Sel. math., New ser. 13 (2007), 57–67

1022-1824/010057-11, *published online* April 5, 2007
DOI 10.1007/s00029-007-0032-x

Selecta Mathematica New Series

On the category of weak Cayley table morphisms between groups

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Abstract. Weak Cayley table functions between groups are generalized conjugacy-preserving homomorphisms, under which products of images are conjugate to images of products. There is a weak Cayley table bijection between two groups iff they have the same 2-characters. In this paper, weak Cayley table functions are augmented to include the specific conjugating elements, leading to the concept of a weak (Cayley table) morphism. If the conjugating elements are chosen subject to a crossed-product condition, then the weak morphisms between groups form a category. The forgetful functor to this category from the category of group homomorphisms is shown to possess a left adjoint. Two weak morphisms are said to be homotopic if they project to the same weak Cayley table function. As a first step in the analysis of the category of weak morphisms, the group of units of the monoid of weak morphisms homotopic to the identity automorphism of a group is described.

Mathematics Subject Classification (2000). 20J15, 20C15.

Keywords. Weak Cayley table, character table, 2-character, Brauer pair.

1. Introduction

The question of how much information about a finite group G is contained in its ordinary character table was formulated by Richard Brauer in [1], together with the question of which information in addition to the character table of G determines G up to isomorphism. For example, for the latter question Brauer suggested that the power maps on conjugacy classes might be enough, but Dade quickly gave an example of a pair of non-isomorphic groups with the same character table and power maps (such a pair is usually called a *Brauer pair*). Recent work revisiting Frobenius' original papers on group characters provided an answer to the second question of Brauer in terms of the "k-characters" of a group ([4]–[6]). It was shown that the 3-character of the regular representation, or the 3-characters of the irreducible representations, determine a group up to isomorphism. The 1-characters coincide with the ordinary characters, so there is a natural question: what information about the group is contained in the 2-characters? Here if χ is a character, the corresponding 2-*character* $\chi^{(2)}$ is defined by

$$\chi^{(2)}(g,h) = \chi(g)\chi(h) - \chi(gh).$$

For an odd prime p, two non-isomorphic non-abelian groups of order p^3 have the same 2-characters, and thus we know that the irreducible 2-characters do not determine a group. The *weak Cayley table* of a group G is the table indexed by the elements of G whose (x, y)-entry is the conjugacy class containing the product xy. Two groups G_1 and G_2 have the same irreducible 2-characters if and only if there is a bijection between them which induces an identification of their weak Cayley tables in the following sense. A function $f : G_1 \to G_2$ is a *weak Cayley table bijection* if it is a bijection which induces a bijection between the conjugacy classes of G_1 and G_2 such that $(xy)^f$ is conjugate to $x^f y^f$ for all x, y in G_1. The paper [7] gives an account of work on the weak Cayley table. In [3] Humphries investigated the group of weak Cayley table bijections from a group G to itself.

If $f : G_1 \to G_2$ is a weak Cayley table bijection, then in particular G_1 and G_2 have the same character table. Often the simplest method to show that a pair of groups has the same character table is to exhibit a weak Cayley table bijection between them. Moreover, whereas character theory can be hard to define in the case of infinite groups, the definition of a weak Cayley table bijection is exactly the same as in the finite case. Just as group isomorphisms are best viewed as invertible group homomorphisms, the weak Cayley table bijections are best viewed as invertible "weak Cayley table functions." A *weak Cayley table function* $f : G \to H$ is a function f from a group G to a group H mapping conjugate elements of G to conjugate elements of H, and such that the image of a product of elements of G is conjugate in H to the product of their images. In addition, f is required to map the identity element of G to the identity element of H. (Note that Humphries introduced the term "weak Cayley table morphism" in [3] for such functions without the condition on the image of the identity. Since he did not make further use of the term, we prefer to reserve it for the context of Definition 2.1. The relevance of the condition on identities is discussed in Remark 2.2(c). The condition turns out to be redundant for bijections [7, p. 378].)

