associative product of a quasigroup §2.2. For example, the dihedral group D_d of degree d may always be obtained (in elementary, combinatorial fashion) as the multiplication group of subtraction of integer residues modulo d [20, Prop. 2.1.4].

1.2.2. *Theme of the work.* The main thrust of our work in these papers is to demonstrate the use of Latin square and quasigroup techniques in a combinatorial approach to the *unit groups* of Clifford algebras. These groups, summarized in Tables 1 and 2, were earlier analyzed in detail by Salingaros [27, 28], and are thus sometimes called “Salingaros vee groups” — compare [1, 2, 23]. As an example, the unit group of the quaternion algebra, the Clifford algebra of the real quadratic form $-x_1^2 - x_2^2$, is the quaternion group Q_8 of order 8.

The fundamental tool that underlies our combinatorial approach is presented in Theorem 3.41:

> Each unit group of a real Clifford algebra appears as a subgroup of the multiplication group of a quasigroup.

In a certain sense, this theorem is the best possible. Note, for example, that the quaternion group Q_8 is not itself the multiplication group of a quasigroup (Corollary 2.12).

1.2.3. *Motivations for the work.* Clifford algebras are usually described by their *gamma matrices* in physical applications (cf. [34, App. E]). Given the pervasiveness of operator methods throughout physics, and the ready availability of linear algebra software packages, one may well ask why any alternative approaches should be considered.

One of the primary motivations for our approach is subsequently to be able to consider (split) octonions in a similar framework, noting that constructs depending on associativity, such as geometric algebras and matrix representations, do not apply to octonions. The recent recognition by Penrose of the importance of the octonions in twistor theory [26] adds some urgency to research in this direction, along with the deep nonassociativity that is suspected to lie at the heart of a complete and satisfactory theory of quantum gravity, for example as indicated in [5, §9.6]. At a more elementary level, one may note the use of the octonions in the classical treatment of Minkowski spacetime, Maxwell's equation, and the wave equation [29, pp.29-30], [30, pp.95-6].

A second motivation for our combinatorial approach is its relative simplicity and directness, when compared against the more elaborate machinery of the linear setting. The reader is invited to compare our combinatorial treatment of the supersymmetry and centers of the unit groups in §3.3.4 with the standard linear-algebraic treatments of the comparable topics in Clifford algebras, such as [31, §5.3], for example.

Inspection of the unit groups of Clifford algebras (Table 2) reveals that central products of groups do not always track the unit groups of tensor products of Clifford algebras construed as superalgebras. For example, Q_8 "should be", but is not, the central square of the cyclic group C_4. If it were, then according to Theorem 2.14 it would be the multiplication group of the group C_4, which instead is just C_4. To handle such situations, we apply the recently introduced combinatorial concept of "signed supergroup" [20, §3.1.2]. Example 3.45 then shows how this new construction creates Q_8 from C_4.

1.3. **Plan of this paper.** Section 2 provides background material on the central products of groups, and on the multiplication groups of quasigroups. In general, the unit groups of real Clifford algebras are *extraspecial* 2-*groups*, whose standard descriptions are given in terms of central products of the groups Q_8 and D_4 [14], [19, §III.13]. The foundational connection with the theory of quasigroups is established by Theorem 2.14, showing that the multiplication group of a group is the central product of that group with itself.

Section 2 presents two descriptions of multiplication groups of groups:

- Proposition 2.13 uses the geometry of the Cayley groupoid of the multiplication group with respect to its generating set of right and left multiplications.
- Proposition 2.17 works with an overdetermined specification of multiplication group elements by chiral pairs that consist of inverted left multiplications and direct right multiplications.

It is convenient to refer to these respective descriptions as the *geometric* and *multivalent* representations.

1.3.1. *Real Clifford algebras, unit groups, supersymmetry.* Section 3 begins with a brief summary of finite-dimensional non-degenerate real quadratic spaces (§3.1) and their Clifford algebras (§3.2). We identify model ordered bases for the quadratic spaces (Definition 3.1). The basis elements are mutually orthogonal, while the quadratic form takes the value 1 on the first p basis elements, and the value -1 on the remaining q basis elements. The approach to the Clifford algebra is combinatorial, working with a signed version of the power set of the model basis of the quadratic space as a replacement for the linear concept of an exterior or Grassmann algebra. The signed power set is closed under the multiplication in the Clifford algebra $\mathrm{Cl}(p,q)$, and forms its *unit group* $E(p,q)$ (Definition 3.8). Tables 1 and 2 provide a convenient overview of the smaller real Clifford algebras and their unit groups. In Table 2, the unit groups are described in traditional terms using the usual mix of central and direct products of cyclic, dihedral, and quaternion groups.

Supersymmetry, which enables the treatment of Clifford algebras as *superalgebras* (Definition 3.11), is the key tool for a full understanding of Clifford algebras, since the Clifford algebra of the direct sum of two quadratic spaces is the *superalgebra tensor product* (Definition 3.37) of the Clifford algebras of the direct summands — (3.22), compare [22, Th. III.3.10], [31, (5.4)]). Based partly on [20] for the combinatorial version, Section 3.3 gives a review of the concept of supersymmetry as it appears in the linear and combinatorial settings, including *supersets* (Definition 3.9), *supergroups* (Definition 3.12, Remark 3.13), and *signed supergroups* (Definition 3.18). The combinatorial analogue of the superalgebra tensor product $\widehat{\otimes}$ is the *signed product* $\widehat{\times}$ of signed supergroups (Definition 3.26).

Once the unit groups of Clifford algebras have been interpreted as signed supergroups (§3.3.3), the apparently random mixes of central and direct products in Table 2 may be replaced uniformly by the signed superproduct (3.23) of Theorem 3.39. As shown in §3.6, the again apparently random mixes of cyclic, dihedral and quaternion groups in Table 2 may be replaced by just two basic building blocks, signed supergroup structures on the Klein four-group and on the cyclic group of order 4 (Proposition 3.43). These are the respective unit groups of the Clifford algebras of the real line with the quadratic forms x^2 and $-x^2$ (Definition 3.44).

Physical applications make unit groups of Clifford algebras especially worthy of attention. Our primary example is the Pauli group of order 16, i.e., the group G_1 of [25, (10.81)], which appears as the unit group of the Clifford algebra of three-dimensional Euclidean space. Abstractly, this Clifford algebra is the algebra $\mathbb{C}_2^2$ of 2×2 complex matrices. The Pauli group is the group generated by the Pauli matrices (3.25). Its connections with quantum information theory are mentioned later [21], but §3.7 considers the *algebraic spinors* that were introduced within Bohm's process approach to quantum theory [6] by Monk and Hiley [24, p.373]. These algebraic spinors are the primitive idempotents of the Clifford algebra. In §3.7.4, we introduce *combinatorial spinors* as an analogue within the Pauli group (3.31): they are the elements of the coset space of a minimal subgroup. Chevalley identifies the quaternion algebra $\mathbb{H}$ as the *space of spinors* for three-dimensional Euclidean space [10, p.121]. Theorem 3.46 identifies the quaternion group Q_8 (the unit group of $\mathbb{H}$) as the space of combinatorial spinors.

1.3.2. *Terminology and notation.* Readers should be aware that names of groups (such as Q_8 or "Pauli group") may be used variously for abstract isomorphism types, or for specific models of these isomorphism types, depending on the context. In general, we default to the algebraic or diagrammatic convention using postfix function application (as with x^2 or $n!$), rather than Euler's prefix notation (as with $\sin\theta$). The book [30] may be used as a general reference for terminology and notation that might not otherwise be explained, such as our use of K_m^n for the space of $m \times n$ matrices over a ring K. Note, however, that we use $\mathbb{Z}/_p$ to denote the set of residues modulo a divisor $p > 1$ (to distinguish from the set $\mathbb{Z}_p$ of p-adic integers for a prime p).

1.4. **Future development.** As a basis for subsequent development of this part of our work, the treatment of quasigroup multiplication groups should be extended to bring in the autotopy groups of quasigroups. Recalling the definition of an *isotopy* from one quasigroup to another as a certain generalization of an isomorphism [29, p.5], an *autotopy* is defined as an isotopy from a quasigroup to itself. The autotopy group of the quasigroup of nonzero octonions is the spin group in dimension eight [33, §3].

2. Central amalgamation and multiplication groups

2.1. **Central amalgamation.** For the following definition, compare [13, Prop. A.19.5].

Definition 2.1. Fix a natural number n. For $1 \leq i \leq n$, consider groups G_i with respective central subgroups $A_i \leq Z(G_i)$. Suppose that there are monomorphisms $\mu_i \colon A \to A_i$ from an abelian group A. Consider the *n-ary unit divisor set*

$$U = \{\, (a_1, \dots, a_n) \in A^n \mid a_1 \cdot \ldots \cdot a_n = 1 \,\}$$

of A. Then in the exact sequence

$$\{1\} \longrightarrow U \xrightarrow{\mu} \prod_{i=1}^{n} G_i \xrightarrow{p} G \longrightarrow \{1\}$$

with

$$\mu\colon (a_1, \dots, a_n) \mapsto (a_1\mu_1, \dots, a_n\mu_n)$$

and

$$p\colon (g_1, \dots, g_n) \mapsto g_1 \cdot \ldots \cdot g_n \,,$$

the group G is described as a *direct product* of the G_i *with amalgamated central subgroups* A_i.

Remark 2.2. (a) If $n = 0$, then the direct product with amalgamated central subgroups is the trivial group $\{1\}$.

