CAYLEY THEOREMS FOR LODAY ALGEBRAS

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Abstract. Loday's notoriously elusive "coquecigrues" are meant to relate to Leibniz algebras in the same various ways that groups relate to Lie algebras. However, with the current approaches based on digroups, deadlock has been reached at the analogues of Lie's Third Theorem. Here, adjoint representations appear in the places where regular representations should be expected. The present work, intended as a stimulus to new approaches to the problem, proposes more symmetrical versions of the algebras involved. The fundamental guiding principle is to maintain both left and right actions on a completely equal footing. A coherent and cumulative series of Cayley theorems gives concrete representations of abstract split versions of semigroups, monoids, and groups, based upon the Galois theory of "symmetries of symmetries". Interpreted within monoidal categories, the new group-like objects we present provide a complete left/right split of Hopf algebra structure. The Cayley embedding appears intrinsically as the left/right symmetric part of the coassociativity diagram.

Contents

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1. Introduction

1.1. **Loday's coquecigrue problem.** In a linear tensor category $\mathcal{L}$ (a monoidal product distributes over coproducts, as in a category of vector spaces), Loday defined *dialgebras* [17] by splitting the multiplication of a not necessarily unital associative algebra into two related left- and right-handed products. A dialgebra corresponds to a semigroup in the monoidal *infinitesimal* category $\mathcal{LM}$ of $\mathcal{L}$-morphisms [18], where the domain of its underlying $\mathcal{L}$-morphism becomes a bimodule over the codomain. In analogous fashion, a Lie algebra in $\mathcal{LM}$ corresponds to a *Leibniz algebra*, where the Jacobi identity and skew-symmetry of a Lie algebra $\mathfrak{g}$ are relaxed to just the derivation property (*Leibniz identity*)

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

of the maps $\mathrm{Ad}_z \colon \mathfrak{g} \to \mathfrak{g}; x \mapsto [x, z]$ for each $z \in \mathfrak{g}$. An infinitesimal category $\mathcal{CM}$ can be defined over more general (i.e., not necessarily linear) tensor categories $\mathcal{C}$. In all cases, the monoidal unit of $\mathcal{CM}$ is the unique morphism $\bot \to \top$ in $\mathcal{C}$ from its initial object $\bot$ to its terminal object $\top$.

In [16, §11], Loday proposed putative structures which he tentatively called *coquecigrues*, whose relations to Leibniz algebras would extend the multifarious relationships of groups to Lie algebras. To this day, they have remained as elusive as Loday's terminology was intended to suggest. Kinyon [12] and several subsequent authors have examined the extent to which *Lie racks*, namely right distributive right quasigroups in a category of smooth maps, might function in a version of Lie's Third Theorem for Leibniz algebras (in the sense of [34, Pt. II, §V.8]). Unfortunately, Lie racks produce adjoint representations where regular representations should appear.

While the search for Loday's coquecigrues may well serve as a first motivation for the current paper, we will defer any detailed discussion of that question to future work, not least on the grounds of space considerations. Suffice it to say that, where Loday in [16, §11] speaks of a single tangent space of a "Leibniz group", the *orbitoid structures* developed in this paper (§2.2, §6.4) will suggest that a "Leibniz group" should support two distinct, interacting foliations. In the Lie group case, these foliations coincide and trivialize to a single leaf comprising the entire Lie group.

1.2. **Left/right splitting of Hopf algebras.** A second motivation for the current paper is the development of a left/right splitting of the concept of a Hopf algebra, to parallel Loday's left/right splitting of semigroup structure in various categories. Current approaches to this question have just taken Hopf algebras in the infinitesimal category $\mathcal{LM}$ of a linear tensor category $\mathcal{L}$, for instance as universal enveloping algebras of Leibniz algebras. Compare [14], [18, §5], and [21, §5], for example. In particular, the connection with Yetter-Drinfel'd modules is noted in [14].

These established approaches will not work within the infinitesimal category $\mathcal{CM}$ of more general tensor categories $\mathcal{C}$, such as categories of sets, topological spaces, or manifolds under smooth maps, where the initial object $\perp$ is the empty set. The counit of a Hopf algebra has the tensor unit as its codomain, but the empty "upstairs" part of the tensor unit $\perp \to \top$ of $\mathcal{CM}$ in these cases could only serve as the codomain of a morphism from $\perp$ itself.

Much of the current paper is dedicated to the solution of this splitting problem, based on a $\mathcal{C}$-morphism that is taken in context as the *adjoint map* $\pi\colon S \to S^\pi$ of Definition 4.6, modeled in the category of sets. Table 1 summarizes the left/right splitting of a Hopf algebra that is achieved, and the specific category diagrams that result are detailed in §4.5 and §6.6.[1] The crucial novelty may be observed in the mutually dual diagrams of §6.6.1 (unitality) and §6.6.2 (counitality), where the usual general monoidal category unitors λ and ρ are accompanied by *bar unitors* $\overline{\lambda}$ and $\overline{\rho}$ (§6.5) that are part of the orbitoid structure.

1.3. **Left/right splitting of semigroups, monoids, and groups.** The third motivation for the current paper is to settle on a consistent and cumulative splitting of the elements of the progression

$$\text{semigroup} \to \text{monoid} \to \text{group} \tag{1.1}$$

of conventional algebraic structures, possibly interpreted literally in the category of sets, or more generally as say non(co)unital bialgebras, (co)unital bialgebras, and Hopf algebras respectively in a linear tensor category. The relationships between the two latter structures in (1.1) include the facts that the set M^* of invertible elements of a monoid M forms a group, and that groups are monoids where each element is invertible. It should also be recalled that French terminology (as in Loday's work) interchanges the terms "semigroup" and "monoid" with respect to the English usage. Thus Kinyon [12] refers to Loday's "dimonoids" as "disemigroups", a precedent that this paper will follow.

The paper works with a progression

$$\text{disemigroup} \to \text{pregrue} \to \text{grue} \tag{1.2}$$

of structures that split their respective counterparts in (1.1). Until quite recently, split counterparts of groups have just been taken to be *digroups* [12, §4], with a single bilateral inversion and a single selected *bar unit* e satisfying the identities

$$x \lhd e = x = e \rhd x \tag{1.3}$$

[1]While this diagrammatic specification of the split Hopf algebra structure takes up more space than compact syntax with Heynemann-Sweedler notation (as in [36, (4.1)–(4.4)], for example), it is much more transparent, particularly where geometric symmetry of the diagrams reflects the logical symmetry and duality of the theory.

[16, §1.2].[2] Digroups without any inversion structure, split counterparts of monoids, are called *dimonoids* [12, §4]. While digroups and dimonoids were allowed to have further bar units beyond the selected one, these additional bar units were not treated on a par with the one that had arbitrarily been selected. Only with the appearance of the *generalized digroups* of [31] were the various bar units finally placed on an equal footing. Generalized digroups also admit separate individual left- and right-handed inversions associated with each of the bar units.

The *pregrues* of (1.2) are disemigroups that are formally specified as heterogeneous or two-sorted algebras $(S, E, \lhd, \rhd)$, comprising a full underlying set S and a set E of bar units (§2.1). The cumulative nature of the sequence (1.2) is initiated by the observation that disemigroups are pregrues $(S, E, \lhd, \rhd)$ in which the sort E is empty. Note that this does not preclude elements of S behaving as bar units according to the property (1.3).

The *grues* of (1.2) are pregrues $(S, E, \lhd, \rhd, I, J)$ endowed with a left inversion I and right inversion J that localize at each bar unit (§2.3, §6.7). Thus grues incorporate the disemigroup properties possessed by generalized digroups into a defined algebraic structure. There is an inversion theory for pregrues (§6.2), and the invertible elements of a pregrue form a grue (Theorem 6.9). Then grues are pregrues where each element is invertible. Remark 2.9 clarifies the relationship between grues and generalized digroups. In [31], inversion properties were imposed axiomatically or syntactically on generalized digroups. Now, these proeprties are given a natural and semantic motivation by the behavior of invertible elements in pregrues.

1.4. **Transformation pregrues, permutation grues.** At first, the classical algebraic structures that appear in (1.1) were abstracted from transformation semigroups, monoids, and groups. Following this model, a fourth motivation for the current paper is to provide left/right split versions of these closed sets of functions that may serve as concrete models of the progression of split algebras in (1.2).

In order to respect the symmetry, both left and right actions on sets need to be involved. If inversion or the antipode of a Hopf algebra is

[2] Turnstiles $\dashv, \vdash$ have previously been the notation of choice for the left- and right-handed products $\lhd, \rhd$ as they appear in (1.3). However, since a turnstile bars access with its horizontal part, the bar unit e in (1.3) would confusingly appear on the side away from the bar of the turnstile. The triangular product symbols of (1.3), which will be used throughout this paper, represent left- and right-handed versions of the multiplication ∇ in a Hopf algebra. Turnstiles will be used for the left- and right-handed convolution products of §6.7.

available, it may be invoked to place both actions on the same side, be it left or right. This is how digroup actions have been handled in the existing literature. However, the requirement for a consistent and cumulative treatment precludes appeal to this option in the grue case, since it would not be available for disemigroups or pregrues. Rather, commuting left and right actions are taken. The progressive concepts of *transformation disemigroups* (§3.3), *transformation pregrues* (§3.4.1) and *permutation grues* (§6.1) are obtained as the canonical concrete models for the algebras of (1.2).

A natural illustration of a transformation pregrue is given (using diagrammatic or algebraic notation, compare §3.1) by the so-called *Yoneda pregrue* $\big(\mathcal{C}(y,y)\times\mathcal{C}(x,y)\times\mathcal{C}(x,x),\mathcal{C}(x,y),\lhd,\rhd\big)$ with

$$(h_1,f_1,g_1)\lhd(h_2,f_2,g_2)=(h_1h_2,f_1h_2,g_1g_2)$$

and

$$(h_1,f_1,g_1)\rhd(h_2,f_2,g_2)=(h_1h_2,g_1f_2,g_1g_2)$$

for an ordered pair (x,y) of objects in a locally small category $\mathcal{C}$. Its grue $\mathcal{C}(y,y)^*\times\mathcal{C}(x,y)\times\mathcal{C}(x,x)^*$ of invertible elements, involving the respective automorphism groups $\mathcal{C}(x,x)^*$ and $\mathcal{C}(y,y)^*$ of x and y, forms a permutation grue. In this example, equal treatment of all the bar units, one of the fundamental principles of our approach, corresponds to the symmetry of the morphism set $\mathcal{C}(x,y)$. No particular morphism is singled out for special attention (the way, say, that the zero morphism might be, if $\mathcal{C}$ were abelian).

1.5. **Cayley theorems.** As reflected in the title, the backbone of the paper is a progressive series (4.2, 4.4, and 6.13) of Cayley theorems, for disemigroups, pregrues, and grues in turn. Previously, respective but unrelated Cayley theorems for dimonoids and digroups had been provided by A.V. Zhuchok [43, Th. 3] and Kinyon [12, Th. 4.8], as well as for generalized digroups by Rodríguez-Neto, Salazar-Díaz, and Velásquez [32, Th. 13].

The cumulative nature of the respective Cayley theorems for the three structures of (1.2) is essential for the workings of the paper. For example, the detailed study of invertible elements in a transformation pregrue (§5.2), paired with the Cayley theorem for pregrues, naturally leads to the identification of the "correct" inversion properties that should be required of a grue (felicitously matching the properties of generalized digroups from [31]). The Cayley embedding for grues then just restricts the Cayley embedding of a pregrue down to its grue of invertible elements.

Cayley theorems, say for monoids and groups, are often just regarded as concrete representations of abstract objects, as a dialogue between applied and pure mathematics. The Cayley theorems for pregrues and grues are more intrinsic. In the context of the adjoint map $S \to S^\pi$ of a pregrue $(S, E, \lhd, \rhd)$, as in Definition 4.6, there are mutually dual maps[3]

$$\alpha\colon S^\pi \times S \times S^\pi \to S; \big(L_\rhd(s_{-1}), s_0, R_\lhd(s_1)\big) \mapsto s_{-1} \rhd s_0 \lhd s_1\,,$$

well-defined by Loday's axiom (2.1) that connects the left- and right-handed products, and

$$\beta\colon S \to S^\pi \times S \times S^\pi; s \mapsto \big(L_\rhd(s), s, R_\lhd(s)\big)\,.$$

These maps form the unique parts of the respective associativity and coassociativity diagrams for pregrues (§4.5.2, §4.5.3) invariant under the left/right symmetry. The first embodies the dialgebra structure, and the second is the embedding in the Cayley theorem.