A detailed study of the weak Cayley table functions demands a specification of the codomain elements that perform the conjugations. In Section 2 of this paper a weak Cayley table function together with an appropriate specification of these conjugating elements is defined to be a *weak Cayley table morphism*, or more briefly just a *weak morphism* (Definition 2.1). The definition includes a crossed-product condition in the sense of [2]—cf. Remark 2.2(d). Theorem 3.3 then shows that under composition, the weak morphisms form a category **Gwp**. It is curious to note that without the crossed-product condition, the weak morphisms would only form a "non-associative category" under composition. Such objects may well justify further consideration. In particular, just as groups are represented by automorphisms in categories (for ordinary linear representations, the category of complex vector spaces; for permutation representations, the category of sets), one might represent quasigroups naturally by automorphisms in such non-associative categories.

Since group homomorphisms are weak Cayley table functions, and may thus be construed as weak morphisms, there is a forgetful functor to **Gwp** from the category **Gp** of group homomorphisms. Theorem 4.1 constructs a left adjoint to this functor. On the other hand, forgetting the specific choice of conjugating elements embodied in a weak morphism yields a projection functor P from **Gwp** to the category **Set** of sets. Two weak morphisms are said to be *homotopic* if they project under P to the same weak Cayley table function. The final section 5 examines the monoid $P_1^{-1}\{\mathrm{id}_G\}$ of weak morphisms that are homotopic to the identity morphism on a group G. As shown by Proposition 5.2(d), this monoid may well contain non-invertible elements. Indeed, Problem 5.1 asks whether each weak Cayley table bijection is the image under the projection functor P of an isomorphism in the category **Gwp**. An invertible element of the monoid $P_1^{-1}\{\mathrm{id}_G\}$ yields a left quasigroup structure on the set G. The left multiplication maps of these various left quasigroups form a group known as the *perturbation group* Γ_G of the group G. Theorem 5.6 analyzes the structure of this group. In turn, the perturbation group Γ_G appears in the structure of the group U_G of units of the monoid $P_1^{-1}\{\mathrm{id}_G\}$, as described by Theorem 5.8. Note that for each weak morphism $\alpha : G \to H$, composition in the category **Gwp** affords a left action of U_G and a right action of U_H on the homotopy class of α.

Throughout the paper, notational conventions and definitions not otherwise explained follow the usage of [8].

2. Weak morphisms

Definition 2.1. Let G and H be groups. Then a *weak* (*Cayley table*) *morphism* $\alpha : G \to H$ consists of a triple $(\alpha_1, \alpha_2, \alpha_3)$ of functions

$$\alpha_1 : G \to H; g \mapsto g^{\alpha_1}$$

and

$$\alpha_i : G^2 \to H; (g_1, g_2) \mapsto \alpha_i(g_1, g_2)$$

for $i = 2,\ 3$ such that:

(1) α_1 maps the identity of G to the identity of H; and
(2) $g_1^{\alpha_1} g_2^{\alpha_1} \alpha_2(g_1, g_2) = \alpha_2(g_1, g_2)(g_1 g_2)^{\alpha_1}$,
(3) $g_1^{\alpha_1} \alpha_3(g_1, g_2) = \alpha_3(g_1, g_2)(g_2^{-1} g_1 g_2)^{\alpha_1}$,
(4) $\alpha_3(g_1, g_2 g_3) = \alpha_3(g_1, g_2)\alpha_3(g_2^{-1} g_1 g_2, g_3)$

for all g_1, g_2, g_3 in G.

Remark 2.2. We use the notation $\pi_2 : G^2 \to G; (g_1, g_2) \mapsto g_2$.

(a) If $f : G \to H$ is a group homomorphism, then the triple $(f, e_H, \pi_2 f)$ is a weak morphism, the second component being the constant map $e_H : G^2 \to H$ whose value is the identity element of H.

(b) Let $J : G \to G; g \mapsto g^{-1}$ be the inversion map on a group G. Then $(J, \pi_2, \pi_2) : G \to G$ is a weak morphism.

(c) Condition (1) of Definition 2.1 is not redundant. Consider the group G of permutations of the set $\{1, 2, 3\}$. Let $\alpha_1 : G \to G$ be the constant map with value (123). Let $\alpha_2 : G^2 \to G$ be the constant map with value (23). Let $\alpha_3 : G^2 \to G$ be the constant map with value (1). Then $(\alpha_1, \alpha_2, \alpha_3)$ satisfies conditions (2)–(4) of Definition 2.1, but not condition (1).