(b) Although the choice of the monomorphisms μ_i in the definition affects the structure of G, that choice is not reflected in the terminology.

In certain situations, Definition 2.1 provides a tighter specification of the direct product G with amalgamated central subgroups. For the second case (b) of the following definition, compare [19, Satz I.9.10].

Definition 2.3. (a) For $1 \le i \le r = 2$, consider groups G_i with finite cyclic centers A_i and embeddings $j_i\colon A_i \to \mathsf{U}(1)$ into the circle group $\mathsf{U}(1)$. Let $A = A_1^{j_1} \cap A_2^{j_2}$ be the intersection of the images of the embeddings. For $i = 1, 2$, define monomorphisms $\mu_i\colon A \to A_i$ so that the diagrams

$$\begin{array}{ccc} A & \hookrightarrow & \mathsf{U}(1) \\ {\scriptstyle \mu_i}\downarrow & \nearrow {\scriptstyle j_i} & \\ A_i & & \end{array}$$

commute.

(b) For $1 \le i \le r$ for some natural number r, consider groups G_i with respective centers A_i, each of which has an isomorphism $\mu_i\colon A \to A_i$ with a fixed abelian group A.

In each of the cases (a), (b), consider the application of Definition 2.1.

(i) The direct product of the groups G_i with amalgamated central subgroups A_i is called the *central product* $G_1 \odot \ldots \odot G_r$ of the groups $G_1, \dots, G_r$.

(ii) If $G_1 = \ldots = G_r = G$ for a certain group G, then $G_1 \odot \ldots \odot G_r$ is written as the *central power* $G^{\odot r}$.

Example 2.4. As an instance of case (a) of Definition 2.3, we have $|D_4 \odot C_4| = 16$.

Definition 2.5. [13, A(19.3)] Consider a group G with subgroups H, K, such that $[H, K] = \{ 1 \}$ and $G = HK$. Then G is described as the *internal central product* of its subgroups H, K.

For the following, compare [13, Prop. A.19.5].

Lemma 2.6. *In the context of Definition 2.3(b)(i), the group* $G_1 \odot G_2$ *is the internal direct product of its subgroups* $G_1 \times \{ 1 \}$ *and* $\{ 1 \} \times G_2$.

2.2. **Quasigroups and their multiplication groups.** Let Q be a set equipped with a binary operation

$$Q \times Q \to Q; (p, q) \mapsto pq$$

of *multiplication*. For an element q of Q, the map

$$R(q)\colon Q \to Q; x \mapsto xq$$

is known as *right multiplication* by q, while the map

$$L(q)\colon Q \to Q; x \mapsto qx$$

is known as *left multiplication* by q. The set Q with its multiplication is a *quasigroup* if

$$\{ L(q), R(q) \mid q \in Q \} \tag{2.1}$$

is a subset of the group $Q!$ of bijective maps from Q to Q (*permutations* on Q) . In this case, the notation $T(q) = L(q)^{-1}R(q)$ is also used. Then the subgroup of $Q!$ generated by (2.1) is called the *multiplication group* $\operatorname{Mlt} Q$ of Q. The respective subgroups $\operatorname{LMlt} Q$ and $\operatorname{RMlt} Q$ generated by $\{ L(q) \mid q \in Q \}$ and $\{ R(q) \mid q \in Q \}$ are known as the *left* and *right* multiplication groups of Q.

Lemma 2.7. *Suppose that* Q *is a quasigroup.*

(a) *The map* $R\colon Q \to \operatorname{Mlt} Q; q \mapsto R(q)$ *is injective.*
(b) *The map* $L\colon Q \to \operatorname{Mlt} Q; q \mapsto L(q)$ *is injective.*

Proof. (a) If Q is empty, the statements are trivial. Otherwise, suppose that $R(q) = R(q')$ for $q, q' \in Q$. Then

$$qL(q) = qq = qR(q) = qR(q') = qq' = q'L(q)\,.$$

Since $L(q)$ is injective, it follows that $q = q'$.

(b) is proved in similar fashion. □

Example 2.8. If Q is a group, then $T(q)\colon x \mapsto q^{-1}xq$ is an inner automorphism. In this case, the maps

$$T\colon Q \to \operatorname{Inn} Q; q \mapsto T(q)\,, \tag{2.2}$$

$$R\colon Q \to \operatorname{RMlt} Q; q \mapsto R(q)\,, \quad \text{and} \tag{2.3}$$

$$L\colon Q \to \operatorname{LMlt} Q; q \mapsto L(q) \tag{2.4}$$

are all group homomorphisms, or an antihomomorphism in the latter case. Indeed, an extended version of Cayley's Theorem states that a (nonempty) quasigroup Q is associative, and thus a group, if and only if $R\colon Q \to Q!; q \mapsto R(q)$ is a quasigroup homomorphism.

For a group Q, the three (anti)homomorphisms (2.2)–(2.4) are surjective. Thus, we often just write $\operatorname{RMlt} Q$ as $R(Q)$ and $\operatorname{LMlt} Q$ as $L(Q)$. By Lemma 2.7, both $R(Q)$ and $L(Q)$ are isomorphic copies of Q.

The following result is an immediate consequence of the definitions of the respective multiplication groups.

Proposition 2.9. *Let Q be a quasigroup. Then its full multiplication group, as well as its right and left multiplication groups, are given by their natural faithful transitive permutation actions on the set Q.*

2.2.1. *Surjective quasigroup homomorphisms.* For the following result, see [29, §2.2].

Proposition 2.10. *Suppose that $\theta\colon P \to Q$ is a surjective quasigroup homomorphism. Then the specifications*

$$R(p) \mapsto R(p\theta) \quad \textit{and} \quad L(p) \mapsto L(p\theta)$$

extend to yield

$$\operatorname{Mlt}\theta\colon \operatorname{Mlt} P \to \operatorname{Mlt} Q \tag{2.5}$$

as a group homomorphism.

Corollary 2.11. *Let Q be a quasigroup. Then the image* $\operatorname{Mlt}\operatorname{Aut} Q$ *of the automorphism group* $\operatorname{Aut} Q$ *of the quasigroup Q under the function* $\operatorname{Mlt}\colon \theta \mapsto \operatorname{Mlt}\theta$ *is a subgroup of the automorphism group* $\operatorname{Aut}\operatorname{Mlt} Q$ *of the multiplication group* $\operatorname{Mlt} Q$.

2.2.2. *The no-go theorem.* The following observation is a special case of [29, Prop. 3.19].

Corollary 2.12. *The quaternion group Q_8 is not the multiplication group of a quasigroup.*

2.3. **Multiplication groups of groups.**

2.3.1. *Structure of the multiplication group of a group.*

Proposition 2.13. *Let G be a group.*

(a) *Each element of* $\mathrm{Mlt}\, G$ *may be written in the form* $T(h)R(g)$ *with* $h, g \in G$.
(b) *In terms of the representations from* (a), *multiplication in* G *is given by*

$$T(h_1)R(g_1)T(h_2)R(g_2) = T(h_1h_2)R\big(g_1T(h_2)g_2\big) \tag{2.6}$$

for $h_i, g_i \in G$.

Proof. (a) Consider $\alpha \in \mathrm{Mlt}\, G$, say with $1\alpha = g$. Since $1\alpha R(g)^{-1} = gR(g)^{-1} = 1$, the mapping $\alpha R(g)^{-1}$ is an element of the stabilizer $\mathrm{Inn}\, G$ of 1 in $\mathrm{Mlt}\, G$. As such, $\alpha R(g)^{-1} = T(h)$ for some $h \in G$.

(b) Since $T(h_2)$ is an automorphism of G, and both $T\colon G \to \mathrm{Aut}\, G$ and $R\colon G \to G!$ are group homomorphisms, we have

$$\begin{aligned} xT(h_1)R(g_1)T(h_2)R(g_2) &= xT(h_1)T(h_2)R\big(g_1T(h_2)\big)R(g_2) \\ &= xT(h_1h_2)R\big(g_1T(h_2)g_2\big) \end{aligned}$$

for any $x \in G$. $\square$

In the following, we make use of Definition 2.3(b).

Theorem 2.14. *Let Q be a group. Then* $\mathrm{Mlt}\, Q \cong Q^{\odot 2}$.

Proof. In the group $\mathrm{Mlt}\, Q$, consider the subsets

$$L(Q) = \{\, L(g) \mid g \in Q \,\} \quad \text{and} \quad R(Q) = \{\, R(g) \mid g \in Q \,\} \,.$$

Each is a subgroup, isomorphic to Q. The respective isomorphisms are

$$R\colon Q \to R(Q); g \mapsto R(g)$$

and

$$L'\colon Q \to L(Q); g \mapsto L(g^{-1}) \tag{2.7}$$

under our chosen convention of algebraic notation. (Euler notation would put the inverses in the isomorphism of $R(Q)$ with Q.) For the first isomorphism, note

$$xR(g)R(h) = xgh = xR(gh)$$

and for the second, note

$$xL'(g)L'(h) = h^{-1}g^{-1}x = (gh)^{-1}x = xL'(gh)$$

with $x, g, h \in Q$. By the associative law, we have

$$xL(g)R(h) = (gx)h = g(xh) = xR(h)L(g) \,,$$

so $[L(Q), R(Q)] = \{1\}$. Since $\operatorname{Mlt} Q$ is generated by the union of $L(Q)$ and $R(Q)$, we then have $\operatorname{Mlt} Q = L(Q)R(Q)$. Thus $\operatorname{Mlt} Q$ is the internal central product of its subgroups $L(Q)$ and $R(Q)$ in the sense of Definition 2.5. The result then follows by Lemma 2.6. □

2.3.2. *Central squares of groups.* No quasigroup theory is involved in the statement of the following corollary of Theorem 2.14.