1.6. **Plan of the paper.** Section 2 summarizes basic definitions of disemigroups, pregrues, and grues. The definitions are presented at two levels: first informally, and then formally as two-sorted algebras (colored operads). The formal definition highlights the importance of the technical requirement of *purity* [23, Def'n. 2.1], forcing a nonempty pregrue to have a nonempty set of bar units. The rudimentary orbitoid structure of pregrues is presented in §2.2.

As a contrast to the splitting of Hopf algebras that eventually emerges in §6.6, a direct comparison is made in §2.4 between grues as two-sorted algebras and the one-sided Hopf algebras of Taft *et al.* (compare [9] and further work of Taft and coauthors referenced there). The analogy establishes an additional precedent for the left/right splitting of the inversion in generalized digroups and grues. The set of bar units of a grue assumes the role played by the tensor unit in the one-sided Hopf algebras.

An inverse to the process of splitting the algebras of (1.1) into the algebras of (1.2) is introduced in §2.5: taking the largest undirected quotient of the split algebra, its *undirected replica*. Undirected replicas of pregrues with $E \neq \emptyset$ are monoids (Lemma 2.16), while undirected replicas of grues with $E \neq \emptyset$ are groups (Proposition 2.17). The term "grue" itself, breaking down as "<u>gr</u>oup-<u>u</u>ndirected, <u>E</u>", is intended to embody two key facts: that the undirected replica of a grue with bar units is a group, and that a grue involves a set E of bar units.

[3]The left and right multiplication notations used in the specifications of α and β follow (2.3) and (2.4).

A recurrent theme is the comparison of the undirected replication $S \to S^v$ with the adjoint map $S \to S^\pi$ of Definition 4.6. The two may differ for a pure pregrue S (Remark 4.7), but must agree when S is a pure grue (Theorem 6.20). Both maps are candidates for interpreting disemigroups as semigroups in infinitesimal categories, as discussed in §2.6.

Section 3 presents the transformation disemigroups and pregrues, as introduced in §1.4. It begins with a careful elementary delineation (§3.1) of the simultaneous use of left and right actions, respectively written in *Eulerian* and in *diagrammatic* or *algebraic* notation. By this device, both right actions (3.3) and left actions (3.4) are given as monoid homomorphisms. Readers should be aware of the conventions used for function composition in algebraic notation (3.1) and Eulerian notation (3.2). The mutual commutation of left and right actions is governed by the Galois theory of §3.2. Transformation disemigroups and pregrues are then defined in §3.3 and §3.4.

Respective Cayley theorems for disemigroups and pregrues appear in Section 4. Working from the pregrue Cayley theorem, the adjoint map $S \xrightarrow{\pi} S^\pi$ of a pregrue is defined in §4.3, and is used to enhance the description of the pregrue orbitoid structure that was begun in §2.2. The properties of the adjoint map as an object of the infinitesimal category of the category of sets are examined in §4.4. The section concludes with the *pregrue diagrams*, representing that fragment of the Hopf algebra splitting discussed in §1.2 that is already available in the pregrue setting. As noted in §1.5, the pregrue Cayley embedding has its place here as the symmetric part of the coassociativity diagram.

The general theory of inversion in pregrues is treated in Section 5, and applied to transformation pregrues. The theory is then invoked in Section 6, through the medium of the pregrue Cayley theorem, to arrive at the abstract definition of a grue. Permutation grues, as introduced in §1.4, appear as the sets of invertible elements of transformation pregrues (Theorem 6.2). Example 6.5 recognizes the *bitorsors* of [5] as permutation grues. Theorem 6.9 shows that the invertible elements of any pregrue (whether pure or not) form a grue. The Cayley theorem for grues (Theorem 6.13) appears in §6.3.

The remainder of Section 6 presents the orbitoid structure (§6.4), bar unitors (§6.5), and split Hopf algebra structure (§6.6) discussed earlier. A coda (§6.7) shows how the convolution structure of a Hopf algebra is split, thereby placing localized versions (Definition 6.34) of grue inversions into a new context.

The final Section 7 provides some concluding remarks and pointers to directions for future development of the work presented here, beyond the primary motivation discussed at the start of this introduction.

2. Disemigroups, pregrues and grues

2.1. Disemigroups and pregrues.

Definition 2.1. A *disemigroup* or *directional semigroup* $(S, \lhd, \rhd)$ is an algebra with two associative multiplications $\lhd, \rhd$, known respectively as the *left* and *right directional multiplications*, such that the *internal associativity*

$$(x \rhd y) \lhd z = x \rhd (y \lhd z) \tag{2.1}$$

and *bar side irrelevance* identities

$$(x \rhd y) \rhd z = (x \lhd y) \rhd z\,, \quad x \lhd (y \lhd z) = x \lhd (y \rhd z)$$

are satisfied.[4] Note that in the products $x \lhd y$ and $y \rhd x$, the variable y is said to be on the *bar side* [17, p.11].

Remark 2.2. (a) The use of the term "disemigroup" in Definition 2.1 follows Kinyon [12], reflecting the standard English usage of the word "semigroup" for associative magmas and "monoid" for unital semigroups. Reflecting the French usage which interchanges these names, e.g. [17, Ex. 1.3(a)], Loday called disemigroups "dimonoids".

(b) A general theory for the splitting of universal algebra operations into *directional* or *directed* operations is presented in [38]. This theory splits semigroups into disemigroups.

Definition 2.3. A *pregrue* $(S, E, \lhd, \rhd)$ is a disemigroup $(S, \lhd, \rhd)$, where the underlying set S contains a set E of *bar units*, such that the *bar unit identities*

$$e \rhd x = x = x \lhd e \tag{2.2}$$

are satisfied for each bar unit e.

Remark 2.4. (a) Note that a pregrue $(S, E, \lhd, \rhd)$ with a pointed set E is a *dimonoid* in the sense of [12, Def'n. 4.1], where the mere existence of a chosen bar unit is required.

(b) A pregrue $(S, E, \lhd, \rhd)$ with $E = \emptyset$ is just a disemigroup.

(c) In terms of universal algebra, a pregrue may be interpreted as a disemigroup equipped with bar units $e, f, \ldots$ being constants that are

[4] In the context of conformal algebras (compare [38, Ex. 2.7]), a referee notes Kolesnikov's terminology of 0-*identities* for the bar side irrelevance identites [13].

selected by nullary operations. Thus a dimonoid is a disemigroup with a single such constant. Dimonoids and disemigroups then constitute the variety of pregrues satisfying the identity $e = f$, whatever the size of the (possibly empty) set E of constants that may be chosen for the language.

(d) An alternative, more satisfactory universal algebra interpretation of a pregrue $(S, E, \lhd, \rhd)$ is as a two-sorted algebra $(S, E, \lhd, \rhd)$ with unary operations $\eta_l, \eta_r \colon E \to S$. In view of Definition 2.10(b) below, where such an interpretation is used for grues, a corresponding two-sorted interpretation of pregrues is presented in Definition 2.10(a). In this context, a pregrue is said to be *pure* if $S = \varnothing$ or $E \neq \varnothing$ [23]. Purity excludes the realization of a nonempty disemigroup as a pregrue according to (b).

2.2. **Orbitoids in pregrues.** Some fundamental properties of bar units in a pregrue are investigated. The notation used in the following proposition is a special case of the general currying of a magma product $A \times A \to A; (x, y) \mapsto x * y$ with *left multiplications*

$$L_*(a) \colon A \to A; y \mapsto a * y \tag{2.3}$$

and *right multiplications*

$$R_*(a) \colon A \to A; x \mapsto x * a \tag{2.4}$$

for each element a of A. Such left and right multiplications will be used throughout the paper.

Proposition 2.5. *Let e be a bar unit of a pregrue $(S, E, \lhd, \rhd)$. Then the maps*

$$L_\lhd(e) \colon (S, \lhd) \to (S, \lhd); s \mapsto e \lhd s \tag{2.5}$$

and

$$R_\rhd(e) \colon (S, \rhd) \to (S, \rhd); s \mapsto s \rhd e \tag{2.6}$$

are semigroup homomorphisms.

Proof. Note

$$(e \lhd s_1) \lhd (e \lhd s_2) = e \lhd (s_1 \lhd e) \lhd s_2 = e \lhd s_1 \lhd s_2$$

for $s_1, s_2 \in S$, so (2.5) is a semigroup homomorphism. A dual proof shows that (2.6) is a semigroup homomorphism. □

The terminology used in the following definition will be motivated by Remark 5.11.

Definition 2.6. Let $(S, E, \lhd, \rhd)$ be a pregrue. Consider $e \in E$.

(a) The image semigroup

$$e \lhd S = \{e \lhd s \mid s \in S\}$$

of (2.5) is called the *right orbitoid* of e in S.

(b) The image semigroup

$$S \rhd e = \{s \rhd e \mid s \in S\}$$

of (2.6) is called the *left orbitoid* of e in S.

Proposition 2.7. *Let $(S, E, \lhd, \rhd)$ be a pregrue. Consider $e \in E$.*

(a) *The right orbitoid $(e \lhd S, \lhd, e)$ of e in S forms a monoid.*

(b) *The left orbitoid $(S \rhd e, \rhd, e)$ of e in S forms a monoid.*

Proof. Note $e \lhd (e \lhd s) = e \lhd s$. Along with the bar unit property of e, this shows that $e \lhd S$ is a monoid. The proof of (b) is dual. $\square$

Proposition 2.8. *Let $(S, E, \lhd, \rhd)$ be a pregrue. For $e, f \in E$, there is a commuting diagram*

$$\begin{array}{ccccccc}
 & S & & = & & S & \\
 {\scriptstyle L_\lhd(f)} \swarrow & & \searrow {\scriptstyle L_\lhd(e)} & & {\scriptstyle R_\rhd(e)} \swarrow & & \searrow {\scriptstyle R_\rhd(f)} \\
f \lhd S & \underset{L_\lhd(e)}{\overset{L_\lhd(f)}{\leftrightarrows}} & e \lhd S & \underset{R_\rhd(e)}{\overset{L_\lhd(e)}{\leftrightarrows}} & S \rhd e & \underset{R_\rhd(f)}{\overset{R_\rhd(e)}{\leftrightarrows}} & S \rhd f
\end{array}$$

of functions, where the arrows are semigroup homomorphisms, and the horizontal arrows are monoid isomorphisms. Furthermore, each pair of adjacent horizontal arrows (left, middle, and right) is mutually inverse.

Proof. The central horizontal arrows form the mutually inverse pair

$$L_\lhd(e) \colon s \rhd e \mapsto e \lhd (s \rhd e) = e \lhd (s \lhd e) = e \lhd s \tag{2.7}$$

and

$$R_\rhd(e) \colon e \lhd s \mapsto (e \lhd s) \rhd e = (e \rhd s) \rhd e = s \rhd e \tag{2.8}$$

of well-defined mappings. Note how bar-side irrelevance intervenes. The equations (2.7) and (2.8) also serve to yield the commutativity of the central trapezoids of the diagram. Then (2.8) preserves e, and is a semigroup homomorphism

$$\begin{array}{ccccc}
(e \lhd s_1, e \lhd s_2) & \overset{\lhd}{\longmapsto} & e \lhd s_1 \lhd e \lhd s_2 & = & e \lhd (s_1 \lhd s_2) \\
{\scriptstyle (R_\rhd(e), R_\rhd(e))} \big\downarrow & & & & \big\downarrow {\scriptstyle R_\rhd(e)} \\
(s_1 \rhd e, s_2 \rhd e) & \overset{\rhd}{\longmapsto} & (s_1 \rhd s_2) \rhd e & = & (s_1 \lhd s_2) \rhd e
\end{array}$$

by bar-side irrelevance. Its inverse (2.7) is also a monoid isomorphism.