(d) In Definition 2.1, consider the conjugation action of G on itself. Then in the language of [2], condition (4) says that α_3 is a crossed product.

Proposition 2.3. *Let $\alpha : G \to H$ be a weak morphism. Then for all g in G, one has the following:*

(i) $[g^{\alpha_1}, \alpha_2(1, g)] = [g^{\alpha_1}, \alpha_2(g, 1)] = 1$*;*

(ii) $(g^{-1})^{\alpha_1} = (g^{\alpha_1})^{-1}$*;*

(iii) *the map $G \to H; x \mapsto \alpha_3(1, x)$ may be chosen as an arbitrary homomorphism;*

(iv) $\alpha_3(g, 1) = 1$.

Proof. Statement (i) follows by specialization of the arguments g_1, g_2 in condition (2) of Definition 2.1. Statement (ii) follows by condition (1) on setting $g_1 = g$ and $g_2 = g^{-1}$ in condition (2)—cf. [7, p. 398]. Statement (iii) is apparent upon specialization of condition (3), the homomorphic property being required for consistency with condition (4). Finally, statement (iv) follows on setting $g_1 = g$ and $g_2 = g_3 = 1$ in condition (4) of Definition 2.1. □

Remark 2.4. In view of Proposition 2.3(iii), one might choose a normalization $\alpha_3(1, g) = 1$ for g in G as part of the requirements of Definition 2.1. However, this would preclude the convenient use of the second projection π_2 in contexts such as Remark 2.2(a) and (b).

3. The category of weak morphisms

Proposition 3.1. *Let $\alpha : G \to H$ and $\beta : H \to K$ be weak morphisms. Then there is a composite weak morphism $\alpha\beta : G \to K$ with components $(\alpha\beta)_1 = \alpha_1\beta_1$,*

$$(\alpha\beta)_2(g_1, g_2) = \beta_2(g_1^{\alpha_1}, g_2^{\alpha_1})\beta_3(g_1^{\alpha_1}g_2^{\alpha_1}, \alpha_2(g_1, g_2)), \tag{3.1}$$

and

$$(\alpha\beta)_3(g_1, g_2) = \beta_3(g_1^{\alpha_1}, \alpha_3(g_1, g_2)) \tag{3.2}$$

for g_1, g_2 in G.

Proof. It must be verified that $\alpha\beta$ satisfies the conditions of Definition 2.1. Condition (1) is immediate. To verify condition (3), note that condition (3) on α implies

$$(g_2^{-1}g_1g_2)^{\alpha_1} = \alpha_3(g_1, g_2)^{-1}g_1^{\alpha_1}\alpha_3(g_1, g_2) \tag{3.3}$$

for g_1, g_2 in G. Condition (3) on β applied to (3.3) then yields

$$(g_2^{-1}g_1g_2)^{\alpha_1\beta_1} = \beta_3(g_1^{\alpha_1}, \alpha_3(g_1, g_2))^{-1}g_1^{\alpha_1\beta_1}\beta_3(g_1^{\alpha_1}, \alpha_3(g_1, g_2))$$

as required, given $(\alpha\beta)_3$ expressed by (3.2). To verify condition (2) on $\alpha\beta$, note that condition (2) on α implies

$$(g_1g_2)^{\alpha_1} = \alpha_2(g_1, g_2)^{-1} g_1^{\alpha_1} g_2^{\alpha_1} \alpha_2(g_1, g_2) \tag{3.4}$$

for g_1, g_2 in G. Applying condition (3) on β to (3.4) gives

$$(g_1g_2)^{\alpha_1\beta_1} = \beta_3(g_1^{\alpha_1} g_2^{\alpha_1}, \alpha_2(g_1, g_2))^{-1} (g_1^{\alpha_1} g_2^{\alpha_1})^{\beta_1} \beta_3(g_1^{\alpha_1} g_2^{\alpha_1}, \alpha_2(g_1, g_2)).$$