Corollary 2.15. *The central product squaring operation $\odot 2$ provides a functor to the (usual) category of group homomorphisms from the category of surjective group homomorphisms.*

Proof. By [29, §2.2], as summarized by Proposition 2.10, taking the (combinatorial) multiplication group yields a functor to the category of groups from the category of surjective quasigroup homomorphisms. □

Remark 2.16. The direct squaring operation $G \mapsto G^2$ on a group G provides an endofunctor of the category of all group homomorphisms. It is interpreted within quasigroup theory as the (restriction to the full subcategory of nonempty quasigroup homomorphisms of the) universal multiplication group functor for the category of associative quasigroups (compare [29, §10.6]).

2.3.3. *Conjugation by inversion.* The proof of Theorem 2.14 made use of the left multiplications (2.7) by inverses of elements of a group Q. Along with the usual right multiplications (with which they commute), they provide a useful (but multivalent) notation for elements of the multiplication group of a group, namely

$$\langle L'(h), R(g)\rangle : Q \to Q; x \mapsto xL'(h)R(g) = h^{-1}xg$$

for h, g in Q. The multivalence arises from the fact that

$$\langle L'(h), R(g)\rangle = \langle L'(hz), R(gz)\rangle$$

for central elements z of Q. At any rate, we do have the following representation of the multiplication group of a group as a corollary of (the proof of) Theorem 2.14.

Proposition 2.17. *Let Q be a group. Then*

$$\{ \langle L'(h), R(g)\rangle \mid h, g \in Q \}$$

is the multiplication group of Q. The formula

$$\langle L'(h_1), R(g_1)\rangle \, \langle L'(h_2), R(g_2)\rangle = \langle L'(h_1h_2), R(g_1g_2)\rangle$$

expresses the product in the multiplication group.

The following lemma makes the connection with Proposition 2.13.

Lemma 2.18. *Let Q be a group. Then*

$$\langle L'(h), R(g)\rangle = T(h)R\left(h^{-1}g\right) \text{ and } T(h)R(g) = \langle L'(h), R(hg)\rangle$$

for elements h, g of Q.

Proof. Note $x\langle L'(h), R(g)\rangle = h^{-1}xg = h^{-1}xh \cdot h^{-1}g = xT(h)R\left(h^{-1}g\right)$ for x in Q. □

On a group Q, write $S\colon Q \to Q; x \mapsto x^{-1}$ for the inversion involution. Then S normalizes $\operatorname{Mlt} Q$ within $Q!$. Indeed, we have

$$xSR(q)S = \left(x^{-1}q\right)^{-1} = q^{-1}x = xL\left(q^{-1}\right)$$

for all $x, q \in Q$, whence $SR(q)S = L\left(q^{-1}\right) = L'(q)$.

Proposition 2.19. *Let Q be a group. Then conjugation by S within the permutation group $Q!$ acts as an automorphism*

$$\sigma\colon \langle L'(h), R(g)\rangle \mapsto \langle L'(g), R(h)\rangle \tag{2.8}$$

of the multiplication group $\operatorname{Mlt} Q$.

Proof. Note

$$xS\langle L'(h), R(g)\rangle S = \left(h^{-1}x^{-1}g\right)S = g^{-1}xh = x\langle L'(g), R(h)\rangle$$

for x in Q. □

We describe a group as *Boolean* if it is an elementary abelian 2-group, i.e., if every non-identity element has order 2.

Corollary 2.20. *If the group Q is not Boolean, then σ does not lie in $\operatorname{Mlt} Q$.*

Proof. Suppose we have elements x, y of Q such that $\sigma = \langle L'(y), R(x)\rangle$. Then, for every $h, g \in Q$,

$$\begin{aligned}\langle L'(g), R(h)\rangle &= \langle L'(h), R(g)\rangle\,\sigma \\ &= \langle L'(h), R(g)\rangle\langle L'(y), R(x)\rangle = \langle L'(hy), R(gx)\rangle\ .\end{aligned}$$

Since $\operatorname{Mlt} Q$ is the central product of $L'(Q)$ and $R(Q)$, we have that

$$\left(L'(hy), R(gx)\right)^{-1}\left(L'(g), R(h)\right) = \left(L'(z_{hg}), R(z_{hg})\right)$$

for a central element z_{hg} of Q that is determined by g and h, whence

$$\left(L'(g), R(h)\right) = \left(L'(hy), R(gx)\right)\left(L'(z_{hg}), R(z_{hg})\right)\ .$$

and the equations

$$gxz_{hg} = h\,, \qquad hyz_{hg} = g \tag{2.9}$$

hold. In particular, $xz_{11} = 1 = yz_{11}$, so $x = z_{11}^{-1} = y$. The conditions (2.9) thus take the form

$$gz_{11}^{-1}z_{hg} = h \quad \text{and} \quad hz_{11}^{-1}z_{hg} = g$$

which imply $g^{-1}h = z_{11}^{-1}z_{hg} = h^{-1}g = (g^{-1}h)^{-1}$: the contradiction that each element $g^{-1}h$ of Q squares to the identity. □

3. Real Clifford algebras, unit groups, supersymmetry

3.1. **Real quadratic spaces.** Suppose V is a finite-dimensional real vector space that is equipped with a non-degenerate quadratic form $Q\colon V \to \mathbb{R}$. We refer to the pair (V, Q) as a (finite-dimensional) *real quadratic space*.

Suppose that $\{ e_1, \dots, e_n \}$ is a basis for V. Then the quadratic form $Q(x_1e_1 + \dots + x_ne_n)$ is a homogeneous polynomial

$$\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} B \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$$

of degree 2, with a real symmetric matrix B of rank n. By diagonalizing B and rescaling, the basis $\{ e_1, \dots, e_n \}$ may be chosen so that the matrix B takes the block diagonal form $I_p \oplus -I_q$. In other words,

$$Q(e_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq p\,; \\ -1 & \text{if } p < i \leq n\,. \end{cases} \tag{3.1}$$

It is sometimes convenient to relabel the set $\{ e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q} \}$ as

$$\{ e_1, \dots, e_p, f_1, \dots, f_q \}\,. \tag{3.2}$$

Note that the numbers p and $q = n - p$ are defined intrinsically, as the respective dimensions of the largest subspaces on which the quadratic form is positive definite and negative definite.

Definition 3.1. Consider a finite-dimensional real quadratic space (V, Q).

(a) The pair (p, q) is known as the *signature* of (V, Q).
(b) Alternatively, if the dimension $n = p + q$ of V is understood, then $p - q$ may also be identified as the *signature* of (V, Q).
(c) An ordered basis $\{ e_1 < \dots < e_n \}$ satisfying (3.1) is described as a *model basis*.

Remark 3.2. In connection with Definition 3.1(a),(b), note that the pairs (p, q) and $(p + q, p - q)$ are bijectively related.

3.2. **Real Clifford algebras.** In this section, we aim to present a combinatorial account of the Clifford algebras of finite-dimensional real vector spaces equipped with a non-degenerate quadratic form. Thus, while Chevalley makes good use of exterior algebras in his classical treatment of Clifford algebras, in particular treating exterior algebras as Clifford algebras of spaces with the degenerate zero quadratic form [10], we will work with signed versions of the power sets of the model bases. Our combinatorial approach facilitates the analysis of the unit groups of Clifford algebras.

3.2.1. *The universality property.* A universality property provides the most efficient specification of the Clifford algebra of a space equipped with a quadratic form [9, Th. 3.1],[10, p.103],[22, III, §3.1]. Within our context, we may present it as follows.

Definition 3.3. Consider a finite-dimensional real quadratic space (V, Q). Then the *Clifford algebra* $\mathrm{Cl}(V, Q)$ is the real algebra, equipped with a linear embedding $j\colon V \to \mathrm{Cl}(V, Q)$ of the vector space V as a subspace of (the underlying vector space of the algebra) $\mathrm{Cl}(V, Q)$, with the following universality property:

> Each linear map $f\colon V \to A; v \mapsto v^f$ from V to (the underlying vector space of) an algebra A, such that
>
> $$\forall\, v \in V\,,\ v^f v^f = Q(v) 1_A\,,$$
>
> extends to an algebra homomorphism $\overline{f}\colon \mathrm{Cl}(V, Q) \to A$ making the diagram
>
> $$\begin{array}{ccc} V & \xrightarrow{\ j\ } & \mathrm{Cl}(V,Q) \\ \Big\| & & \Big\downarrow{\scriptstyle \overline{f}} \\ V & \xrightarrow[\ f\ ]{} & A \end{array} \tag{3.3}$$
>
> commute.

The following result is an immediate consequence of the universality property — compare [22, §III.3.5].

Lemma 3.4. *Each orthogonal transformation of the quadratic space (V, Q) lifts to an automorphism of the Clifford algebra $\mathrm{Cl}(V, Q)$.*

Definition 3.5. The automorphism $\overline{-j}$ of $\mathrm{Cl}(V,Q)$ defined by the instance

$$\begin{array}{ccc} V & \xrightarrow{\ j\ } & \mathrm{Cl}(V,Q) \\ \| & & \downarrow{\scriptstyle \overline{-j}} \\ V & \xrightarrow[-j]{} & \mathrm{Cl}(V,Q) \end{array}$$

of (3.3) is called the *involution automorphism*.