All the arrows on the left and the right hand sides of the diagram are (co)restrictions of the semigroup homomorphisms of Proposition 2.5.

The bar unit properties imply that $L_\triangleleft(f) \circ L_\triangleleft(e) = L_\triangleleft(f)$ and $R_\triangleright(e)R_\triangleright(f) = R_\triangleright(f)$ for $e, f \in E$. Here, the respective function composition notations of (3.2) and (3.1) are employed. These equations in turn imply that the adjacent horizontal arrows on the left and on the right hand side of the diagram are mutually inverse, and preserve units. □

2.3. **Grues.** At an informal level, a *grue* $(S, E, \triangleleft, \triangleright, I, J)$ is defined to be a pregrue $(S, E, \triangleleft, \triangleright)$ equipped with both a *left inversion* operation $I_e \colon S \to S$ and a *right inversion* operation $J_e \colon S \to S$ for each bar unit e, such that the conditions

$$\forall\, e \in E\,,\ \forall\, x \in S\,,\ e = x^{I_e} \triangleleft x \text{ and } x \triangleright x^{J_e} = e \tag{2.9}$$

are satisfied.

Remark 2.9. The concept of a *generalized digroup* [31, Def'n. 5, Prop. 4] corresponds to the disemigroup reduct[5] $(S, \triangleleft, \triangleright)$ of a grue, with a nonempty set of bar units and respective unilateral inversions appearing as disemigroup properties.

More formal, universal-algebraic definitions of pregrues and grues are as follows.

Definition 2.10. (a) A *pregrue* is a two-sorted algebra $(S, E, \triangleleft, \triangleright)$ such that $(S, \triangleleft, \triangleright)$ is a disemigroup, where there are unary operations $\eta_l, \eta_r \colon E \to S$, and the bar unit identities

$$e^{\eta_l} \triangleright x = x = x \triangleleft e^{\eta_r} \tag{2.10}$$

are satisfied for $x \in S$ and $e \in E$. Normally, $\eta_l = \eta_r$, but the distinction in the labeling will provide a self-duality of the subsequent theory.

(b) A *grue* $(S, E, \triangleleft, \triangleright, I, J)$ is a pregrue $(S, E, \triangleleft, \triangleright)$ endowed with a *left inversion* operation $I \colon E \times S \to S; (e, x) \mapsto x^{I_e}$ and a *right inversion* operation $J \colon S \times E \to S; (x, e) \mapsto x^{J_e}$, along with projections $\epsilon_l \colon E \times S \to E; (e, x) \mapsto e$ and $\epsilon_r \colon S \times E \to E; (x, e) \mapsto e$, such that the identities

$$x^{I_e} \triangleleft x = (e, x)^{\epsilon_l \eta_l} \quad \text{and} \quad x \triangleright x^{J_e} = (x, e)^{\epsilon_r \eta_r} \tag{2.11}$$

hold.

[5]Specifically, discarding the formal specification of E and the inversion operations (cf. [40, p.287]

Example 2.11. Let E be a set. Then $(E, E, \lhd, \rhd, \lhd, \rhd)$ is a pure grue with

$$y \rhd x = x = x \lhd y \tag{2.12}$$

for all $x, y \in E$.

(a) Note $\eta_l = 1_E = \eta_r$, so that (2.12) reduces to (2.10).

(b) Turning to Definition 2.10(b), one has

$$x^{I_e} \lhd x = (e,x)^I \lhd x = e \lhd x \lhd x = e$$

and

$$x \rhd x^{J_e} = x \rhd (x,e)^J = x \rhd x \rhd e = e$$

for $e, x \in E$.

(c) For each element e of E, its right and left orbitoids are $\{e\}$.

Example 2.12. Let $\big(G, \cdot, e_G, {}^{-1}\big)$ be a group. Then $(G, \{e_G\}, \cdot, \cdot, I, J)$ is a pure grue with

$$I \colon \{e_G\} \times G \to G; (e_G, x) \mapsto x^{-1}$$

and

$$J \colon G \times \{e_G\} \to G; (x, e_G) \mapsto x^{-1}$$

for all $x \in G$.

(a) Note $\eta_l = \eta_r = \big(\{e_G\} \hookrightarrow G\big)$.

(b) In terms of Definition 2.10(b), one has

$$x^{I_{e_G}} \cdot x = (e_G, x)^I \cdot x = x^{-1} \cdot x = e_G$$

and

$$x \cdot x^{J_{e_G}} = x \cdot (x, e_G)^J = x \cdot x^{-1} = e_G$$

for $x \in G$.

(c) The full group G is the right and left orbitoid of e_G.

Remark 2.13. (a) The universal algebraic definitions of pregrues and grues immediately admit the application of standard (multi-sorted) universal algebra [23] to pregrues and grues. For example, a *pregrue homomorphism*

$$f \colon (S, E, \lhd, \rhd) \to (S', E', \lhd, \rhd) \tag{2.13}$$

consists of functions $f_S \colon S \to S'$ and $f_E \colon E \to E'$ which respect the pregrue operations. In particular, it is a disemigroup homomorphism $f_S \colon (S, \lhd, \rhd) \to (S' \lhd, \rhd)$. A *grue homomorphism*

$$f \colon (S, E, \lhd, \rhd, I, J) \to (S', E', \lhd, \rhd, I, J)$$

is a pregrue homomorphism (2.13) that respects (commutes with) the inversions. Alternatively, one may define the homomorphisms (2.13) as being the functions whose graphs are subpregrues or subgrues of $(S, E, \lhd, \rhd) \times (S', E', \lhd, \rhd)$.

In [31], universal-algebraic properties of generalized digroups, such as the First Isomorphism Theorem, are obtained by means of group representations of the generalized digroups involved. Now, since the definition of a generalized digroup implies purity of the corresponding grues, such results emerge in standard fashion from the multi-sorted techniques of [23].

(b) Usually, the pregrue operations $\eta_l, \eta_r \colon E \to S$ are left implicit: each element of E is just associated with its images e^{η_l}, e^{η_r}, and the formal bar unit identities (2.10) will revert to their informal counterparts (2.2). Note, however, that the Yoneda pregrue presented in §1.4 has nontrivial $\eta_l, \eta_r \colon f \mapsto (1_y, f, 1_x)$ for each morphism $f \colon x \to y$.

(c) Likewise, in normal practice, the grue operations $\epsilon_l \colon E \times S \to E$ and $\epsilon_r \colon S \times E \to E$ are surpressed, and the formal inversion properties (2.11) revert to (2.9).

(d) A grue with no bar units is just a disemigroup, as in Remark 2.4(b). Purity, as in Remark 2.4(d), would only allow the absence of bar units from the empty grue.

2.4. **Pregrues, grues and one-sided Hopf algebras.** Structures on pregrues and grues, as introduced in the previous section, may be placed in context by comparison with the one-sided Hopf algebras of Taft *et al.* [7, 9, 15, 24, 28]. Working in a category of vector spaces over a field with tensor product, or in a more general (strict) symmetric monoidal category, the key definitions of the antipodes in one- or two-sided Hopf algebras are as follows.

Definition 2.14. Suppose that $(S, \nabla, \Delta, \eta, \epsilon)$ is a left- or right-unital and counital bimagma with a morphism $\nu \colon S \to S$.

(a) The morphism $\nu \colon S \to S$ is a *left antipode* if $\epsilon\eta = \Delta(\nu \otimes 1_S)\nabla$.
(b) The morphism $\nu \colon S \to S$ is said to be a *right antipode* if $\epsilon\eta = \Delta(1_S \otimes \nu)\nabla$.
(c) The morphism $\nu \colon S \to S$ is an *antipode* if it is both a left and a right antipode.

Now, for comparison, the formal inversion identities (2.11) of a grue may be presented as the commuting diagrams

(2.14)
$$\begin{array}{ccccc} & & E\otimes S\otimes S & \xrightarrow{I\otimes 1_S} & S\otimes S \\ & \nearrow^{1_E\otimes\Delta} & & & \searrow^{\lhd} \\ E\otimes S & & \xrightarrow{\quad\epsilon_l\quad} \; E \; \xrightarrow{\quad\eta_l\quad} & & S \end{array}$$

and

(2.15)
$$\begin{array}{ccccc} S\otimes E & & \xrightarrow{\quad\epsilon_r\quad} \; E \; \xrightarrow{\quad\eta_r\quad} & & S \\ & \searrow_{\Delta\otimes 1_E} & & & \nearrow_{\rhd} \\ & & S\otimes S\otimes E & \xrightarrow[1_S\otimes J]{} & S\otimes S \end{array}$$

in the symmetric Cartesian monoidal category of sets. Here, we are using the tensor product symbol for the Cartesian product, and the diagonal embedding $\Delta\colon S\to S\otimes S$.

These same diagrams may be interpreted in the symmetric monoidal category of vector spaces over a field E, implicitly incorporating its natural unitor isomorphisms

(2.16) $$\lambda_S\colon E\otimes S\to S \quad \text{and} \quad \rho_S\colon S\otimes E\to S\,.$$

For left- or right-unital bimagmas $(S,\lhd,\Delta,\eta_l,\epsilon_l)$ and $(S,\rhd,\Delta,\eta_r,\epsilon_r)$, the diagrams (2.14) and (2.15) then reproduce the respective left and right antipode conditions of Definition 2.14(a),(b) for one-sided Hopf algebras.

In similar fashion, the formal pregrue bar-unit identities (2.10) may be expressed by the commuting diagrams

(2.17)
$$\begin{array}{ccc} E\otimes S & \xrightarrow{\eta_l\otimes 1_S} & S\otimes S \\ & \searrow_{\lambda_S} & \downarrow{\scriptstyle\rhd} \\ & & S \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S\otimes E & \xrightarrow{1_S\otimes\eta_r} & S\otimes S \\ & \searrow_{\rho_S} & \downarrow{\scriptstyle\lhd} \\ & & S \end{array}$$

in the symmetric Cartesian monoidal category of sets, interpreting ρ_S and λ_S as the projections to the factor S in the respective Cartesian products $S\otimes E$ and $E\otimes S$. Note that these interpretations of ρ_S and λ_S do not coincide with their usual unitor interpretations, the analogues of (2.16) in the Cartesian monoidal category of sets, unless E is a singleton. For example, they reduce to $\varnothing\hookrightarrow S$ if E is empty.

Once again, the diagrams (2.17) may also be interpreted within the symmetric monoidal category of vector spaces over a field E, with λ_S and ρ_S as the unitors (2.16). Now, they represent the respective left and right unital conditions [9, Def'n. 2.1(a)] that are part of the structure of left-, right-, or two-sided Hopf algebras.

Finally, of course, commuting of the diagram

$$\begin{array}{ccc} S\otimes S\otimes S & \xrightarrow{\rhd\otimes 1_S} & S\otimes S \\ {\scriptstyle 1_S\otimes\lhd}\big\downarrow & & \big\downarrow{\scriptstyle\lhd} \\ S\otimes S & \xrightarrow[\rhd]{} & S \end{array}$$

captures the mixed associative law.

2.5. **Undirected replicas.** A disemigroup $(S,\lhd,\rhd)$ is *undirected* if the identity $x\lhd y = x\rhd y$ is satisfied. Thus undirected disemigroups are just *iterated semigroups*, semigroups in which the multiplication appears twice as a fundamental operation (cf. [29, p.60]). Conversely, the congruence υ generated by the set

$$\{(x\lhd y, x\rhd y)\mid x,y\in S\}$$

of pairs of elements of a given disemigroup $(S,\lhd,\rhd)$ yields a natural projection

$$\operatorname{nat}\upsilon_S\colon (S,\lhd,\rhd)\to(S^\upsilon,\cdot,\cdot);x\mapsto x^\upsilon \tag{2.18}$$

to the *semigroup replica* or *undirected replica* S^υ of the disemigroup S. The congruence υ itself is known as the *undirected replica congruence* (compare [37, p.316], [40, §IV.2.1]).