Expansion of the middle term $(g_1^{\alpha_1} g_2^{\alpha_1})^{\beta_1}$ of the right hand side of this equation using condition (2) on β then yields the required condition (2) on $\alpha\beta$, with $(\alpha\beta)_2$ being specified by (3.1). Finally, for g_i in G, equation (3.2) and Definition 2.1 give

$$\begin{aligned}
&(\alpha\beta)_3(g_1, g_2g_3) = \beta_3(g_1^{\alpha_1}, \alpha_3(g_1, g_2g_3))\\
&\quad = \beta_3(g_1^{\alpha_1}, \alpha_3(g_1, g_2)\alpha_3(g_2^{-1}g_1g_2, g_3))\\
&\quad = \beta_3(g_1^{\alpha_1}, \alpha_3(g_1, g_2))\beta_3(\alpha_3(g_1, g_2)^{-1} g_1^{\alpha_1} \alpha_3(g_1, g_2), \alpha_3(g_2^{-1}g_1g_2, g_3))\\
&\quad = \beta_3(g_1^{\alpha_1}, \alpha_3(g_1, g_2))\beta_3((g_2^{-1}g_1g_2)^{\alpha_1}, \alpha_3(g_2^{-1}g_1g_2, g_3))\\
&\quad = (\alpha\beta)_3(g_1, g_2)(\alpha\beta)_3(g_2^{-1}g_1g_2, g_3),
\end{aligned}$$

verifying condition (4) on $\alpha\beta$. □

Proposition 3.2. *For weak morphisms $\alpha : G \to H$, $\beta : H \to K$ and $\gamma : K \to L$, the associative law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds.*

Proof. The equality $(\alpha\beta.\gamma)_1 = (\alpha.\beta\gamma)_1$ just represents the associativity $(\alpha_1\beta_1)\gamma_1 = \alpha_1(\beta_1\gamma_1)$ of functional composition, while $(\alpha\beta.\gamma)_3 = (\alpha.\beta\gamma)_3$ follows easily from (3.2). Now for elements x, y, z of G, the compositions (3.1) and (3.2) reduce $(\alpha\beta.\gamma)_2(x, y)$ to the product of $\gamma_2(x^{\alpha_1\beta_1}, y^{\alpha_1\beta_1})$ with

$$\gamma_3(x^{\alpha_1\beta_1} y^{\alpha_1\beta_1}, \beta_2(x^{\alpha_1}, y^{\alpha_1})\beta_3(x^{\alpha_1} y^{\alpha_1}, \alpha_2(x, y))), \tag{3.5}$$

and $(\alpha.\beta\gamma)_2(x, y)$ to the product of $\gamma_2(x^{\alpha_1\beta_1}, y^{\alpha_1\beta_1})$ with

$$\gamma_3(x^{\alpha_1\beta_1} y^{\alpha_1\beta_1}, \beta_2(x^{\alpha_1}, y^{\alpha_1}))\gamma_3((x^{\alpha_1} y^{\alpha_1})^{\beta_1}, \beta_3(x^{\alpha_1} y^{\alpha_1}, \alpha_2(x, y))). \tag{3.6}$$

Since β satisfies Definition 2.1(2), the latter term of (3.6) may be rewritten as

$$\gamma_3(\beta_2(x^{\alpha_1}, y^{\alpha_1})^{-1} x^{\alpha_1\beta_1} y^{\alpha_1\beta_1} \beta_2(x^{\alpha_1}, y^{\alpha_1}), \beta_3(x^{\alpha_1} y^{\alpha_1}, \alpha_2(x, y))).$$

The equality between (3.5) and (3.6) then follows since γ satisfies Definition 2.1(4). □

Theorem 3.3. *There is a locally small category* **Gwp** *whose object class is the class of all groups, such that for groups G and H, the morphism class* $\mathbf{Gwp}(G, H)$ *is the set of all weak morphisms from G to H. The identity morphism at a group G is the weak morphism $\iota_G = (\mathrm{id}_G, e_G, \pi_2)$, while the composition of weak morphisms is given by Proposition* 3.1.