Remark 3.6. The terminology of Definition 3.5 is justified by the equation $-(-j) = j$.

3.2.2. *The construction.* This section will deliver our combinatorial construction of the Clifford algebra $\mathrm{Cl}(p,q)$ of the quadratic space (V,Q) as in Definition 3.1 with the model basis of Definition 3.1(c). Write each of the ordered subsets $I = \{ e_{i_1} < \ldots < e_{i_r} \}$ of the model basis as $e_I = e_{i_1} \cdots e_{i_r}$. Thus, the power set of the model basis is the set

$$\mathcal{P}\left(\{ e_1 < \ldots < e_n \}\right) = \{ e_{i_1} \cdots e_{i_r} \mid 0 \leq r \leq n \} \ , \tag{3.4}$$

with the empty subset Ø written conventionally as 1.

An (associative) algebra structure, with 1 as the identity element, is defined on the 2^n-dimensional real linear span $\mathrm{Cl}(p,q)$ of the power set $\mathcal{P}\left(\{ e_1 < \cdots < e_n \}\right)$ by

$$\begin{cases} e_i e_j = Q(e_i)1 & \text{if } i = j; \\ e_j e_i = -e_i e_j & \text{if } i \neq j. \end{cases} \tag{3.5}$$

It is sometimes convenient to write 1 as e_0. The improper subset

$$\omega = e_1 e_2 \cdots e_n \tag{3.6}$$

is known as the *volume element* of $\mathrm{Cl}(p,q)$. The embedding

$$j \colon V \to \mathrm{Cl}(p,q)$$

that is the linear extension of the insertion

$$\{ e_1 < \cdots < e_n \} \to \mathcal{P}\left(\{ e_1 < \cdots < e_n \}\right); e_i \mapsto e_i$$

satisfies the universality property of Definition 3.3, with

$$\overline{f} \colon \mathrm{Cl}(p,q) \to A; e_{i_1} \cdots e_{i_r} \mapsto e_{i_1}^f \cdots e_{i_r}^f$$

as the uniquely defined algebra homomorphism that extends a linear transformation $f \colon V \to A$.

As a consequence of (3.5), it is clear that the set

$$E = \pm \mathcal{P}\left(\{ e_1 < \cdots < e_n \}\right) \tag{3.7}$$

is closed under multiplication. If I is a subset of $\{ e_1 < \ldots < e_n \}$ of cardinality r, then the elements $\pm e_I$ of (3.7) are said to have *degree* r. In particular, if $\{ e_{j_1}, \ldots, e_{j_r} \}$ is an r-element subset of $\{ e_1 < \ldots < e_n \}$ which is not necessarily in its ordered form $\{ e_{i_1} < \ldots < e_{i_r} \}$, then the product $e_{j_1} \cdot e_{j_2} \cdots e_{j_r}$ taken within the algebra $\mathrm{Cl}(p,q)$ evaluates to $\sigma e_{i_1} \ldots e_{i_r}$, where σ is the sign of the permutation $e_{j_k} \mapsto e_{i_k}$ of the set $\{ e_{i_1}, \ldots, e_{i_r} \} = \{ e_{j_1}, \ldots, e_{j_r} \}$.

Lemma 3.7. *Consider the set E of* (3.7)*.*

(a) *It forms a group under multiplication.*
(b) *It is invariant under the action of the involution automorphism of Definition 3.5.*

Proof. (a) Invertibility of the elements $\pm I$ of (3.7) is established by induction on the degree. Note that the elements ± 1 of degree zero square to 1, and are thus invertible.

Now, for $0 \leq r < n$, suppose that all the elements s of degree at most r are invertible. Consider an element se_j of degree at most $r+1$. Then $se_j \cdot e_j s^{-1} = sQ(e_j)s^{-1} = Q(e_j) = \pm 1$, so that se_j is invertible.

(b) For $0 \leq r \leq n$, the equation

$$e_{j_1} \cdots e_{j_r} \mapsto (-e_{j_1}) \cdots (-e_{j_r}) = (-1)^r e_{j_1} \cdots e_{j_r} \tag{3.8}$$

describes the action of the involution automorphism on E. □

Definition 3.8. Consider a signature (p,q), with $p+q=n$.

(a) The algebra structure (3.5) on the real span of (3.4) constitutes the *real Clifford algebra* $\mathrm{Cl}(p,q)$, of dimension 2^n.
(b) The group E or $E(p,q) = \pm \mathcal{P}(\{ e_1 < \ldots < e_n \})$ of order 2^{n+1} is called the *unit group* of $\mathrm{Cl}(p,q)$.

Tables 1 and 2 display some of the smaller real Clifford algebras and their unit groups, respectively. Although we will not make full use of the tables in this paper, the information they embody will provide a good background against which to view the main thrust of the paper.

	$p+q$									Sign
$p-q$	8	7	6	5	4	3	2	1	0	
-8	$\mathbb{R}_{16}^{16}$									$+$
-7		$\mathbb{R}_8^8 \oplus \mathbb{R}_8^8$								
-6	$\mathbb{R}_{16}^{16}$		$\mathbb{R}_8^8$							$-$
-5		$\mathbb{C}_8^8$		$\mathbb{C}_4^4$						
-4	$\mathbb{H}_8^8$		$\mathbb{H}_4^4$		$\mathbb{H}_2^2$					$+$
-3		$\mathbb{H}_4^4 \oplus \mathbb{H}_4^4$		$\mathbb{H}_2^2 \oplus \mathbb{H}_2^2$		$\mathbb{H} \oplus \mathbb{H}$				
-2	$\mathbb{H}_8^8$		$\mathbb{H}_4^4$		$\mathbb{H}_2^2$		$\mathbb{H}$			$-$
-1		$\mathbb{C}_8^8$		$\mathbb{C}_4^4$		$\mathbb{C}_2^2$		$\mathbb{C}$		
0	$\mathbb{R}_{16}^{16}$		$\mathbb{R}_8^8$		$\mathbb{R}_4^4$		$\mathbb{R}_2^2$		$\mathbb{R}$	$+$
1		$\mathbb{R}_8^8 \oplus \mathbb{R}_8^8$		$\mathbb{R}_4^4 \oplus \mathbb{R}_4^4$		$\mathbb{R}_2^2 \oplus \mathbb{R}_2^2$		$\mathbb{R} \oplus \mathbb{R}$		
2	$\mathbb{R}_{16}^{16}$		$\mathbb{R}_8^8$		$\mathbb{R}_4^4$		$\mathbb{R}_2^2$			$-$
3		$\mathbb{C}_8^8$		$\mathbb{C}_4^4$		$\mathbb{C}_2^2$				
4	$\mathbb{H}_8^8$		$\mathbb{H}_4^4$		$\mathbb{H}_2^2$					$+$
5		$\mathbb{H}_4^4 \oplus \mathbb{H}_4^4$		$\mathbb{H}_2^2 \oplus \mathbb{H}_2^2$						
6	$\mathbb{H}_8^8$		$\mathbb{H}_4^4$							$-$
7		$\mathbb{C}_8^8$								
8	$\mathbb{R}_{16}^{16}$									$+$

TABLE 1. Real Clifford algebras $\mathrm{Cl}(p,q)$, indexed by the dimension $p+q$ and the signature $p-q$ [12, Table 1.1].

3.3. Superalgebras and supergroups.

3.3.1. *Superspaces and supersets.* Superspace terminology (cf. [31]) lays the basis for a smoother treatment of Clifford algebras. Here, we also consider the parallel superset terminology (cf. [20]).

Definition 3.9. Consider a vector space V and a set S.

	$p+q$							Sign
$p-q$	6	5	4	3	2	1	0	
-6	$D_4^{\odot 3}$							$-$
-5		$D_4^{\odot 2} \odot C_4$						
-4	$D_4^{\odot 2} \odot Q_8$		$D_4 \odot Q_8$					$+$
-3		$D_4 \odot Q_8 \times C_2$		$Q_8 \times C_2$				
-2	$D_4^{\odot 2} \odot Q_8$		$D_4 \odot Q_8$		Q_8			$-$
-1		$D_4^{\odot 2} \odot C_4$		$D_4 \odot C_4$		C_4		
0	$D_4^{\odot 3}$		$D_4^{\odot 2}$		D_4		C_2	$+$
1		$D_4^{\odot 2} \times C_2$		$D_4 \times C_2$		$C_2 \times C_2$		
2	$D_4^{\odot 3}$		$D_4^{\odot 2}$		D_4			$-$
3		$D_4^{\odot 2} \odot C_4$		$D_4 \odot C_4$				
4	$D_4^{\odot 2} \odot Q_8$		$D_4 \odot Q_8$					$+$
5		$D_4^{\odot 2} \times C_2$						
6	$D_4^{\odot 2} \odot Q_8$							$-$

TABLE 2. Real unit groups, indexed by the dimension $p+q$ and the signature $p-q$ [27, Table III].

(a) The vector space V becomes a *superspace* when equipped with a specified direct sum decomposition

$$V = V_0 \oplus V_1 \tag{3.9}$$

in which the respective summands are identified as the *even part* V_0 and *odd part* V_1.