Lemma 2.15. *Let $\theta\colon S\to T$ be a disemigroup homomorphism from a disemigroup S to an iterated semigroup T. Then*

$$\begin{array}{ccc} & S^\upsilon & \\ {\scriptstyle\operatorname{nat}\upsilon_S}\nearrow & & \searrow{\scriptstyle\theta^\upsilon} \\ S & \xrightarrow[\theta]{} & T \end{array} \qquad \textit{with} \qquad \begin{array}{ccc} & s^\upsilon & \\ {\scriptstyle\operatorname{nat}\upsilon_S}\nearrow & & \searrow{\scriptstyle\theta^\upsilon} \\ s & \xmapsto[\theta]{} & s^\theta \end{array}$$

factorizes θ.

Note that the formal repetition of the multiplication in an iterated semigroup is often suppressed, and one refers simply to semigroups. The usual proof of equality between left and right units in a monoid yields the following.

Lemma 2.16. *Let $(S,E,\lhd,\rhd)$ be a pregrue. Let υ be the undirected replica congruence of the disemigroup $(S,\lhd,\rhd)$.*

(a) *Any two elements of E are related by υ.*
(b) *If E is empty, then the undirected replica $(S,E,\lhd,\rhd)^\upsilon$ is a semigroup.*

(c) *If E is nonempty, then the undirected replica $(S,E,\lhd,\rhd)^\upsilon$ is a monoid $(S^\upsilon,\cdot,1_{S^\upsilon})$, with $E\subseteq 1_{S^\upsilon}$.*

Proof. (a): For $e,f\in E$, one has $f^\upsilon=(e\rhd f)^\upsilon=(e\lhd f)^\upsilon=e^\upsilon$.

(b): If E is empty, Remark 2.4(b) applies.

(c): For $x\in S$ and $e\in E$, one has

$$e^\upsilon\cdot x^\upsilon=(e\rhd x)^\upsilon=x^\upsilon=(x\lhd e)^\upsilon=x^\upsilon\cdot e^\upsilon$$

according to the bar-unit identities. □

In the context of Lemma 2.16(c), Proposition 3.21 below shows that one may have E as a proper subset of the congruence class 1_{S^υ}.

Proposition 2.17. *Suppose that $(S,E,\lhd,\rhd,I,J)$ is a grue with a nonempty set E of bar units. Then the undirected replica S^υ of the pregrue $(S,E,\lhd,\rhd)$ forms a group $(S^\upsilon,\cdot,^{-1},1_{S^\upsilon})$ within which the grue inversions I or J induce the group inversion.*

Proof. The condition

$$\forall\ e\in E\,,\ \forall\ x\in S\,,\ e=x^{I_e}\lhd x \text{ and } x\rhd x^{J_e}=e$$

of (2.9) replicates to

$$\forall\ x^\upsilon\in S^\upsilon\,,\ 1_{S^\upsilon}=(x^{I_e})^\upsilon\cdot x^\upsilon \text{ and } x^\upsilon\cdot(x^{J_e})^\upsilon=1_{S^\upsilon}$$

in the monoid $(S^\upsilon,\cdot,1_{S^\upsilon})$ of Lemma 2.16(c), for each element e of E. In particular, each element x^υ of S^υ is invertible. Thus the monoid S^υ coincides with its group of units, where inversion is given by

$$(x^\upsilon)^{-1}=(x^{I_e})^\upsilon=(x^{J_e})^\upsilon \tag{2.19}$$

for each $x^\upsilon\in S^\upsilon$ and $e\in E$. □

2.6. **Semigroups in infinitesimal categories.** Let $(\mathbf{V},\otimes,I)$ be a symmetric monoidal category, having finite coproducts (including the initial object $\perp$) that are preserved by the monoidal product. Such categories will often be described simply as *tensor* categories. The *infinitesimal category* $\mathbf{V}\mathcal{M}$ of $\mathbf{V}$ has $\mathbf{V}$-morphisms $p\colon U\to D$ as its objects, with U "upstairs" and D "downstairs" [18]. The morphisms $f\colon p^1\to p^2$ of $\mathbf{V}\mathcal{M}$ have $\mathbf{V}$-morphisms $f^/\colon U^1\to U^2$ upstairs and $f_/\colon D^1\to D^2$ downstairs, with $p^1f_/=f^/p^2$. It is often convenient to suppress the decoration of the symbol f here.

The infinitesimal category is endowed with a symmetric monoidal structure $(\mathbf{V}\mathcal{M},\square,\perp\to I)$. Downstairs, $\square=\otimes$. Upstairs, $p^1\square p^2$ has

$U^1 \otimes D^2 + D^1 \otimes U^2$, with

$$p^1 \Box p^2 = \begin{cases} p^1 \otimes 1_{D^2} \colon U^1 \otimes D^2 \to D^1 \otimes D^2\,; \\ 1_{D^1} \otimes p^2 \colon D^1 \otimes U^2 \to D^1 \otimes D^2\,. \end{cases}$$

This structure has been motivated nicely in terms of chain complexes [14, Rem. 2.2]. An alternative approach uses simplicial objects, with a pair of morphisms from upstairs to downstairs [21].

Let $\mu\colon \pi \Box \pi \to \pi$ be a semigroup structure in $\mathbf{V}\mathcal{M}$ on an object $p\colon S \to T$. Downstairs, $\mu_{/}\colon T \otimes T \to T; t_1 \otimes t_2 \mapsto t_1 \cdot t_2$ is a semigroup in $\mathbf{V}$. Decompose the upstairs $\mathbf{V}$-morphism

$$\mu^{/}\colon (S \otimes T + T \otimes S) \to S$$

as $\mu_{\lhd}\colon S \otimes T \to S$ and $\mu_{\rhd}\colon T \otimes S \to S$. In the literature, these morphisms are recognized as creating a T-bimodule structure on S, and $p\colon S \to T$ is characterized as a T-bimodule morphism $\pi\colon {}_TS_T \to {}_TT_T$ [14, §2.2] [18, §2.3]. The following reformulation shows how bimodule properties are encoded in universal-algebraic terms by the identities of a disemigroup (cf. [17, Ex. 2.2(d)],[26],[30]).

Proposition 2.18. *Under the products*

$$x \lhd y = (x \otimes yp)\mu_{\lhd} \quad \textit{and} \quad x \rhd y = (xp \otimes y)\mu_{\rhd}\,,$$

the $\mathbf{V}$*-morphism* $p\colon S \to T$ *is a homomorphism*

$$p\colon (S, \lhd, \rhd) \to (T, \cdot, \cdot)$$

of disemigroups in $\mathbf{V}$*.*

Proof. The mixed associativities for the respective right and left module structures translate to the associativities of the two products. Internal associativity corresponds to the bimodule property. Finally, the barside irrelevance is equivalent to the fact that $p\colon S \to T$ is a morphism of bimodules. □

Conversely, a $\mathbf{V}$-morphism $p\colon S \to T$ that is a homomorphism from a disemigroup S in $\mathbf{V}$ to an iterated semigroup T in $\mathbf{V}$ yields a semigroup structure in the infinitesimal category $\mathbf{V}\mathcal{M}$. For a given disemigroup S, the projection (2.18) from S to its undirected replica represents one possible choice of infinitesimal category morphism. For pregrues, the adjoint maps of Definition 4.6 will turn out to be natural candidates.

3. Transformation disemigroups and pregrues

3.1. **Left and right actions.** Suppose that X is a set. In order to give a coherent and concurrent account of left and right semigroup, monoid, and group actions on the set X, it will be convenient to use

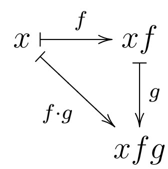
Eulerian functional notation (as in $\sin\theta$) for left actions, and algebraic or diagrammatic notation (as in $n!$ or x^{-1}) for right actions.

An elementary review of the parallel use of these conventions will prepare for subsequent work. Readers are reassured that our parallel use will not be as confusing as the juxtaposition of the notation $\sin^2\theta$ for the square of the sine of an angle θ against the notation $\sin^{-1}\theta$ for $\arcsin\theta$, where $\csc\theta$ might have been expected!

3.1.1. *Transfomation monoids.* Write $(X^X, \cdot, 1_X)$, or simply X^X, for the monoid of transformations of X acting on the right; and $({}^X X, \circ, 1_X)$, or just ${}^X X$, for the monoid of transformations of X acting on the left.

Consider functions $f\colon X \to X; x \mapsto xf$ (instead of xf, we may also write x^f) and $g\colon X \to X$. The composition or product in X^X, denoted by $\cdot$ or mere juxtaposition, is specified by the commutative diagram

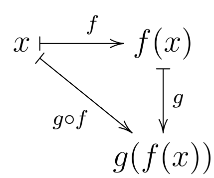
(3.1)



for each element x of X. The functions and their composition are written with *algebraic* or *diagrammatic* notation.

Now consider functions $f\colon X \to X; x \mapsto f(x)$ and $g\colon X \to X$. Instead of $f(x)$, we may also write fx. Thus the functions are written in *Eulerian* notation. The composition or product in ${}^X X$, denoted by $\circ$, is specified by the commutative diagram

(3.2)



for each element x of X. Taken abstractly, the monoids ${}^X X$ and X^X are mutual opposites. By the Yoneda Lemma [20], each action is the monoid of endomorphisms of the other.

3.1.2. *Left and right monoid actions.* Monoid actions are construed as universal algebras.

Definition 3.1. Suppose that $(M, \cdot, 1_M)$ is a monoid.

(a) A *right M-set* (X, M) is a set X that is equipped with a unary operation $m\colon X \to X; x \mapsto xm$ for each element m of M, such that the *mixed associative law*

$$(xm_1)m_2 = x(m_1 \cdot m_2)$$

and the *unital law* $x1_M = x$ are satisfied (for all $x \in X$ and) for all m_1, m_2 in M.

(b) A *left M-set* (X, M) is a set X that is equipped with a unary operation $m\colon X \to X; x \mapsto mx$ for each element m of M, such that the *mixed associative law*

$$m_2(m_1x) = (m_2 \cdot m_1)x$$

and the *unital law* $x1_M = x$ are satisfied (for all $x \in X$ and) for all m_1, m_2 in M.

Lemma 3.2. *Suppose that* $(M, \cdot, 1_M)$ *is a monoid.*

(a) *Given a right* M-set (X, M)*, define*

$$R(m)\colon x \mapsto xm$$

for each element m of M, so that $xR(m) = xm$ for $x \in X$. Then

$$R\colon M \to X^X; m \mapsto R(m) \tag{3.3}$$

is a monoid homomorphism.

(b) *Given a left* M-set (X, M)*, define*

$$L(m)\colon x \mapsto mx$$

for each element m of M, so that $L(m)x = mx$ for $x \in X$. Then

$$L\colon M \to {}^XX; m \mapsto L(m) \tag{3.4}$$

is a monoid homomorphism.

Proof. (a) The mixed associative law yields

$$xR(m_1)R(m_2) = (xm_1)m_2 = x(m_1 \cdot m_2) = xR(m_1 \cdot m_2)$$

for all $x \in X$ and for all m_1, m_2 in M. Thus $R(m_1)R(m_2) = R(m_1 \cdot m_2)$. Together with $R(1_M) = 1_X$, this makes (3.3) a monoid homomorphism.

(b) The mixed associative law yields

$$L(m_1) \circ L(m_2)x = m_1(m_2x) = (m_1 \cdot m_2)x = L(m_1 \cdot m_2)x$$

for all $x \in X$ and for all m_1, m_2 in M. Thus $L(m_1) \circ L(m_2) = L(m_1 \cdot m_2)$. In conjunction with $L(1_M) = 1_X$, this ensures that (3.4) is a monoid homomorphism. □

Proposition 3.3. *Suppose that* $(M, \cdot, 1_M)$ *is a monoid.*

(a) *Right M-sets are equivalent to monoid homomorphisms* (3.3).

(b) *Left M-sets are equivalent to monoid homomorphisms* (3.4).