Proof. Consider a weak morphism $\alpha : G \to H$. By (3.1), one has

$$(\alpha\iota_H)_2(g_1, g_2) = e_H(g_1, g_2)\pi_2(g_1^{\alpha_1} g_2^{\alpha_1}, \alpha_2(g_1, g_2)) = \alpha_2(g_1, g_2).$$

By (3.2), one has

$$(\alpha\iota_H)_3(g_1, g_2) = \pi_2(g_1^{\alpha_1}, \alpha_3(g_1, g_2)) = \alpha_3(g_1, g_2).$$

Thus $\alpha\iota_H = \alpha$. Again by (3.1), one has

$$(\iota_G\alpha)_2(g_1, g_2) = \alpha_2(g_1, g_2)\alpha_3(g_1 g_2, e_G(g_1, g_2)) = \alpha_2(g_1, g_2),$$

the latter equation holding by statement (iv) of Proposition 2.3. By (3.2), one has

$$(\iota_G\alpha)_3(g_1, g_2) = \alpha_3(g_1, \pi_2(g_1, g_2)) = \alpha_3(g_1, g_2).$$

Thus $\iota_G\alpha = \alpha$. The partial associativity of the composition in **Gwp** is given by Proposition 3.2. □

Corollary 3.4. *There is a forgetful functor U : **Gp** $\to$ **Gwp** from the category of (homomorphisms between) groups, with morphism part $U : (f : G \to H) \mapsto (f, e_H, \pi_2 f)$.*

Proof. Compare Remark 2.2(a). Verification of the functoriality is straightforward. □

4. The adjunction

Let G be a group. Let W be the free group on the disjoint union $G + G^2 + G^2$ of the set G with two copies of G^2. Let $\eta'_1 : G \to W$ insert the generators from G. For $i = 2,\ 3$, let $\eta'_i : G^2 \to W$ insert the generators from the $(i-1)$-th copy of G^2. Let GF be the quotient of W obtained by imposing the relations

(1) $1_G^{\eta'_1} = 1_W$;
(2) $\forall g_1, g_2 \in G,\ g_1^{\eta'_1} g_2^{\eta'_1} \eta'_2(g_1, g_2) = \eta'_2(g_1, g_2)(g_1 g_2)^{\eta'_1}$;
(3) $\forall g_1, g_2 \in G,\ g_1^{\eta'_1} \eta'_3(g_1, g_2) = \eta'_3(g_1, g_2)(g_2^{-1} g_1 g_2)^{\eta'_1}$;
(4) $\forall g_i \in G,\ \eta'_3(g_1, g_2 g_3) = \eta'_3(g_1, g_2)\eta'(g_2^{-1} g_1 g_2, g_3)$

corresponding to the respective conditions of Definition 2.1. Let η_i for $1 \le i \le 3$ denote the composite of η'_i with the projection from W to GF. Note that η_G or

$$(\eta_1, \eta_2, \eta_3) : G \to GFU \tag{4.1}$$

is a weak morphism.

Theorem 4.1. *The forgetful functor U : **Gp** $\to$ **Gwp** has a left adjoint F : **Gwp** $\to$ **Gp**.*

Proof. Let $\alpha : G \to H$ be a weak morphism. There is a unique homomorphism from W to H defined by $g_1^{\eta'_1} \mapsto g_1^{\alpha_1}$ and $\eta'_i(g_1, g_2) \mapsto \alpha_i(g_1, g_2)$ for $i = 2, 3$ and $g_1,\ g_2$ in G. This homomorphism factorizes through a unique homomorphism $\overline{\alpha} : GF \to H$.

It will now be verified that $(\eta_1, \eta_2, \eta_3)(\overline{\alpha}, e_H, \pi_2\overline{\alpha}) = (\alpha_1, \alpha_2, \alpha_3)$. For g in G, one has

$$g^{\eta_1 \overline{\alpha}} = g^{\alpha_1} \tag{4.2}$$

by the definition of $\overline{\alpha}$. For $g_1,\ g_2$ in G, (3.1) yields

$$(\eta\overline{\alpha}^U)_2(g_1, g_2) = \eta_2(g_1, g_2)^{\overline{\alpha}} = \alpha_2(g_1, g_2), \tag{4.3}$$

while (3.2) gives

$$(\eta\overline{\alpha}^U)_3(g_1, g_2) = \eta_3(g_1, g_2)^{\overline{\alpha}} = \alpha_3(g_1, g_2). \tag{4.4}$$