(a′) The set S becomes a *superset* when equipped with a specified disjoint union decomposition

$$S = S_0 \uplus S_1 \tag{3.10}$$

in which the respective uniands are identified as the *even part* S_0 and *odd part* S_1.

(b) As a superspace according to (3.9), V has a *superdimension* defined as $\operatorname{sdim} V = \dim V_0 | \dim V_1$.

(b′) As a superset according to (3.10), S has a *supercardinality* or *superorder* that is defined as $\operatorname{scrd} V = \#S_0|\#S_1$. Here, $\#X$ is used for the usual cardinality $|X|$ of a set X, to avoid an excess of vertical strokes.

(c) Elements x of V_0 are described as being *homogeneous* of *even parity* $|x| := 0$.

(c′) Elements x of S_0 are described as having *even parity*: $|x| := 0$.

(d) Elements x of V_1 are described as being *homogeneous* of *odd parity* $|x| := 1$.

(d′) Elements x of S_1 are described as having *odd parity*: $|x| := 1$.

(e) If W_0, W_1 are respective subspaces within the even and odd summands V_0, V_1 from (3.9), then $W = W_0 \oplus W_1$ is said to be a *supersubspace* or *subsuperspace* of V.

(e′) If T_0, T_1 are respective subsets of the even and odd uniands S_0, S_1 from (3.10), then $T = T_0 \uplus T_1$ is said to be a *supersubset* or *subsuperset* of S.

Remark 3.10. (a) The primed parts of Definition 3.9 are designed to emphasize the parallel between supersets and superspaces, the latter just being a linearization of the former. Away from this context, one may equivalently define a superset S to be the domain of a *parity function*

$$p\colon S \to \mathbb{Z}/_2; x \mapsto |x| . \tag{3.11}$$

Then, we have S_r as the inverse image $p^{-1}\{ r \}$ for $r = 0, 1$.

(b) The use of the term "superset" in Definition 3.9(a') conveniently matches the term "superspace" in Definition 3.9(a). The clumsier term $\mathbb{Z}/_2$-*graded set*, which is equivalent by (3.11), may be invoked as needed to disambiguate from the dual of the term "subset".

Definition 3.11. Suppose that (the underlying vector space of) an algebra A is a superspace $A = A_0 \oplus A_1$. Suppose that whenever x and y are homogeneous elements of A, then their product $x \cdot y$ is homogeneous of parity $|x \cdot y| = |x| + |y|$ with addition modulo 2. Then A is said to be a *superalgebra*.

Definition 3.12. Suppose that (the underlying set of) a semigroup S is a superset $S = S_0 \uplus S_1$. Suppose that whenever x and y are elements of S, then their product $x \cdot y$ has $|x \cdot y| = |x| + |y|$ with addition modulo 2. Then S is said to be a *supersemigroup*.

Remark 3.13. (a) The condition $|x \cdot y| = |x| + |y|$ of Definition 3.13 says that the parity function (3.11) is a semigroup homomorphism. Thus, within a supersemigroup, the even part forms a subsemigroup.

(b) As applied in Definition 3.12 to semigroups, the prefix "super-" will equally be applied to monoids and groups. Note, however, that our use of the term "supergroup" should not be confused with the (nevertheless related) notions of a "finite super-group" as in [8, §2.2], or of an "affine algebraic supergroup" as in [32, §8.5]. Compare Remark 3.23.

Lemma 3.14. *Suppose that a set S is a superset $S = S_0 \uplus S_1$.*

(a) *If S is a supermonoid, then its identity element is even.*
(b) *If x is an element of a supergroup S, then $|x^{-1}| = |x|$.*

Proof. Both parts follow from the requirement that the parity function be a homomorphism of the structures involved. □

Example 3.15. The algebra $\mathbb{C}$ of complex numbers becomes a superalgebra of superdimension $1|1$ with $\mathbb{C}_0 = \mathbb{R}$ and $\mathbb{C}_1 = i\mathbb{R}$. The cyclic group C_4 generated by i then becomes a supergroup $C_4 = \{\pm 1\} \uplus \{\pm i\}$ of supercardinality $2|2$.

Example 3.16. Under each of the decompositions:

$$\begin{array}{lll} (i) & \mathbb{H}_0 = \mathbb{R} \oplus \mathbf{i}\mathbb{R}\,, & \mathbb{H}_1 = \mathbf{j}\mathbb{R} \oplus \mathbf{k}\mathbb{R}\,; \\ (j) & \mathbb{H}_0 = \mathbb{R} \oplus \mathbf{j}\mathbb{R}\,, & \mathbb{H}_1 = \mathbf{k}\mathbb{R} \oplus \mathbf{i}\mathbb{R}\,; \\ (k) & \mathbb{H}_0 = \mathbb{R} \oplus \mathbf{k}\mathbb{R}\,, & \mathbb{H}_1 = \mathbf{i}\mathbb{R} \oplus \mathbf{j}\mathbb{R}\,, \end{array}$$

the quaternion algebra $\mathbb{H}$ becomes a superalgebra of superdimension $2|2$. Note that these three superalgebras are distinct, even though they have the same underlying algebra. Their structure breaks the full symmetry on the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ that is present in $\mathbb{H}$ as an algebra.

There are parallel decompositions

$$\begin{array}{ll} (i) & Q_8 = \{\pm 1, \pm\mathbf{i}\} \uplus \{\pm\mathbf{j}, \pm\mathbf{k}\}\ ; \\ (j) & Q_8 = \{\pm 1, \pm\mathbf{j}\} \uplus \{\pm\mathbf{k}, \pm\mathbf{i}\}\ ; \\ (k) & Q_8 = \{\pm 1, \pm\mathbf{k}\} \uplus \{\pm\mathbf{i}, \pm\mathbf{j}\} \end{array}$$

of the quaternion group Q_8, endowing it with respective supergroup structures of supercardinality $4|4$.

Example 3.17. As an immediate illustration of the use of the supergroup concept, one may note how it captures a relationship between the quaternion group Q_8 and the dihedral group D_4: According to (3.28) and (3.29) below, the two groups each appear as the even subgroup in different supergroup structures on the 16-element Pauli group.

3.3.2. *Signed supergroups.* The supergroups presented in Examples 3.15 and 3.16 have a special form.

Definition 3.18. Consider a super(semi)group or supermonoid $S = S_0 \uplus S_1$ of supercardinality $2k|2l$. Suppose that S has a subsuperset $T = T_0 \uplus T_1$ of supercardinality $k|l$ whose (even or odd) elements are written as $x = +x = (+1)x$, such that each element of S is of the form $+x$ or $-x = (-1)x$, with parities $|+x| = |-x|$. Further, suppose that

$$(\sigma x)(\sigma' x') = (\sigma\sigma')xx' \tag{3.12}$$

in S, for $\sigma, \sigma' \in \{\pm 1\}$ and $x, x' \in T$. Then the super(semi)group or supermonoid

$$S = S_0 \uplus S_1 \quad \text{with} \quad S_0 = \{\pm x \mid x \in T_0\} \quad \text{and} \quad S_1 \{\pm x \mid x \in T_1\} \tag{3.13}$$

is said to be *signed*, *with respect to* the subsuperset T as a *transversal*.

Lemma 3.19. *In Definition 3.18, the super(semi)group S is specified from its transversal superset T as the superset* (3.13) *equipped with the multiplication* (3.12)*.*

Lemma 3.20. *The smallest signed supersemigroup is $\emptyset = \emptyset \uplus \emptyset$, the empty supersemigroup, with supercardinality $0|0$ and transversal $\emptyset = \emptyset \uplus \emptyset$.*

Lemma 3.21. *The smallest signed supermonoid or supergroup is the cyclic group $C_2 = \{\pm 1\} \uplus \emptyset$, with supercardinality $2|0$ and transversals $\{1\} \uplus \emptyset$ or $\{-1\} \uplus \emptyset$.*

Proposition 3.22. *Suppose that $G = G_0 \oplus G_1$ is a finite supergroup. Suppose that the even element z is a central involution in the underlying group G. Let $T = (T \cap G_0) \uplus (T \cap G_1)$ be a transversal to the subgroup $\{1, z\}$ of G. For each element t of T, set $t = +t$ and $zt = -t$. Then the supergroup G is a signed supergroup with respect to the transversal supersubset T.*

Remark 3.23. In the context of Proposition 3.22, the pair (G, z) is a "finite super-group" in the sense of [8, §2.2].

Definition 3.24. Suppose that $T = T_0 \uplus T_1$ and $T' = T'_0 \uplus T'_1$ are supersets, with respective supercardinalities $r|s$ and $r'|s'$. Then the superset

$$T \widehat{\times} T' = [(T_0 \times T'_0) \uplus (T_1 \times T'_1)] \uplus [(T_0 \times T'_1) \uplus (T_1 \times T'_0)] \tag{3.14}$$

of supercardinality $(rr' + ss')|(rs' + sr')$ is the *superproduct* of the supersets T and T'.