Proof. The passage from a right M-set to the homomorphism (3.3) is captured by Lemma 3.2(a). In the converse direction, the M-set (X, M) which corresponds to such a monoid homomorphism is defined by $xm = xR(m)$ for $x \in X$ and $m \in M$. The equivalence between left M-sets and monoid homomorphisms (3.4) is similarly described. □

3.1.3. *Group actions.* Write $(X!, \cdot, 1_X)$, or simply $X!$, for the group of permutations of X acting on the right; and $(!X, \circ, 1_X)$, or simply $!X$, for the group of permutations of X acting on the left. Thus $X!$ is the group of units of X^X, while $!X$ is the group of units of ${}^X X$.

The abstract groups $!X$ and $X!$ are mutual opposites. By the Yoneda Lemma, each action is the group of automorphisms of the other.

If G is a group, right G-sets correspond to group homomorphisms $R\colon G \to X!$, while left G-sets will correspond to group homomorphisms $L\colon G \to !X$.

3.2. **Commutants.**

3.2.1. *Polarities.* [40, Example III.3.3.2(c)]

Suppose that $\alpha \subseteq I \times J$ is a relation between sets I and J. For a subset S of I, define

$$S' := \{j \in J \mid \forall\, i \in S\,,\ (i, j) \in \alpha\}\,.$$

For a subset T of J, define

$$T' := \{i \in I \mid \forall\, j \in T\,,\ (i, j) \in \alpha\}\,.$$

Note that $s \in S$ implies $\forall\, j \in S'\,,\ (s, j) \in \alpha$, so that $s \in S''$. Thus $S \subseteq S''$, and similarly $T \subseteq T''$.

A subset T of J is *closed* if $T = S'$ for some $S \subseteq I$. Similarly, a subset S of I is *closed* if $S = T'$ for some $T \subseteq J$. Then the containment-reversing (or "antitone") functions

$$2^I \to 2^J; S \mapsto S' \quad\text{and}\quad 2^J \to 2^I; T \mapsto T' \tag{3.5}$$

restrict to mutual bijections (described as *Galois correspondences* in general, or *polarities* in the present context) between the respective sets of closed subsets of the sets I and J.

3.2.2. *Mutual commutativity.* Let X be an object of a locally small category $\mathbf{C}$. Consider the monoid $\mathbf{C}(X, X)$ of endomorphisms of the object X. The symmetric *mutual commutativity* relation κ is defined on $\mathbf{C}(X, X)$ by

$$(\theta, \varphi) \in \kappa \ \Leftrightarrow\ \theta\varphi = \varphi\theta \tag{3.6}$$

for θ, φ in $\mathbf{C}(X, X)$. For a set T of endomorphisms of X, the set

$$T' := \{\varphi \in \mathbf{C}(X, X) \mid \forall\ \theta \in T\ ,\ (\theta, \varphi) \in \kappa\}$$

is known as the *commutant* of the subset T. In particular, $\{1_X\}$ and $\varnothing' = \mathbf{C}(X, X)$ are mutual commutants.

Remark 3.4. The "commutant" terminology is standard in functional analysis, for example in the name of the Double Commutant Theorem [10, §5.3].

Lemma 3.5. *The sets* $\mathbf{C}(X, X)$ *and* $\mathbf{C}^{\mathsf{op}}(X, X)$ *coincide. Then for a subset* T *of these sets, the commutants* T' *in* $\mathbf{C}(X, X)$ *and* $\mathbf{C}^{\mathsf{op}}(X, X)$ *coincide.*

Proof. The definition (3.6) is self-dual. □

Lemma 3.6. *For* $T \subseteq \mathbf{C}(X, X) = \mathbf{C}^{\mathsf{op}}(X, X)$, *the commutant* T' *is a submonoid of* $\mathbf{C}(X, X)$ *and* $\mathbf{C}^{\mathsf{op}}(X, X)$.

Proof. Certainly the identity 1_X commutes with each element of T. Now consider an element θ of T, and elements φ_i of T' for $i = 1, 2$. Then $\theta\varphi_1\varphi_2 = \varphi_1\theta\varphi_2 = \varphi_1\varphi_2\theta$, as required. □

Remark 3.7. More generally, if the category $\mathbf{C}$ is enriched over a symmetric, monoidal category $\mathbf{V}$ or $(\mathbf{V}, \otimes, I)$, then the commutant T' is a submonoid in $\mathbf{V}$ of the monoid $\mathbf{C}(X, X)$ in $\mathbf{V}$. In Lemma 3.6, the enriching category is the Cartesian monoidal category $(\mathbf{Set}, \times, \top)$.

Example 3.8. Let $\mathbf{C}$ be a category of vector spaces over a field K. Consider a permutation representation of a group G on a set Q, with a corresponding linear representation space $X = KQ$ in $\mathbf{C}$ affording a group representation $\rho \colon G \to \mathbf{C}(X, X)$ with image T. Then the subset T' of $\mathbf{C}(X, X)$ is the *centralizer ring* or *Vertauschungsring* $V(G, Q)$ of Wielandt [1, §2.1], [37, §6.3], [41].

If M is a monoid, the notation M^* denotes the group of units (i.e., invertible elements) of M. In particular, if X is an object of a locally small category $\mathbf{C}$, then $\mathbf{C}(X, X)^*$ is the group of automorphisms of the object X.

Corollary 3.9. *For* $T \subseteq \mathbf{C}(X, X)^*$, *the intersection* $T' \cap \mathbf{C}(X, X)^*$ *is the centralizer subgroup* $C_{\mathbf{C}(X,X)^*}(T)$ *of* T *in the automorphism group* $\mathbf{C}(X, X)^*$ *of the object* X *of* $\mathbf{C}$.

3.2.3. *Symmetry-breaking.* Let X be a set, thus an object of the locally small category $\mathbf{Set}$ of sets. The endomorphism monoid $\mathbf{Set}(X, X)$ of X, as an instance of the monoids $\mathbf{C}(X, X)$ considered in §3.2.2, is the set of functions from X to X. The monoid $\mathbf{Set}(X,X)$ is the monoid of transformations of X as studied in §3.1.1.

One of the fundamental considerations for the work of this paper is to respect the left/right symmetry that is inherent to disemigroups. This symmetry is broken by any particular choice of convention for functional notation, say Eulerian or algebraic as discussed in §3.1. The notation $\mathbf{Set}(X,X)$ will be used "in the abstract", independently of any particular choice that a reader may prefer, or independently of the default option of algebraic notation adopted in this paper.

Following §3.1, the abstract monoid $\mathbf{Set}(X,X)$ will be implemented concretely as X^X to act on X from the right, or ${}^XX$ to act on X from the left. Similarly, the abstract group $\mathbf{Set}(X,X)^*$ will be implemented concretely as $X!$ to act on X from the right, or $!X$ to act on X from the left.

3.2.4. *Sets with unary operations.* Let X be a set. Let T be a set of functions from X to X. One then has the "universal algebra" (X,T) as the carrier set X equipped with the set T of unary operations on X. Following the default of algebraic notation that was recalled in §3.2.3, $T \subseteq X^X$. Then there are respective commutants $T' \subseteq {}^XX$ and $T'' \subseteq X^X$. These commutants feature prominently in the subsequent constructions. In particular, the mixed associative law

$$(hx)g = h(xg) =: hxg$$

holds for $h \in T'$, $x \in X$, and $g \in T''$. In Theorem 3.10 below, this mixed associative law yields the internal associative law of a disemigroup.

3.3. **Transformation disemigroups of a set with operations.**

3.3.1. *The transformation disemigroup.*

Theorem 3.10. *Let (X,T) consist of a set X with a set T of unary operations on X. Consider $T'' \subseteq X^X$ and $T' \subseteq {}^XX$. Define:*

$$(g_1,x_1,h_1) \lhd (g_2,x_2,h_2) = (g_1 \cdot g_2, x_1g_2, h_1 \circ h_2)\,; \tag{3.7}$$

$$(g_1,x_1,h_1) \rhd (g_2,x_2,h_2) = (g_1 \cdot g_2, h_1x_2, h_1 \circ h_2) \tag{3.8}$$

on $T''XT' := T'' \times X \times T'$. Then $(T''XT', \lhd, \rhd)$ is a disemigroup.

Proof. Consider elements (g_i,x_i,h_i) of $T''XT'$ with $1 \leq i \leq 3$.

(a) First,

$$\begin{aligned}(g_1, x_1, h_1) \rhd \big((g_2, x_2, h_2) \lhd (g_3, x_3, h_3)\big) \\ &= (g_1, x_1, h_1) \rhd (g_2 \cdot g_3, x_2 g_3, h_2 \circ h_3) \\ &= (g_1 \cdot g_2 \cdot g_3, h_1 x_2 g_3, h_1 \circ h_2 \circ h_3)\end{aligned}$$

and

$$\begin{aligned}\big((g_1, x_1, h_1) \rhd (g_2, x_2, h_2)\big) \lhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2, h_1 x_2, h_1 \circ h_2) \lhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2 \cdot g_3, h_1 x_2 g_3, h_1 \circ h_2 \circ h_3)\,,\end{aligned}$$

confirming the validity of the internal associative law.

(b) Next,

$$\begin{aligned}(g_1, x_1, h_1) \rhd \big((g_2, x_2, h_2) \rhd (g_3, x_3, h_3)\big) \\ &= (g_1, x_1, h_1) \rhd (g_2 \cdot g_3, h_2 x_3, h_2 \circ h_3) \\ &= (g_1 \cdot g_2 \cdot g_3, h_1 h_2 x_3, h_1 \circ h_2 \circ h_3)\end{aligned}$$

and

$$\begin{aligned}\big((g_1, x_1, h_1) \rhd (g_2, x_2, h_2)\big) \rhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2, h_1 x_2, h_1 \circ h_2) \rhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2 \cdot g_3, (h_1 \circ h_2) x_3, h_1 \circ h_2 \circ h_3)\,.\end{aligned}$$

The commuting of (3.2) then confirms the associativity of the right directional multiplication. In similar fashion, the associativity of the left directional multiplication is verified.

(c) Finally,

$$\begin{aligned}\big((g_1, x_1, h_1) \rhd (g_2, x_2, h_2)\big) \rhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2, h_1 x_2, h_1 \circ h_2) \rhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2 \cdot g_3, (h_1 \circ h_2) x_3, h_1 \circ h_2 \circ h_3)\end{aligned}$$

and

$$\begin{aligned}\big((g_1, x_1, h_1) \lhd (g_2, x_2, h_2)\big) \rhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2, x_1 g_2, h_1 \circ h_2) \rhd (g_3, x_3, h_3) \\ &= (g_1 \cdot g_2 \cdot g_3, (h_1 \circ h_2) x_3, h_1 \circ h_2 \circ h_3)\,,\end{aligned}$$

confirming the validity of the first bar side irrelevance identity. The validity of the second bar side irrelevance identity follows similarly. $\square$

3.3.2. *Transformation disemigroups.*

Corollary 3.11. *In the context of Theorem 3.10, suppose that U is a subsemigroup of T'', and V a subsemigroup of T'. Define the set $UXV := U \times X \times V$. Then $(UXV, \lhd, \rhd)$ is a subdisemigroup of $(T''XT', \lhd, \rhd)$.*

Corollary 3.12. *In the context of Theorem 3.10, suppose that G is a subsemigroup of $T'' \times T'$. Set*

$$XG := \{(g, x, h) \in T''XT' \mid (g, h) \in G\}\,. \tag{3.9}$$

Then $(XG, \lhd, \rhd)$ is a subdisemigroup of $(T''XT', \lhd, \rhd)$.

Definition 3.13. Let T be a set of transformations of a set X.

(a) The disemigroup $(T''XT', \lhd, \rhd)$ of Theorem 3.10 is described as **the** *transformation disemigroup* of the set (X, T) with unary operations from T.
(b) Let U be a subsemigroup of T'', and let V be a subsemigroup of T'. Then the disemigroup $(UXV, \lhd, \rhd)$ of Corollary 3.11 is described as **a** (*balanced*) *transformation disemigroup* on the set (X, T) with unary operations from T.
(c) Let G be a subsemigroup of $T'' \times T'$. Then the disemigroup $(XG, \lhd, \rhd)$ as appearing in Corollary 3.12 is described as **a** *transformation disemigroup* on the structure (X, T).