On the other hand, the final equations in the lines (4.2)–(4.4) specify the homomorphism $\overline{\alpha} : GF \to H$ uniquely. □

Corollary 4.2. *Let $\prod_{i\in I} H_i$ be the product (in* **Gp***) of a family of groups, equipped with projections $p_i : \prod_{j\in I} H_j \to H_i$ for each i in I. Then the group $\prod_{i\in I} H_i$, equipped with projections $p_i^U : \prod_{j\in I} H_j \to H_i$ for each i in I, is the product in* **Gwp** *of the family of groups.*

Proof. The right adjoint U : **Gp** → **Gwp** creates products. □

Corollary 4.3. *For a group G, the weak morphism* (4.1) *is the component at G of the unit of the adjunction of Theorem 4.1.*

As a dual to Corollary 4.3, note that the component at a group G of the counit of the adjunction of Theorem 4.1 is the group homomorphism $\varepsilon_G : GF \to G$ given by

$$g_1^{\eta_1} \mapsto g_1, \quad \eta_2(g_1, g_2) \mapsto 1, \quad \eta_3(g_1, g_2) \mapsto g_2$$

for $g_1,\ g_2$ in G.

5. Homotopy

Let P : **Gwp** → **Set** be the functor to the category of sets projecting each weak morphism $\alpha : G \to H$ to its first component $\alpha_1 : G \to H$. The image of the functor P, a subcategory $\mathbf{Gwp}^P$ of **Set**, is called the *category of weak Cayley table functions*. The basic open problem concerning the relation between weak Cayley table bijections and the categorical considerations of this paper is the following.

Problem 5.1. Is each weak Cayley table bijection the projection under P of a weak isomorphism?

The following concept may help to put Problem 5.1 into context. Two parallel weak morphisms $\alpha, \beta : G \to H$ are said to be *homotopic* if $\alpha_1 = \beta_1$. The proposition below shows that the homotopy class of an isomorphism may contain weak morphisms which are not isomorphisms. In other words, there may be non-invertible weak morphisms that project under P to a weak Cayley table bijection.

Proposition 5.2. *Let G be a group.*

(a) *The homotopy class $P_1^{-1}\{\mathrm{id}_G\}$ of the identity $\iota_G = (\mathrm{id}_G, e_G, \pi_2)$ forms a monoid.*

(b) *For each α in $P_1^{-1}\{\mathrm{id}_G\}$, one has*

$$\alpha_2(x,y) \in \mathrm{C}_G(xy) \quad \textit{and} \quad \alpha_3(x,y) \in \mathrm{C}_G(x)y$$

for all x, y in G.

(c) *For elements α, β of $P_1^{-1}\{\mathrm{id}_G\}$, one has $\alpha\beta = \iota_G$ if and only if*

$$\beta_3(x, \alpha_3(x,y)) = y \tag{5.1}$$

and

$$\beta_2(x,y) = \beta_3(xy, \alpha_2(x,y))^{-1} \tag{5.2}$$

for all x, y in G.

(d) *If G is non-trivial, then the monoid $P_1^{-1}\{\mathrm{id}_G\}$ is not a group.*

Proof. (a) is an immediate consequence of the functoriality of P.
(b) follows from Definition 2.1(2),(3).
(c) follows from (3.1) and (3.2), along with the definition of ι_G.
(d): For α in $P_1^{-1}\{\mathrm{id}_G\}$ to be invertible, (5.1) shows that

$$\widehat{\alpha}_x : G \to G; y \mapsto \alpha_3(x,y) \tag{5.3}$$

must biject for each x in G. On the other hand, Proposition 2.3(iii) shows that the homomorphism $\widehat{\alpha}_1 : G \to G$ may be chosen arbitrarily, and in particular need not biject if G is non-trivial. □

Equation (5.1) shows that each invertible weak morphism α homotopic to the identity map on a group G yields a left quasigroup structure $(G, \alpha_3, (\alpha^{-1})_3)$ on the underlying set G of the group. The maps (5.3) are the left multiplications in the left quasigroup. The following definition gives a different description of these maps. For a set X, let $X!$ denote the group of permutations of X. For a subgroup H of a group G, let $H\backslash G$ denote the set $\{Hx \mid x \in G\}$ of right cosets of H.