Lemma 3.25. *In the context of Definition 3.24, consider $t \in T$ and $t' \in T'$. Then the addition*

$$|(t, t')| := |t| + |t'|$$

in $\mathbb{Z}/_2$ *recovers the partition* (3.14)*. Thus, in terms of* (3.11)*, the parity function of the superproduct takes the sum of the respective parity functions of the components.*

Definition 3.26. Suppose that $S = S_0 \uplus S_1$ and $S' = S'_0 \uplus S'_1$ are signed super(semi)groups, with respective transversals $T = T_0 \uplus T_1$ and $T' = T'_0 \uplus T'_1$. Their *signed (super)product* $S \widehat{\times} S'$ is specified according to Lemma 3.19 as the signed super(semi)group with transversal superset $T \widehat{\times} T'$ as a domain for the multiplication

$$(t \otimes t')(u \otimes u') = (-1)^{|t'|\cdot|u|}(tu \otimes t'u') \tag{3.15}$$

defined in terms of the componentwise products tu in S and $t'u'$ in S'. Here, ordered pairs $(t, t') \in T \times T'$ appearing in the uniands of the right hand side of (3.14) are written in the tensor notation $t \otimes t'$, subject to the identifications

$$(-t) \otimes t' = t \otimes (-t') = -(t \otimes t') \tag{3.16}$$

in the signed super(semi)group $S \widehat{\times} S'$.

Example 3.45 below will serve to provide an illustration of how the signed product of signed groups actually works. The following two lemmas justify the claims implicit in the wording of Definition 3.26.

Lemma 3.27. *The signed product* $S \widehat{\times} S'$ *of signed supersemigroups* S, S' *is a signed supersemigroup.*

Proof. The required associativity follows from a direct computation using (3.15). ☐

Lemma 3.28. *Let* S *and* S' *be signed groups. Suppose that* $t \in S$ *and* $u \in S'$*. Then*

$$(t \otimes u)^{-1} = (-1)^{|t|\cdot|u|} \left(t^{-1} \otimes u^{-1}\right)$$

is an inverse of $t \otimes u$ *in the signed product supersemigroup* $S \widehat{\times} S'$*. Thus, the signed product of supergroups is a supergroup.*

Proof. The verification is again a direct computation using (3.15). ☐

3.3.3. *Real Clifford algebras as superalgebras.* For a signature (p, q), with $p + q = n$, consider the real Clifford algebra $\mathrm{Cl}(p, q)$. The action of the involutive automorphism $\overline{-j}$ (Definition 3.5) decomposes the (underlying vector space of the) Clifford algebra as the direct sum

$$\mathrm{Cl}(p, q) = \mathrm{Cl}(p, q)_0 \oplus \mathrm{Cl}(p, q)_1 \tag{3.17}$$

of eigenspaces $\mathrm{Cl}(p, q)_j$ for the two respective eigenvalues $(-1)^j$ with $j \in \{0, 1\}$. The eigenspace decomposition (3.17) provides an instance of (3.9), making $\mathrm{Cl}(p, q)$ a superspace. Intersection of the direct sum decomposition (3.17) with the unit group $E(p,q)$ provides a disjoint union decomposition

$$E(p,q) = E(p,q)_0 \uplus E(p,q)_1 \tag{3.18}$$

of spanning sets, making $E(p,q)$ a supergroup. Note that the even and odd elements of E are signed subsets of the model basis of respective even and odd degrees. Indeed, as a supergroup (3.18), the unit group is a signed supergroup with respect to the power set (3.4), or more precisely the *power superset*

$$\mathcal{P}\left(\{\, e_1 < \ldots < e_n \,\}\right) = \mathcal{P}\left(\{\, e_1 < \ldots < e_n \,\}\right)_0 \uplus \mathcal{P}\left(\{\, e_1 < \ldots < e_n \,\}\right)_1 \,,$$

with the *even powerset*

$$\mathcal{P}\left(\{\, e_1 < \ldots < e_n \,\}\right)_0 = \{\, e_{i_1} \cdots e_{i_r} \mid 0 \le r \le n \,,\ 2 \mid r \,\}$$

and the *odd powerset*

$$\mathcal{P}\left(\{\, e_1 < \ldots < e_n \,\}\right)_1 = \{\, e_{i_1} \cdots e_{i_r} \mid 1 \le r \le n \,,\ 2 \nmid r \,\} \,,$$

as a transversal.

By the construction of §3.2.2, it follows that $\mathrm{Cl}(p,q)$, when equipped with the decomposition (3.17), is a superalgebra. In particular, $\mathrm{Cl}(p,q)_0$ is a subalgebra of $\mathrm{Cl}(p,q)$, and likewise, $E(p,q)_0$ is a subgroup of $E(p,q)$. The superdimension of $\mathrm{Cl}(p,q)$ is $2^{n-1}|2^{n-1}$. Similarly, the supercardinality of $E(p,q)$ is $2^n|2^n$.

3.3.4. *Supercenters and centers.* The following two definitions provide respective definitions of supercommutativity within superalgebras and signed super(semi)groups.

Definition 3.29. Let A be a superalgebra.

(a) Homogeneous elements x, y of A *supercommute* if

$$x \cdot y = (-1)^{|x|\cdot|y|} y \cdot x \,. \tag{3.19}$$

(b) The *supercenter* of A is defined as the linear span of the set of homogeneous elements of A which supercommute with each homogeneous element of A.

Definition 3.30. Let S be a signed super(semi)group.

(a) Elements x, y of S *supercommute* if

$$x \cdot y = (-1)^{|x|\cdot|y|} y \cdot x \,. \tag{3.20}$$

(b) The *supercenter* of S is defined as the set of elements c of S which supercommute with every element of S: in other words, $|c||x| = 0$ for all x in S.

Lemma 3.31. *Let S be a signed supergroup. Then even elements of its supercenter are central.*

Proof. Suppose that z is an even element of the supercenter. Then by (3.20), we have $x \cdot z = (-1)^{|x|\cdot 0} z \cdot x = z \cdot x$ for each element x of S. □

Lemma 3.32. *Let (V, Q) be a finite-dimensional real quadratic space of signature (p, q), equipped with a model basis $\{ e_1 < \ldots < e_n \}$ as in Definition 3.1(c). Let I be a nonempty ordered subset of the model basis. Suppose that $I = \{ e_{i_1} < \ldots < e_{i_r} \}$. Then in the unit group $E(p, q)$, we have*

$$e_I e_j = -(-1)^r e_j e_I \quad \textit{and} \quad e_I e_k = (-1)^r e_k e_I \tag{3.21}$$

for $e_j \in I$, $e_k \notin I$ and $e_I = e_{i_1} \cdots e_{i_r}$.

Proof. Apply induction on the positive integer r, using (3.5). □

Proposition 3.33. *Consider a signature (p, q).*

(a) *The supercenter of the unit supergroup $E(p, q)$ is $\{ \pm 1 \}$.*
(b) *The supercenter of the superalgebra $\mathrm{Cl}(p, q)$ is $\mathbb{R}$.*

Proof. (a) Take the notation of Lemma 3.32. If e_I were supercentral, we would have

$$e_I e_j = (-1)^r e_j e_I$$

upon setting $x = e_I$ and $y = e_j$ in (3.20). But this contradicts (3.21) from Lemma 3.32.

(b) See [31, Prop. 5.3.1(i)]. □

Proposition 3.34. *Consider a signature (p, q), with $p + q = n$.*

(a) *If n is even, the center of the unit supergroup $E(p, q)$ is $\{ \pm 1 \}$, isomorphic to C_2 as an abstract group.*
(b) *If $n = 2m + 1$ is odd, the volume element ω of (3.6) is an odd central element of the unit supergroup $E(p, q)$. Then the center of the unit supergroup $E(p, q)$ has supercardinality $2|2$ as a supersubset of $E(p, q)$.*
(c) *For even $n = 2m$ or odd $n = 2m + 1$, we have $\omega^2 = (-1)^{m+q}$.*
(d) *Consider odd $n = 2m + 1$.*
  (i) *If $m + q$ is even, then the center $\{ \pm 1 \} \times \{ \pm \omega \}$ of the unit supergroup $E(p, q)$ is isomorphic to $C_2 \times C_2$ as an abstract group.*
  (ii) *If $m + q$ is odd, then the center $\langle \omega \rangle$ of the unit supergroup $E(p, q)$ is isomorphic to C_4 as an abstract group.*

Proof. Take the notation of Lemma 3.32. If e_I is central, then

$$e_I e_l = e_l e_I$$

for each element e_l of the model basis. This is incompatible with (3.21), unless I has cardinality n and n is odd. In the latter case, e_I is the volume element ω. Thus (a) and (b) are proved.

The proof of (c) involves use of the *triangular numbers*

$$T(l) = l + (l-1) + \ldots + 2 + 1 = (l+1)l/2$$

for positive integers l. For even $n = 2m$, so that $n-1 = 2m-1$ and

$$T(n-1) = 2m(2m-1)/2 = m(2m-1) = 2m^2 - m\,,$$

or for odd $n = 2m+1$, so that $n-1 = 2m$ and

$$T(n-1) = (2m+1)2m/2 = (2m+1)m = 2m^2 + m\,,$$

the equation $(-1)^{T(n-1)} = (-1)^m$ holds.