Remark 3.14. The use of the articles in Definition 3.13 is similar to their usage in distinguishing **the** permutation group $\mathbf{Set}(X, X)^*$ on a set X from its subgroups G, whereby such a subgroup G is described as **a** permutation group on the set X.

3.3.3. *Balancing a transformation disemigroup.*

Lemma 3.15. *Let T be a set of transformations of a set X. Let G be a subsemigroup of $T'' \times T'$. Define*

$$U_G = \{g \mid (g, h) \in G\} \quad \textit{and} \quad {}_GV = \{h \mid (g, h) \in G\}\,. \tag{3.10}$$

(a) *The set U_G is a subsemigroup of T'', and the set ${}_GV$ is a subsemigroup of T'.*
(b) *The set G is a subsemigroup of $U_G \times {}_GV$.*

Definition 3.16. Suppose that T is a set of transformations of a set X. Suppose that G is a subsemigroup of $T'' \times T'$. Then the balanced transformation disemigroup $(U_G X_G V, \lhd, \rhd)$ is called the *balancing* of the transformation disemigroup $(XG, \lhd, \rhd)$ on the structure (X, T).

3.3.4. *Undirected replicas.* Here, we examine the undirected replicas of transformation disemigroups, making use of concepts from §2.5. The terminal object $\top$ of the category of sets is taken as $\top = \{*\}$.

Proposition 3.17. *Suppose that (X, T) consists of a nonempty set X with a set T of unary operations on X. Consider $T'' \subseteq X^X$ and $T' \subseteq {}^X X$. Then the projection*

$$\pi\colon T'' \times X \times T' \to T'' \times \top \times T'; (g, x, h) \mapsto (g, *, h)$$

presents the undirected replica of the disemigroup $(T''XT', \lhd, \rhd)$ as the iterated monoid $(T'', \cdot, \cdot) \times (T', \circ, \circ)$.

Proof. From the definitions (3.7) and (3.8), it is immediate that π is a disemigroup homomorphism. The undirected replica congruence υ will now be shown to be the kernel congruence $\ker \pi$ of the projection π.

Let (g, x, h) be an element of $T''XT'$. Let y be an element of X. Recalling that 1_X is an element of the commutant monoids T'' and T', we have

$$\begin{aligned}(3.11)\qquad &\big((g, x, h), (g, y, h)\big) \\ &\qquad = \big((g, x, 1) \lhd (1, y, h), (g, x, 1) \rhd (1, y, h)\big) \in \upsilon\,,\end{aligned}$$

so $\ker \pi \subseteq \upsilon$. Conversely, for elements $(g, x, h), (g', y, h')$ of $T''XT'$, one has

$$\begin{aligned}&\big((g, x, h) \lhd (g', y, h'), (g, x, h) \rhd (g', y, h')\big) \\ &\qquad = \big((g \cdot g', xg', h \circ h'), (g \cdot g', hy, h \circ h')\big) \in \ker \pi\,,\end{aligned}$$

so $\upsilon \subseteq \ker \pi$. □

Corollary 3.18. *Suppose that U is a submonoid of T'', and that V is a submonoid of T'. Then the projection*

$$(3.12)\qquad \pi\colon U \times X \times V \to U \times \top \times V; (g, x, h) \mapsto (g, *, h)$$

presents the undirected replica of the disemigroup $(UXV, \lhd, \rhd)$ as the iterated monoid $(U, \cdot, \cdot) \times (V, \circ, \circ)$.

Proof. The proof carries through as in the theorem. In particular, since U and V are monoids, the computation (3.11) is available to show that $\ker \pi \subseteq \upsilon$. □

Corollary 3.18 may break down for general subsemigroups U of T'' and V of T'.

Proposition 3.19. *Let X be a set, with an element ∞. Consider the constant map $c_\infty\colon X \to X$ with image ∞, an idempotent element of* **Set**(X, X)*. Set $U = V = \{c_\infty\}$. Then the disemigroup $(UXV, \lhd, \rhd)$ is undirected.*

Proof. Note

(3.13)
$$(c_\infty, x, c_\infty) \lhd (c_\infty, y, c_\infty) = (c_\infty, \infty, c_\infty) = (c_\infty, x, c_\infty) \rhd (c_\infty, y, c_\infty)$$

for elements x, y of X. □

Corollary 3.20. *If $|X| > 1$, the projection* (3.12) *does not present the undirected replica 1_{UXV}.*

An extension of the construction of Proposition 3.19 addresses an issue that arose in connection with Lemma 2.16(c).

Proposition 3.21. *In the context of Proposition 3.19, consider the subdisemigroup*

$$S = \{(1_X, \infty, 1_X)\} \cup \{(c_\infty, x, c_\infty) \mid x \in X\}$$

of the transformation disemigroup of $(X, \{c_\infty\})$.

(a) *The disemigroup S is an iterated semigroup.*
(b) *The only bar unit of S is $(1_X, \infty, 1_X)$.*
(c) *Setting $E = \{(1_X, \infty, 1_X)\}$, the undirected replica $(S, E, \lhd, \rhd)^v$ of the pregrue $(S, E, \lhd, \rhd)$ is a monoid $(S^v, \cdot, 1_{S^v})$, with E as a proper subset of 1_{S^v} if $|X| > 1$.*

Proof. Note

(3.14)
$$(1_X, \infty, 1_X) \rhd (c_\infty, x, c_\infty) = (c_\infty, x, c_\infty) = (c_\infty, x, c_\infty) \lhd (1_X, \infty, 1_X)\,,$$
$$(1_X, \infty, 1_X) \rhd (1_X, \infty, 1_X) = (1_X, \infty, 1_X) = (1_X, \infty, 1_X) \lhd (1_X, \infty, 1_X)\,,$$

and

(3.15)
$$(c_\infty, x, c_\infty) \rhd (1_X, \infty, 1_X) = (c_\infty, \infty, c_\infty) = (1_X, \infty, 1_X) \lhd (c_\infty, x, c_\infty)$$

for any x in X.

(a): Follows by (3.13), (3.14), and (3.15).

(b): By (3.14), $(1_X, \infty, 1_X)$ is a bar unit. By (3.15), it is the only bar unit. Note $(1_X, \infty, 1_X) = (c_\infty, \infty, c_\infty)$ if and only if $|X| = 1$.

(c): By (a), the monoid $(S^v, \cdot, 1_{S^v})$ is trivial. Thus $1_{S^v} = S$. By (b), S properly contains E if $|X| > 1$. □

The example constructed in Proposition 3.21 deserves a special name, as it will play a significant role later in distinguishing grue properties from those of more general pregrues.

Definition 3.22. Let (X, ∞) be a pointed set, with $|X| > 1$.

(a) The pregrue $(S, E, \triangleleft, \triangleright)$ of Proposition 3.21 is described as an *infinity pregrue.*
(b) Elements of $X \smallsetminus \{\infty\}$ are said to be *finite.*

3.4. **Transformation pregrues of a set with operations.**

3.4.1. *Transformation pregrues.*

Proposition 3.23. *Let (X, T) consist of a set X with a set T of unary operations on X. Consider $T'' \subseteq X^X$ and $T' \subseteq {}^X X$. Define*

$$E_X = \{(1_X, x, 1_X) \mid x \in X\}$$

in the transformation disemigroup $T''XT'$. Then $(T''XT', E_X, \triangleleft, \triangleright)$ is a pregrue.

Proof. One has

$$\begin{aligned} &(g, x, h) \triangleleft (1_X, e, 1_X) = (g \cdot 1_X, x1_X, h \circ 1_X) = (g, x, h) \\ \text{and} \quad &(1_X, e, 1_X) \triangleright (g, x, h) = (1_X \cdot g, 1_X x, 1_X \circ h) = (g, x, h) \end{aligned}$$

for each element e of X and element (g, x, h) of $T''XT'$. □

Corollary 3.24. *In the context of Proposition 3.23, suppose that U, V are respective submonoids of T'', T'. Then $(UXV, E_X, \triangleleft, \triangleright)$ is a subpregrue of $(T''XT', E_X, \triangleleft, \triangleright)$.*

Corollary 3.25. *In the context of Proposition 3.23, suppose that G is a submonoid of $T'' \times T'$. Take XG as in* (3.9). *Then $(XG, E_X, \triangleleft, \triangleright)$ is a subpregrue of $(T''XT', \triangleleft, \triangleright)$. Furthermore, the projection π or*

$$\pi_{XG} \colon XG \to G; (g, x, h) \mapsto (g, h) \tag{3.16}$$

is a pregrue homomorphism from XG to the iterated monoid G.

Definition 3.26. Consider the context of Proposition 3.23.

(a) The pregrue $(T''XT', E_X, \triangleleft, \triangleright)$ is **the** *transformation pregrue* of the set (X, T) with unary operations from T.
(b) Let U be a submonoid of T'', and let V be a submonoid of T'. Then the pregrue $(UXV, E_X, \triangleleft, \triangleright)$ of Corollary 3.24 is described as **a** (*balanced*) *transformation pregrue* on (X, T).
(c) Suppose that G is a submonoid of $T'' \times T'$. Then the dimonoid $(XG, E_X, \triangleleft, \triangleright)$ appearing in Corollary 3.25 is described as **a** *transformation pregrue* on the set (X, T).

3.4.2. *Bar units.*

Proposition 3.27. *Let (X,T) be a set X equipped with a set T of unary operations. Then $E_X = \{1_X\} \times X \times \{1_X\}$ is the full set of bar units of each transformation pregrue XG on the structure (X,T).*

Proof. Consider an element (a,y,z) of XG. It is a bar unit iff

$$\forall\ (g,x,h) \in\ XG\,,\ (a,y,z) \rhd (g,x,h) = (ag,zx,zh) = (g,x,h) \tag{3.17}$$

and

$$\forall\ (g,x,h) \in\ T''XT'\,,\ (g,x,h) \lhd (a,y,z) = (ga,xa,hz) = (g,x,h)\,. \tag{3.18}$$

Since identity elements of monoids are unique, the first components of (3.17) and (3.18) are satisfied iff $a = 1_X$, while their third components are satisfied iff $z = 1_X$. Now, under the first condition, the middle component of (3.18) is satisfied, while the middle component of (3.17) is satisfied under the second condition. □

Consider a balanced transformation pregrue UXV on (X,T). By Corollary 3.18, the projection (3.16) is the undirected replication of $(UXV,\lhd,\rhd)$. The set E_X of bar units of UXV may then be understood conceptually as follows.

Proposition 3.28. *The set of bar units of the balanced pregrue UXV is the preimage, under the replication, of the identity element of the replica monoid.*

4. Cayley theorems

This section presents the Cayley theorems for disemigroups and pregrues.

4.1. **Cayley's Theorem for disemigroups.** Soppose that $(S,\lhd,\rhd)$ is a disemigroup. Define

$$R_\lhd(s)\colon S \to S; x \mapsto x \lhd s$$

and

$$L_\rhd(s)\colon S \to S; x \mapsto s \rhd x$$

for each element s of S. The associative laws for $(S,\lhd)$ and $(S,\rhd)$ imply that the maps

$$R_\lhd\colon S \to S^S \quad \text{and} \quad L_\rhd\colon S \to {}^S S$$

are semigroup homomorphisms. The respective images of these two homomorphisms are the subsemigroups $R_\lhd(S)$ of S^S and $L_\rhd(S)$ of ${}^S S$.

Lemma 4.1. *For all s_1, s_2 in S, the mutual commutativity*

$$L_{\triangleright}(s_1)R_{\triangleleft}(s_2) = R_{\triangleleft}(s_2)L_{\triangleright}(s_1)$$

holds.

Proof. For each element x of S, the relation

$$xL_{\triangleright}(s_1)R_{\triangleleft}(s_2) = (s_1 \triangleright x) \triangleleft s_2 = s_1 \triangleright (x \triangleleft s_2) = xR_{\triangleleft}(s_2)L_{\triangleright}(s_1)$$

holds by internal associativity. □

Consider the set S equipped with the set $R_{\triangleleft}(S)$ of transformations. By the antitone nature of the functions (3.5), $R_{\triangleleft}(S) \subseteq R_{\triangleleft}(S)''$, while $L_{\triangleright}(S) \subseteq R_{\triangleleft}(S)'$ by Lemma 4.1.