Definition 5.3. Let G be a group. Then a *perturbation* of G is a map $\theta : x \mapsto \theta_x$ with domain G such that

(1) $\theta_x \in \prod_{X \in \mathrm{C}_G(x)\backslash G} X!$ and
(2) $(yz)\theta_x = y\theta_x . z\theta_{y^{-1}xy}$

for all x, y, z in G. Such a map θ is said to *perturb* G.

Remark 5.4. Let θ perturb a group G with elements x and y.

(a) The map θ_x restricts to an automorphism of $\mathrm{C}_G(x)$. In particular, a perturbation of an abelian group A is just an indexed collection of automorphisms of A.

(b) By Definition 5.3(2), knowledge of θ_x implies knowledge of $\theta_{y^{-1}xy}$. Thus a perturbation is specified completely by its values on a set of representatives for the conjugacy classes of G. These various values, in turn, are independent of each other.

Proposition 5.5. *The set* Γ_G *of all perturbations of a group* G *forms a group under the multiplication* $(\theta, \varphi) \mapsto (x \mapsto \theta_x \varphi_x)$.

Proof. Let θ and φ be perturbations. Then for x, y, z in G, one has $(yz)\theta_x\varphi_x = (y\theta_x.z\theta_{y^{-1}xy})\varphi_x = y\theta_x\varphi_x.z\theta_{y^{-1}xy}\varphi_{v^{-1}xv}$ with $v = y\theta_x$. However, $v^{-1}xv = y^{-1}xy$ by Definition 5.3(1) for θ. □

The structure of the group Γ_G of perturbations of a group G is described as follows.

Theorem 5.6. *Let* G *be a group, and let* $\{g_i \mid 0 \leq i < s\}$ *be a set of representatives for the conjugacy classes of* G, *with* $g_0 = 1$. *For each* $0 \leq i < s$, *let* n_i *be the cardinality of the conjugacy class of* g_i, *and let* $\mathrm{Aut}(\mathrm{C}_G(g_i))$ *act diagonally on the power* $\mathrm{C}_G(g_i)^{n_i-1}$. *Then the group* Γ_G *of perturbations of* G *is isomorphic to the product*

$$\prod_{0\leq i<s} \mathrm{C}_G(g_i)^{n_i-1} \rtimes \mathrm{Aut}(\mathrm{C}_G(g_i)) \tag{5.4}$$

of split extensions.

Proof. Consider a particular representative $g \in \{g_i \mid 0 \leq i < s\}$, with conjugacy class of cardinality m. Let $\{x_1, \dots, x_m\}$ be a set of representatives of the right cosets of $\mathrm{C}_G(g)$ in G, with $x_1 = 1$.

For a perturbation θ, denote the restriction of θ_g to $\mathrm{C}_G(g)$ by $\overline{\theta}_g$. By Remark 5.4(a), $\overline{\theta}_g$ is an automorphism of $\mathrm{C}_G(g)$. For perturbations θ and φ, suppose that $x_i\theta_g = c_ix_i$ and $x_i\varphi_g = d_ix_i$ with c_i, d_i in $\mathrm{C}_G(g)$. The permutation θ_g of G is specified completely by the m-tuple $(\overline{\theta}_g, c_2, \dots, c_m)$, since for $x = cx_i \in \mathrm{C}_G(g)x_i$, one has $x\theta_g = (cx_i)\theta_g = c\theta_g.x_i\theta_{c^{-1}gc} = c\overline{\theta}_g.c_ix_i$. Moreover, $x_i(\theta_g\varphi_g) = (c_ix_i)\varphi_g = (c_i\overline{\varphi}_g.d_i)x_i$, so that Γ_G maps homomorphically to the product (5.4).

Conversely, consider an element of the product (5.4) whose component at g is $(\overline{\theta}_g, c_2, \dots, c_m)$. For an element $x = cx_i \in \mathrm{C}_G(g)x_i$ of G, define $x\theta_g = c\overline{\theta}_g.c_ix_i$. These specifications, for the various conjugacy class representatives g, completely specify a unique perturbation θ in accordance with Remark 5.4(b). □

Proposition 5.7. *Let* G *be a group. If* α *is an invertible element of the monoid* $P_1^{-1}\{\mathrm{id}_G\}$, *then* $\widehat{\alpha}$ *given by its values* (5.3) *is a perturbation of* G.