Consider $I = \{\, e_1, \ldots, e_p, e_{p+1}, \ldots . e_n \,\} = \{\, e_1, \ldots, e_p, f_1, \ldots . f_q \,\}$. We have $\omega^2 = e_I^2 = e_1 \cdots e_n \cdot e_1 \cdots e_n = (-1)^{T(n-1)} e_1^2 \cdots e_p^2 \cdot e_{p+1}^2 \cdots e_n^2 = (-1)^m e_1^2 \cdots e_p^2 \cdot f_1^2 \cdots f_q^2 = (-1)^{m+q}$. Here, for the third equality, the second case of (3.5) is invoked $n-1$ times to bring the second e_1 left to the first, $n-2$ times to bring the second e_2 left to the first, and so on. This completes the proof of (c). The statements of (d) then follow. □

Proposition 3.35. [31, Prop. 5.3.1(ii), (iii)] *Consider a signature (p,q), with $p+q=n$.*

(a) *If n is even, the center of the Clifford algebra $Cl(p,q)$ is $\mathbb{R}$.*
(b) *If n is odd, the center of the Clifford algebra $Cl(p,q)$ is $\mathbb{R} \oplus \mathbb{R}\omega$.*

3.3.5. *Relationships between real Clifford algebras.* There are several relationships that connect the real Clifford algebras $\mathrm{Cl}(p,q)$ of various signatures (p,q) — cf. [10, p.104, §II.1.4], [22, III, §3.5]. Each of these relationships induces a corresponding relationship between the unit groups $E(p,q)$.

Based on the real linear extension f of the embedding

$$\{\, e_1, \ldots, e_p, f_1, \ldots, f_q \,\} \hookrightarrow \{\, e_1, \ldots, e_{p'}, f_1, \ldots, f_{q'} \,\}$$

for $p \le p', q \le q'$ (where the notation of (3.2) is being used), the following result is a consequence of the universality property (3.3). Alternatively, it may be obtained directly from the construction of §3.2.2.

Lemma 3.36. *If $p \le p'$ and $q \le q'$, then* $\mathrm{Cl}(p,q)$ *is a subalgebra of* $\mathrm{Cl}(p',q')$.

The following definition is needed for the statement of Theorem 3.39 below.

Definition 3.37. Suppose that $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are superalgebras. Their (*superalgebra*) *tensor product* is the superspace

$$A \widehat{\otimes} B = \big((A_0 \otimes B_0) \oplus (A_1 \oplus B_1)\big) \oplus \big((A_0 \otimes B_1) \oplus (A_1 \oplus B_0)\big)$$

equipped with the product defined as

$$(a \otimes b)(a' \otimes b') = (-1)^{|b|\cdot|a'|}(aa' \otimes bb')$$

for homogeneous elements a, a' of A and b, b' of B.

Remark 3.38. Suppose that the superalgebras A, B in Definition 3.37 are finite-dimensional real Clifford algebras. Then their superalgebra tensor product corresponds directly to the signed product of their unit groups, as presented in Definition 3.26.

Theorem 3.39. [22, Th. III.3.10], [31, (5.4)] *There are isomorphisms*

$$\mathrm{Cl}(p+p', q+q') \cong \mathrm{Cl}(p,q) \widehat{\otimes} \mathrm{Cl}(p',q') \tag{3.22}$$

and

$$E(p+p', q+q') \cong E(p,q) \widehat{\times} E(p',q') \tag{3.23}$$

for all natural numbers p, q, p', q'.

Theorem 3.40. [22, Th. III.3.21] *There are isomorphisms*

$$\mathrm{Cl}(p+8, q) \cong \mathrm{Cl}(p, q+8) \cong \mathrm{Cl}(p,q)^{16}_{16} \tag{3.24}$$

for all natural numbers p, q.

The property expressed by Theorem 3.40 may be described as *Bott periodicity*. For actual periodicity, (3.24) is replaced by

$$\mathrm{Cl}(p+8, q) \approx \mathrm{Cl}(p, q+8) \approx \mathrm{Cl}(p,q)\,.$$

Here, $\approx$ denotes the relationship of *Morita equivalence*, which in the current setting may simply be taken as the symmetric and transitive closure of the relationship between an algebra A and the algebra A^r_r of $r \times r$ matrices over A, for any positive integer r.

3.4. **Unit groups and multiplication groups.**

Theorem 3.41. *Suppose that E is the unit group of the real Clifford algebra* $\mathrm{Cl}(p,q)$*. Then E is a subgroup of the multiplication group of a quasigroup. In particular, suppose that $r = \lceil \max\{p/2, q/2\}\rceil$. Then E is a subgroup of* $\mathrm{Mlt}(D_4^{\odot r})$*.*

Proof. Since $p, q \leq 2r$, we have $\mathrm{Cl}(p,q)$ as a subalgebra of $\mathrm{Cl}(2r,2r)$ according to Lemma 3.36. Thus, E is a subgroup of the unit group of $\mathrm{Cl}(2r,2r)$, which is the extraspecial group $D_4^{\odot 2r}$ of order 2^{4r+1} (cf. [27, Table III]). By Theorem 2.14, this group is $\mathrm{Mlt}\, D_4^{\odot r}$. □

Remark 3.42. Within the context of Theorem 3.41, the group $\mathrm{Mlt}\, D_4^{\odot r}$ is not necessarily the smallest quasigroup multiplication group that contains the given unit group E. For example, as illustrated in Table 4 and described in [21, Th. 2.1], the unit group D_4 of $\mathrm{Cl}(2,0)$ is already a quasigroup multiplication group. The particular quasigroup multiplication group $\mathrm{Mlt}\, D_4^{\odot r}$ chosen in the proof of Theorem 3.41 is taken for simplicity, to avoid any case analysis.

3.5. **The sign of an algebra.** By Proposition 3.34(c), the square of the volume element ω of (3.6) is either 1 or -1 in a real Clifford algebra $\mathrm{Cl}(p,q)$. The *sign* of the algebra, tabulated in the final columns of Tables 1, 2 for even signatures, is respectively taken as positive or negative.

Basis element	Quaternion	Degree
e_0	1	0
f_1	$\mathbf{i}$	1
f_2	$\mathbf{j}$	1
$\omega = f_1 f_2$	$\mathbf{k}$	2

TABLE 3. Basic elements from the quaternion group Q_8 within the real Clifford algebra $\mathrm{Cl}(0,2)$. Here, the parities correspond to the supergroup structure (*k*) given in Example 3.16.

As displayed in Table 1, the negative type real Clifford algebra $\mathrm{Cl}(0,2)$ is isomorphic to the quaternion algebra $\mathbb{H}$, as a four-dimensional real algebra. As displayed in Table 2, its unit group is $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, the quaternion group Q_8. We may take a Clifford algebra basis as shown in Table 3, noting $f_1^2 = f_2^2 = -1$, $f_1 f_2 + f_2 f_1 = 0$, and $\omega^2 = \mathbf{k}^2 = -1$. Here, the superalgebra $\mathrm{Cl}(0,2)$ corresponds to the decomposition (k) of $\mathbb{H}$ from Example 3.16.

Table 4 shows unit generators for the negative and positive type real Clifford algebras $\mathbb{R}^2_2$. Together with their negatives, these elements form 2-dimensional real representations of the dihedral group D_4 of degree 4 and order 8. In the negative type, the even-degree elements $\pm\omega$ are the unique elements of order 4 in D_4. In the positive type, the odd-degree elements $\pm f_2$ (at which the quadratic form takes the value -1) are the unique elements of order 4 in D_4.

	0	1		2
D_4^- (2,0)	$e_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$e_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	$e_2 = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$	$\omega = e_1 e_2 = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$
Mlt (C_4)	$R(\pm 1)$	$L(\pm 1)$	$L(\pm i)$	$\omega = R(\pm i)$
D_4^+ (1,1)	$e_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$e_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	$f_2 = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$	$\omega = e_1 f_2 = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$
Mlt (C_4)	$R(\pm 1)$	$L(\pm 1)$	$R(\pm i)$	$\omega = L(\pm i)$

TABLE 4. Basic elements forming parts of dihedral groups D_4 in the negative type real Clifford algebra $\mathrm{Cl}(2,0)$ and positive type $\mathrm{Cl}(1,1)$. The multiplication group interpretations of D_4 refer to [21, Remark 2.2].

3.6. **Iterative construction.** The relationships between real Clifford algebras $\mathrm{Cl}(p,q)$ and their unit groups $E(p,q)$ that were presented in §3.3.5 suggest the possibility of direct iterative specifications of these structures, making use of superalgebra tensor products and signed products of signed supergroups, as an extension and replacement of the use of the central product and direct product of groups in Table 2. For convenient reference, the base cases are recorded in the following.

Proposition 3.43. (a) *The real Clifford superalgebra* $\mathrm{Cl}(0,0) \cong \mathbb{R}$ *has the supergroup* $C_2 = \{\pm 1\} \uplus \emptyset$ *as its unit signed supergroup, with multiplication table*

	1
1	1

on its transversal $\{1\} \uplus \emptyset$. *(Compare* [20, Def'n. 4.17(a)]*.)*

(b) *The real Clifford superalgebra* $\mathrm{Cl}(1,0) \cong \mathbb{R} \oplus \mathbb{R}$ *has the supergroup* $C_2 \times C_2 = \{\pm(1,0)\} \uplus \{\pm(0,1)\}$ *as its unit signed supergroup, with multiplication table*

	$(1,0)$	$(0,1)$
$(1,0)$	$(1,0)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,0)$

on its transversal $\{(1,0)\} \uplus \{(0,1)\}$. *(Compare* [20, Def'n. 4.17(b)]*.)*

(c) *The real Clifford superalgebra* $\mathrm{Cl}(0,1) \cong \mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ *carries the supergroup* $C_4 = \{\pm 1\} \uplus \{\pm i\}$ *as its unit signed supergroup, with multiplication table*

	1	i
1	1	i
i	i	-1

on its transversal $\{1\} \uplus \{i\}$. *(Compare* [20, Def'n. 4.17(c)]*.)*

Definition 3.44. (a) Define V_0 as the zero-dimensional real quadratic space $(\{0\}, \{0\} \hookrightarrow \mathbb{R})$.