Theorem 4.2 (Disemigroup Cayley Theorem)**.** *Suppose that $(S, \triangleleft, \triangleright)$ is a disemigroup. There is an injective disemigroup homomorphism*

$$\beta \colon S \to R_{\triangleleft}(S)SL_{\triangleright}(S); s \mapsto (R_{\triangleleft}(s), s, L_{\triangleright}(s)) \tag{4.1}$$

from S to the transformation disemigroup $R_{\triangleleft}(S)SL_{\triangleright}(S)$ on $\big(S, R_{\triangleleft}(S)\big)$ as specified in Definition 3.13(b).

Proof. If S is empty, then $\beta = 1_{\varnothing}$. Otherwise, for elements s_1, s_2 of S, we have

$$\begin{aligned}(R_{\triangleleft}(s_1),&s_1, L_{\triangleright}(s_1)) \triangleright (R_{\triangleleft}(s_2), s_2, L_{\triangleright}(s_2)) \\ &= (R_{\triangleleft}(s_1) \cdot R_{\triangleleft}(s_2), L_{\triangleright}(s_1)s_2, L_{\triangleright}(s_1) \circ L_{\triangleright}(s_2)) \\ &= (R_{\triangleleft}(s_1 \triangleleft s_2), s_1 \triangleright s_2, L_{\triangleright}(s_1 \triangleright s_2)) \\ &= (R_{\triangleleft}(s_1 \triangleright s_2), s_1 \triangleright s_2, L_{\triangleright}(s_1 \triangleright s_2))\,,\end{aligned}$$

since $s_1 \triangleright (s_2 \triangleright s) = (s_1 \triangleright s_2) \triangleright s$ by the associativity of the right directional multiplication in S, while

$$(s \triangleleft s_1) \triangleleft s_2 = s \triangleleft (s_1 \triangleleft s_2) = s \triangleleft (s_1 \triangleright s_2)$$

for $s \in S$ by the associativity of the left directional multiplication in S and bar side irrelevance. Therefore, with respect to right directional multiplication, the embedding (4.1) is a homomorphism. With respect to left directional multiplication, the proof is similar. □

4.2. **Cayley's Theorem for pregrues.** The bar unit identities (2.2) may be cast in the following form.

Lemma 4.3. *Let $(S, E, \triangleleft, \triangleright)$ be a pregrue. Then*

$$R_{\triangleleft}(e) = L_{\triangleright}(e) = 1_S$$

for each bar unit $e \in E$.

For a pregrue $(S, E, \lhd, \rhd)$, set

$$R_\lhd = \{1_S\} \cup R_\lhd(S) \quad \text{and} \quad L_\rhd = \{1_S\} \cup L_\rhd(S) \tag{4.2}$$

to obtain submonoids $R_\lhd$ of S^S and $L_\lhd$ of ${}^S S$. Lemma 4.3 shows that the identity transformation on S already lies in $R_\lhd(S)$ and $L_\rhd(S)$ if the pregrue has bar units. On the other hand, the iterated semigroup of positive integers under addition, for example, has no bar units, but may still be construed as an impure pregrue $(\mathbb{Z}^+, \varnothing, +, +)$. In this case, $R_+ = L_+ = R_+(\mathbb{N})$.

Theorem 4.4 (Pregrue Cayley Theorem)**.** *Let $(S, E, \lhd, \rhd)$ be a pregrue. Then the disemigroup homomorphism*

$$\beta\colon S \to R_\lhd S L_\rhd; s \mapsto (R_\lhd(s), s, L_\rhd(s)) \tag{4.3}$$

obatined by codomain extension of (4.1) *becomes an injective pregrue homomorphism if its codomain is taken as the transformation pregrue $(R_\lhd S L_\rhd, E_S, \lhd, \rhd)$ on $\big(S, R_\lhd(S)\big)$.*

Proof. For a bar unit e of the pregrue $(S, E, \lhd, \rhd)$, note that

$$\beta\colon e \mapsto (R_\lhd(e), e, L_\rhd(e)) = (1_S, e, 1_S)$$

by Lemma 4.3. Thus $e\beta \in E_S$. □

The Pregrue Cayley Theorem 4.4 may be compared with the Cayley Theorem for dimonoids that appeared as [43, Th. 3].

Remark 4.5. If $E = \varnothing$, then (4.3) essentially reduces to (4.1) when one disregards any codomain extension. According to Remark 2.4(b), this case of Theorem 4.4 then subsumes Theorem 4.2.

4.3. **The adjoint map of a pure pregrue.** Suppose that $(S, E, \lhd, \rhd)$ is a pure pregrue. Consider the isomorphic copy S^β of S produced by the injective homomorphism (4.3) of the Cayley Theorem for pregrues. Consider the projection

$$S^\beta \to (R_\lhd, \cdot) \times (L_\rhd, \circ); \big(R_\lhd(s), s, L_\rhd(s)\big) \mapsto \big(R_\lhd(s), L_\rhd(s)\big)$$

with image monoid $\big\{\big(R_\lhd(s), L_\rhd(s)\big) \,\big|\, s \in S\big\}$.

Definition 4.6. Let $(S, E, \lhd, \rhd)$ be a pure pregrue. (Thus S and E are either both empty, or both nonempty.)

(a) The *adjoint map* of the empty pregrue $\varnothing$ is the monoidal unit $\pi\colon \varnothing \to \{1_\varnothing\}$ of the infinitesimal category $(\mathbf{Set}\mathcal{M}, \Box, \bot \to \top)$. It is convenient to write $\varnothing^\pi = \{(1_\varnothing, 1_\varnothing)\}$.

(b) If S is nonempty, its *adjoint map* is defined to be the surjective pregrue homomorphism

$$\pi\colon S \to \big\{\big(R_{\triangleleft}(s), L_{\triangleright}(s)\big) \,\big|\, s \in S\big\}; s \mapsto \big(R_{\triangleleft}(s), L_{\triangleright}(s)\big)\,.$$

Its codomain, a(n iterated) monoid, is written simply as S^{π}.

Remark 4.7. For an infinity pregrue $(S, E, \triangleleft, \triangleright)$ as in Definition 3.22, the adjoint map

$$S \to S^{\pi}\colon (1_X, \infty, 1_X) \mapsto (1_X, 1_X), (c_\infty, x, c_\infty) \mapsto (c_\infty, c_\infty)$$

is distinct from the undirected replication $1_S\colon S \to S$.

Now consider the transformation pregrue $(SS^{\pi}, E_S, \triangleleft, \triangleright)$ on the structure $(S, R_{\triangleleft})$ for any pure pregrue S, and the corestricted version

$$\beta\colon S \to SS^{\pi}; s \mapsto (R_{\triangleleft}(s), s, L_{\triangleright}(s))$$

of the Cayley embedding (4.3).

Proposition 4.8. *Let e be a bar unit of a pregrue $(S, E, \triangleleft, \triangleright)$.*

(a) *The right orbitoid of e is $e \triangleleft S = eR_{\triangleleft}$.*
(b) *The left orbitoid of e is $S \triangleright e = L_{\triangleright}e$.*

Proof. The respective equalities follow by

(4.4)
$$(e \triangleleft s)\beta = (1_S, e, 1_S) \triangleleft \big(R_{\triangleleft}(s), s, L_{\triangleright}(s)\big) = \big(R_{\triangleleft}(s), eR_{\triangleleft}(s), L_{\triangleright}(s)\big)$$

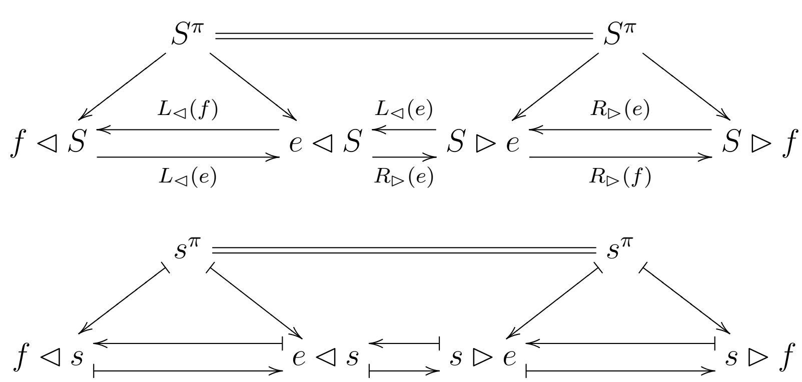
and

(4.5)
$$(s \triangleright e)\beta = \big(R_{\triangleleft}(s), s, L_{\triangleright}(s)\big) \triangleright (1_S, e, 1_S) = \big(R_{\triangleleft}(s), L_{\triangleright}(s)e, L_{\triangleright}(s)\big)$$

for $s \in S$, along with the injectivity of β. □

In a nonempty pure pregrue $(S, E, \triangleleft, \triangleright)$, the properties expressed in Proposition 2.8 may be adapted and strengthened to the following.

Proposition 4.9. *For $e, f \in E$, there is a commuting diagram*



of monoid isomorphisms.

Proof. With $e \in E$ fixed, and imaging $e \lhd S$ under β with the Cayley embedding, the map

$$S^\pi \to e \lhd S; s^\pi \mapsto e \lhd s$$

becomes the well-defined monoid isomorphism

$$S^\pi \to (e \lhd S)\beta; \big(R_\lhd(s), L_\rhd(s)\big) \mapsto \big(R_\lhd(s), eR_\lhd(s), L_\rhd(s)\big)$$

according to (4.4). The map $S^\pi \to S \rhd e; s^\pi \mapsto s \rhd e$ images to

$$S^\pi \to (S \rhd e)\beta; \big(R_\lhd(s), L_\rhd(s)\big) \mapsto \big(R_\lhd(s), L_\rhd(s)e, L_\rhd(s)\big)$$

in similar fashion, according to (4.5). □

Proposition 4.10. *Suppose that $(S, E, \lhd, \rhd)$ is a pure pregrue. Then S^π is the graph of a monoid isomorphism from $R_\lhd$ to $L_\rhd$. In particular, $R_\lhd \cong S^\pi \cong L_\rhd$.*

Proof. If S is empty, the result is true according to the convention of Definition 4.6(a). Now suppose that $s, t \in S$ and $e \in E$. Then

$$\begin{aligned} R_\lhd(s) = R_\lhd(t) \ &\Rightarrow \ e \lhd s = e \lhd t \ \overset{(2.8)}{\Rightarrow} \ s \rhd e = t \rhd e \\ &\Rightarrow \ (s \rhd e)\beta = (t \rhd e)\beta \\ &\Rightarrow \ \big(R_\lhd(s), eR_\lhd(s), L_\rhd(s)\big) = \big(R_\lhd(t), eR_\lhd(t), L_\rhd(t)\big) \\ &\Rightarrow \ L_\rhd(s) = L_\rhd(t)\,, \end{aligned}$$

and *vice versa* similarly. □

4.4. **The tetraset of a pure pregrue.** The adjoint map $S \xrightarrow{\pi} S^\pi$ of a pure pregrue $(S, E, \lhd, \rhd)$ is an object of the infinitesimal category $(\mathbf{Set}\mathcal{M}, \Box, \bot \to \top)$. This section reviews the structure of π in $\mathbf{Set}\mathcal{M}$. Standard symmetric monoidal category notation will be used for the Cartesian monoidal structure on **Set**. Thus the direct product of sets A and B is written as $A \otimes B$, and an ordered pair (a, b) inside it is written as $a \otimes b$. For example, the exchange of components is $\tau \colon a \otimes b \to b \otimes a$. The notation supports the immediate reinterpretation of the results of this section in other symmetric monoidal categories.