Proof. Satisfaction of Definition 5.3(1) by $\widehat{\alpha}$ follows from Proposition 5.2 and its proof. Condition (2) of Definition 5.3 for $\widehat{\alpha}$ is an immediate consequence of condition (4) of Definition 2.1 for α. □

Theorem 5.8. *Let* G *be a group, and let* U_G *be the group of units* $P_1^{-1}\{\mathrm{id}_G\}^*$ *of the homotopy class* $P_1^{-1}\{\mathrm{id}_G\}$.

(a) U_G *contains a normal subgroup*

$$U_2 = \{(\mathrm{id}_G, \alpha_2, \pi_2) \mid \forall x, y \in G,\ \alpha_2(x, y) \in \mathrm{C}_G(xy)\}$$

isomorphic to $\prod_{(x,y)\in G^2} \mathrm{C}_G(xy)$.

(b) *U_G contains a subgroup*

$$U_3 = \{\alpha = (\mathrm{id}_G, e_G, \alpha_3) \mid \widehat{\alpha} \in \Gamma_G\}$$

isomorphic to the perturbation group Γ_G.

(c) *U_G is the semidirect product of the subgroup U_2 by the subgroup U_3. The action of U_3 on U_2 is given by*

$$(\mathrm{id}_G, e_G, \beta_3)^{-1}(\mathrm{id}_G, \alpha_2, \pi_2)(\mathrm{id}_G, e_G, \beta_3) = (\mathrm{id}_G, \alpha_2', \pi_2) \tag{5.5}$$

with $\alpha_2' : (x, y) \mapsto \alpha_2(x, y)\widehat{\beta}_{xy}$.

Proof. (a): The isomorphism is given by

$$(\mathrm{id}_G, \alpha_2, \pi_2) \mapsto ((x, y) \mapsto \alpha_2(x, y)^{-1}).$$

Then for $\alpha \in U_2$, $\beta \in U$, and $x, y \in G$, one has

$$(\alpha\beta)_3(x, y) = \beta_3(x, \pi_2(x, y)) = y\widehat{\beta}_x$$

and

$$(\beta^{-1}.\alpha\beta)_3(x, y) = (\alpha\beta)_3(x, \beta_3^{-1}(x, y)) = y\widehat{\beta}_x^{-1}\widehat{\beta}_x = y,$$

so U_2 is normal in U. If now $\beta \in U_3$, then

$$(\alpha\beta)_2(x, y) = e_G(x, y)\beta_3(xy, \alpha_2(x, y)) = \alpha_2(x, y)\widehat{\beta}_{xy}$$

and

$$\begin{aligned}(\beta^{-1}.\alpha\beta)_2(x, y) &= (\alpha\beta)_2(x, y).(\alpha\beta)_3(xy, \beta_2^{-1}(x, y)) \\ &= \alpha_2(x, y)\widehat{\beta}_{xy}.\mathrm{id}_G\widehat{\beta}_{xy} = \alpha_2(x, y)\widehat{\beta}_{xy},\end{aligned}$$

the last equality holding by Remark 5.4(a). Thus (5.5) is verified.
(b): The isomorphism is given by the map $(\mathrm{id}_G, e_G, \alpha_3) \mapsto \widehat{\alpha}$ and its inverse $\widehat{\alpha} \mapsto (\mathrm{id}_G, e_G, (x, y) \mapsto y\widehat{\alpha}_x)$.
(c): Certainly $U_2 \cap U_3$ is trivial. Consider a general element $\alpha = (\mathrm{id}_G, \alpha_2, \alpha_3)$ of U. Then $\alpha = (\mathrm{id}_G, e_G, \alpha_3)(\mathrm{id}_G, \alpha_2, \pi_2)$. Thus $U = U_3.U_2$. □

Remark 5.4(a) yields the following special case of Theorem 5.8.

Corollary 5.9. *For a finite abelian group A of order n, the group of units U_A is isomorphic to the semidirect product $A^{n^2} \rtimes \mathrm{Aut}(A)^n$.*

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