(b) Define V_+ as the real quadratic space $(\mathbb{R}, Q\colon x \mapsto x^2)$.

(c) Define V_- as the real quadratic space $(\mathbb{R}, Q\colon x \mapsto -x^2)$.

According to Theorem 3.39, we then have

$$\mathrm{Cl}(p,q) = \big(\mathbb{R} \oplus \mathbb{R}\big)^{\widehat{\otimes} p} \,\widehat{\otimes}\, \big(\mathbb{R} \oplus \mathbb{R}i\big)^{\widehat{\otimes} q}$$

as a superalgebra, and

$$E(p,q) = \big(C_2 \times C_2\big)^{\widehat{\times} p} \,\widehat{\times}\, \big(C_4\big)^{\widehat{\times} q}$$

as a signed supergroup, on the real quadratic space

$$V_+^p \oplus V_-^q$$

of signature (p,q), for $p+q>0$.

Example 3.45. [20, §4.6.1] As an illustration, we may consider how the iterative construction recovers $Q_8 = C_4 \widehat{\times} C_4$ from the signed supergroup C_4 and its transversal $T = \{1\} \uplus \{i\}$ in Proposition 3.43(c), using the signed product as specified in Definition 3.26. We obtain the following multiplication table on the superproduct $T \widehat{\times} T$ of the transversal T with itself:

	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$1 \otimes 1$	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$i \otimes i$	$i \otimes i$	$-(-1 \otimes -1)$	$i \otimes -1$	$-(-1 \otimes i)$
$1 \otimes i$	$1 \otimes i$	$-(i \otimes -1)$	$1 \otimes -1$	$-(i \otimes i)$
$i \otimes 1$	$i \otimes 1$	$-(-1 \otimes -1)$	$i \otimes i$	$-1 \otimes 1$

in which the even element $i \otimes i$ represents $\mathbf{k}$, and the respective odd elements $i \otimes 1$ and $1 \otimes i$ represent $\mathbf{i}$ and $\mathbf{j}$ — compare Example 3.16(k). As sample products in Q_8 obtained using the table, we have

$$\mathbf{j} \cdot \mathbf{i} = (1 \otimes i)(i \otimes 1) \overset{(3.15)}{=} (-1)^{|i| \cdot |i|}(i \otimes i) = -(i \otimes i) = -\mathbf{k}$$

and

$$\begin{aligned} \mathbf{j} \cdot \mathbf{k} &= (1 \otimes i)(i \otimes i) \overset{(3.15)}{=} (-1)^{|i| \cdot |i|}(i \otimes -1) \\ &= -(i \otimes -1) \overset{(3.16)}{=} -(-1)(i \otimes 1) = \mathbf{i} \end{aligned}$$

according to Definition 3.26.

The approach taken in this section compares with other iterative constructions, like the explicit faithful matrix representations of real Clifford algebras $\mathrm{Cl}(p, q)$ at the computational level in [15, Th. 7(3°)]. At a more sophisticated level, there are the deformations from abelian groups to (generalized) Clifford algebras, for instance as presented in [4]. Interesting features of that work include use of a Cayley-Dickson process as an iterative step, and treatment of coalgebras on an equal footing.

3.7. **The Pauli group and combinatorial spinors.**

3.7.1. *The Pauli group.* Consider the Pauli matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.25}$$

along with the 2×2 identity matrix I [25, §2.1.3]. The *Pauli group* is described as consisting of the set

$$G_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} \tag{3.26}$$

of complex 2×2 matrices [25, (10.81)]. (The suffix 1 in G_1 refers to the fact that (3.26) is the Pauli group for a single qubit. For the Pauli group on two qubits, compare [3].) The matrix equations

$$-YX = XY = iZ\,, \ -ZY = YZ = iX\,, \ -XZ = ZX = iY \tag{3.27}$$

and $X^2 = Y^2 = Z^2 = 1$ summarize the multiplication in G_1.

The elements of the Pauli group form a real spanning set for the 8-dimensional real algebra $\mathbb{C}_2^2$ of 2×2 complex matrices. Explicitly, G_1 is the unit group of the real Clifford algebras $\mathrm{Cl}(3,0)$ and $\mathrm{Cl}(1,2)$, as displayed in the analogue Table 5 of Table 4. Note that in each case, the volume element ω squares to -1, according to Proposition 3.34(c). In terms of (3.18), we have the respective decompositions

$$E(3,0) = G_1 = \{\pm I, \pm iX, \pm iY, \pm iZ\} \uplus \{\pm X, \pm Y, \pm Z, \pm iI\} \tag{3.28}$$

and

$$E(1,2) = G_1 = \{\pm I, \pm iX, \pm Y, \pm Z\} \uplus \{\pm X, \pm iY, \pm iZ, \pm iI\} \tag{3.29}$$

of the Pauli group as a supergroup. Thus, the even subgroup $E(3,0)_0$ is Q_8 (compare [21, Cor. 3.2]), while the even subgroup $E(1,2)_0$ is D_4. (The latter statement is verified by noting that iX, of order 4, is inverted when conjugated by the involution Y.) Indeed, as the internal central product of its commuting subgroups $\langle iI \rangle$ and $E(1,2)_0$, the Pauli group constitutes the central product $C_4 \odot D_4$, according to Lemma 2.6 — compare Example 2.4.

	0	1			2			3
$\mathrm{Cl}(3,0)$	I	X	Y	Z	iX	iY	iZ	$\omega = iI$
$\mathrm{Cl}(1,2)$	I	X	iY	iZ	$-iX$	$-Y$	$-Z$	$\omega = -iI$

TABLE 5. Basic elements forming parts of Pauli groups G_1 as unit groups in the real Clifford algebras $\mathrm{Cl}(3,0)$ and $\mathrm{Cl}(1,2)$.

3.7.2. *Biquaternions and the space of spinors.* For three-dimensional Euclidean space, consider the Clifford algebra $\mathrm{Cl}(3,0) = \mathbb{C}_2^2$. Its even part $\mathrm{Cl}(3,0)_0$ is the central simple quaternion algebra $\mathbb{H}$ [10, p.121]. Each element of $\mathrm{Cl}(3,0)$ may be written uniquely as a sum $q+\omega r$ with quaternions q, r. Historically, *biquaternions* were defined to be sums of this type, albeit with varying specifications of the square of ω as $0, 1$, or -1 according to the underlying geometry [11, p.386], [16, §669].

To within equivalence, the central simple quaternion algebra $\mathbb{H}$ has a unique (say right) representation, which may be modeled by the right regular representation of the algebra $\mathbb{H}$ on its underlying 4-dimensional real space. In Chevalley's terminology, this is the *space of spinors* [10, p.121] for 3-dimensional Euclidean space (compare [26, (7.4)]).

3.7.3. *Algebraic spinors.* Monk and Hiley provide an approach to the space of spinors that is purely internal to the Clifford algebra $\mathrm{Cl}(3,0)$, in the guise of *algebraic spinors* [24, p.373]. Adopting their approach, we take the primitive idempotent

$$\tfrac{1}{2}(1+Z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.30}$$

of the Clifford algebra $\mathrm{Cl}(3,0)$, and interpret the space of spinors as the (right) ideal that it generates.

3.7.4. *Combinatorial spinors.* Consider the signed supergroup $E(3,0)$, isomorphic to the Pauli group at the level of abstract groups, as the home for a combinatorial version of the algebraic spinors of Monk and Hiley. Their primitive idempotent (3.30) is interpreted combinatorially as the barycenter of the subgroup $\langle Z\rangle = \{\, I, Z \,\}$ of the Pauli group. While the algebraic spinors are the elements of the (for us, right) ideal generated by the primitive idempotent, our *combinatorial spinors* are the elements of the homogeneous space

$$\langle Z\rangle \backslash E(3,0) = \{\, \langle Z\rangle\, p \mid p \in E(3,0) \,\} \tag{3.31}$$

of cosets of the subgroup $\langle Z\rangle$. Then, taking $Q_8 = E(3,0)_0$ inside the Pauli group $E(3,0)$, as discussed above and displayed in the first row of the body of Table 5, we may match Chevalley's identification of $\mathbb{H}$ as the space of spinors for 3-dimensional Euclidean space with the following identification of Q_8 as the corresponding set of combinatorial spinors.

Theorem 3.46. *Consider the space* (3.31) *of combinatorial spinors.*

(a) *The map*

$$\lambda_Z \colon Q_8 \to \langle Z\rangle \backslash E(3,0); q \mapsto \langle Z\rangle\, q$$

is bijective.

(b) *The map λ_Z induces a quaternion group structure on the space $\langle Z\rangle\backslash E(3,0)$.*

Proof. (a) Since $|Q_8| = 8 = 16/2 = |\langle Z\rangle\backslash E(3,0)|$, it suffices to show that λ_Z is injective. Indeed, for $q_1, q_2 \in E(3,0)_0$, we have

$$\langle Z\rangle\, q_1 = \langle Z\rangle\, q_2 \;\Rightarrow\; q_1 q_2^{-1} \in \langle Z\rangle \cap E(3,0)_0 = \{\, I \,\} \;\Rightarrow\; q_1 = q_2$$

as required.

(b) The equation $q_1^{\lambda_Z} \cdot q_2^{\lambda_Z} = (q_1 q_2)^{\lambda_Z}$ serves to define the quaternion group structure on $\langle Z\rangle\backslash E(3,0)$. □

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