Full use will be made of the three isomorphic incarnations of S^π that are presented in Proposition 4.10: $R_\lhd$, $L_\rhd$, and the graph S^π of the isomorphism between them. Thus, with the usual multiplication $\nabla \colon S^\pi \otimes S^\pi \to S^\pi$ on the monoid S^π, the diagonal comultiplication may be implemented by $\Delta \colon S^\pi \to S^\pi \otimes S^\pi; s^\pi \mapsto L_\rhd(s) \otimes R_\lhd(s)$, as in the proof of Lemma 4.14 below. Note that the diagonal is the only counital comultiplication available in $(\mathbf{Set}, \times, \top)$ [39, Lemma 3.9]. In Lemma 4.11 below, associativity and unitality for (a) and (b) are easily verified, as are the dual coassociativity and counitality for (c) and (d). They also feature in the grue diagrams of §6.6.

In general, given a certain monoid M, the axioms for a left or right *co-M-set* are dual to the axioms for a left or right M-set. In particular, the coaction $\Delta\colon M \to M \otimes M$ and counit $\varepsilon\colon M \to \top$ make M a ("tautological") co-M-set.

Lemma 4.11. (a) *The action $\alpha_r\colon S \otimes S^\pi \to S; s_0 \otimes R_\lhd(s_1) \mapsto s_0 \lhd s_1$ endows S with the structure of a right S^π-set.*

(b) *The action $\alpha_l\colon S^\pi \otimes S \to S; L_\rhd(s_1) \otimes s_0 \mapsto s_1 \rhd s_0$ endows S with the structure of a left S^π-set.*

(c) *The coaction $\beta_r\colon S \to S \otimes S^\pi; s \mapsto s \otimes R_\lhd(s)$ endows S with the structure of a right co-S^π-set.*

(d) *The coaction $\beta_l\colon S \to S^\pi \otimes S; s \mapsto L_\rhd(s) \otimes s$ endows S with the structure of a left co-S^π-set.*

Lemma 4.12. (a) *The actions α_l and α_r commute:*

$$\begin{array}{ccc} S^\pi \otimes S \otimes S^\pi & \xrightarrow{1_{S^\pi} \otimes \alpha_r} & S^\pi \otimes S \\ {\scriptstyle \alpha_l \otimes 1_{S^\pi}} \downarrow & \searrow^{\alpha} & \downarrow {\scriptstyle \alpha_l} \\ S \otimes S^\pi & \xrightarrow[\alpha_r]{} & S\,. \end{array}$$

(b) *The coactions β_l and β_r commute:*

$$\begin{array}{ccc} S^\pi \otimes S \otimes S^\pi & \xleftarrow{1_{S^\pi} \otimes \beta_r} & S^\pi \otimes S \\ {\scriptstyle \beta_l \otimes 1_{S^\pi}} \uparrow & \nwarrow^{\beta} & \uparrow {\scriptstyle \beta_l} \\ S \otimes S^\pi & \xleftarrow[\beta_r]{} & S\,. \end{array}$$

Here, the diagonal is the Cayley embedding of S in the transformation pregrue $(R_\lhd S L_\rhd, E_S, \lhd, \rhd)$.

Proof. (a) For $s_{-1}, s_0, s_1 \in S$, one has

$$\begin{array}{ccccc} L_\rhd(s_{-1}) \otimes s_0 \otimes R_\lhd(s_1) & & \xmapsto{1_{S^\pi} \otimes \alpha_r} & & L_\rhd(s_{-1}) \otimes (s_0 \lhd s_1) \\ {\scriptstyle \alpha_l \otimes 1_{S^\pi}} \downarrow & & & & \downarrow {\scriptstyle \alpha_l} \\ (s_{-1} \rhd s_0) \otimes R_\lhd(s_1) & \xmapsto[\alpha_r]{} & (s_{-1} \rhd s_0) \lhd s_1 & = & s_{-1} \rhd (s_0 \lhd s_1) \end{array}$$

by the internal associativity.

(b) Note

$$\begin{array}{ccc}
L_{\triangleright}(s)\otimes s\otimes R_{\triangleleft}(s) & \xleftarrow{\;1_{S^\pi}\otimes\beta_r\;} & L_{\triangleright}(s)\otimes s \\
\uparrow{\scriptstyle \beta_l\otimes 1_{S^\pi}} & \nwarrow{\scriptstyle \beta} & \uparrow{\scriptstyle \beta_l} \\
s\otimes R_{\triangleleft}(s) & \xleftarrow[\;\beta_r\;]{} & s\,.
\end{array}$$

for $s \in S$. ☐

Definition 4.13. Suppose that $(S, E, \triangleleft, \triangleright)$ is a pure pregrue. Then the diagonal

$$\alpha\colon S^\pi \otimes S \otimes S^\pi \to S; L_{\triangleright}(s_{-1}) \otimes s_0 \otimes R_{\triangleleft}(s_1) \mapsto s_{-1} \triangleright s_0 \triangleleft s_1$$

of Lemma 4.12(a), well-defined by internal associativity, is called the (*Cayley*) *action* or *coembedding* of the pregrue.

Lemma 4.14. *The diagram*

$$\begin{array}{ccccc}
S\otimes S^\pi & \xrightarrow{\;\alpha_r\;} & S & \xrightarrow{\;\beta_l\;} & S^\pi\otimes S \\
\downarrow{\scriptstyle \beta_l\otimes\Delta} & & & & \uparrow{\scriptstyle \nabla\otimes\alpha_r} \\
(S^\pi\otimes S)\otimes(S^\pi\otimes S^\pi) & & \xrightarrow[\;1_{S^\pi}\otimes\tau\otimes 1_{S^\pi}\;]{} & & (S^\pi\otimes S^\pi)\otimes(S\otimes S^\pi)
\end{array}$$

commutes.

Proof. For $s_0, s_1 \in S$, note

$$\begin{array}{ccc}
s_0\otimes R_{\triangleleft}(s_1) & \longmapsto & s_0\triangleleft s_1 \\
\Big\downarrow & & \Big\downarrow \\
 & & L_{\triangleright}(s_0\triangleleft s_1)\otimes(s_0\triangleleft s_1) \\
 & & \Big\| \\
 & & L_{\triangleright}(s_0)\circ L_{\triangleright}(s_1)\otimes(s_0\triangleleft s_1) \\
\Big\downarrow & & \Big\uparrow \\
L_{\triangleright}(s_0)\otimes s_0\otimes L_{\triangleright}(s_1)\otimes R_{\triangleleft}(s_1) & \longmapsto & L_{\triangleright}(s_0)\otimes L_{\triangleright}(s_1)\otimes s_0\otimes R_{\triangleleft}(s_1)
\end{array}$$

using the barside irrelevance for the equality. ☐

Definition 4.15. (a) [11, §2] Let M be a monoid in a symmetric monoidal category $(\mathbf{V}, \otimes, I)$. Let S be an object of $\mathbf{V}$ equipped with:

(H1) An associative, unital right action $\alpha_r\colon S \otimes M \to S$;
(H1) An associative, unital left action $\alpha_l\colon M \otimes S \to S$;
(H2) A coassociative, counital right coaction $\beta_r\colon S \to S \otimes M$;
(H2) A coassociative, counital left coaction $\beta_l\colon S \to M \otimes S$,

such that:

(H1′) The actions α_l and α_r commute: $(\alpha_l \otimes 1_M)\alpha_r = (1_M \otimes \alpha_r)\alpha_l$;
(H2′) The coactions β_l and β_r commute: $\beta_r(\beta_l \otimes 1_M) = \beta_l(1_M \otimes \beta_r)$;
(H3) For $m, n \in \{l, r\}$, α_m is a homomorphism of coactions β_n. In detail, substituting M for S^π, the diagrams of §4.5.4 commute.

Then S is an *M-tetramodule.*

(b) An M-tetramodule in the Cartesian monoidal category $(\mathbf{Set}, \times, \top)$ will be called an *M-tetraset.*

Remark 4.16. (a) For $m = r$ and $n = l$, along with $M = S^\pi$, the condition (H3) is illustrated by the diagram of Lemma 4.14. For $m = n = r$, the instance of τ in the proof of the analogous result is merely the usual swap $R_\triangleleft(s_0) \otimes R_\triangleleft(s_1) \mapsto R_\triangleleft(s_1) \otimes R_\triangleleft(s_0)$.

(b) Using Heyneman-Sweedler notation, the four conditions comprising (H3) are written out as [36, (4.1)–(4.4)].

(c) With $M = S^\pi$, the condition (H1′) is illustrated by the diagram of Lemma 4.12(a), while the condition (H2′) is illustrated by the outer square of the diagram of Lemma 4.12(b).

(d) Tetramodules appear in the literature under many different names, such as "bidimodules" [6], "Hopf bimodules" [19], "two-sided two-cosided Hopf modules" [33],"4-modules" [35], "bicovariant bimodule" [42], etc.

Theorem 4.17. *Let $(S, E, \triangleleft, \triangleright)$ be a pure pregrue with adjoint map $\pi \colon S \to S^\pi$. Then S is an S^π-tetraset.*

Proof. Apply Lemmas 4.11–4.14. Remaining cases of (H3) are handled by analogy with Lemma 4.14. □

In infinitesimal linear categories, it transpires that tetramodules are equivalent to bimonoids [14, §2.2], [18, §5.1] (cf. §2.6). On the other hand, if one takes the infinitesimal category $\mathbf{Set}\mathcal{M}$ as defined in §2.6, Theorem 4.17 at best yields the following.

Corollary 4.18. *In the infinitesimal category $(\mathbf{Set}\mathcal{M}, \square, \bot \to \top)$, the adjoint map $\pi \colon S \to S^\pi$ gives an associative, coassociative, and unital bimagma.*

The obstruction to counitality arises because the tensor unit $\varnothing \hookrightarrow \top$ of $\mathbf{Set}\mathcal{M}$ cannot in general serve as the codomain of a counit ε with domain π.

4.5. **Pregrue diagrams.** This section collates disemigroup structural features of a pure pregrue $(S, E, \lhd, \rhd)$ in diagrammatic fashion. Just as disemigroups represent a semigroup structure that has been split into two halves, these diagrams split diagrams of a bisemigroup structure (H, ∇, Δ) into two halves. The left/right symmetry of disemigroups is displayed by reflection symmetry of the diagrams. In most cases, the symmetrically invariant part of the diagrams will have a special role to play, including a depiction of the Cayley Theorems.

The diagrams are formulated in a locally small symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, which for present purposes is interpreted as the Cartesian $(\mathbf{Set}, \otimes, \top)$. The notations of Definition 2.10(a) and §4.4 are used, along with the bisemigroup structure (S^π, ∇, Δ) of S^π in $(\mathbf{Set}, \otimes, \top)$.

The diagrams, along with their counterparts in §6.6, will lead to the specification of mathematical objects that are left/right splits of Hopf algebras (compare Table 1). The general picture (which we refrain from presenting in the interests of space and concreteness) would replace $S \xrightarrow{\pi} S^\pi$ with a general $\mathbf{V}$-morphism.

4.5.1. *The equivariance diagram.*

Proposition 4.19. *Suppose that $(S, E, \triangleleft, \triangleright)$ is a pure pregrue. Then the diagram*

$$\begin{array}{ccccc}
S^\pi\otimes S & \xrightarrow{1_{S^\pi}\otimes\pi} & S^\pi\otimes S^\pi & \xleftarrow{\pi\otimes 1_{S^\pi}} & S\otimes S^\pi \\
{\scriptstyle\alpha_l}\downarrow & & {\scriptstyle\nabla}\downarrow & & \downarrow{\scriptstyle\alpha_r} \\
S & \xrightarrow{\quad\pi\quad} & S^\pi & \xleftarrow{\quad\pi\quad} & S \\
{\scriptstyle\beta_l}\downarrow & & {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\beta_r} \\
S^\pi\otimes S & \xrightarrow[1_{S^\pi}\otimes\pi]{} & S^\pi\otimes S^\pi & \xleftarrow[\pi\otimes 1_{S^\pi}]{} & S\otimes S^\pi
\end{array}$$

commutes.

Proof. The commuting of the upper right square is verified by the chase

$$\begin{array}{ccccc}
R_\triangleleft(s_0)\otimes R_\triangleleft(s_1) & & \xleftarrow{\qquad\pi\otimes 1_{S^\pi}\qquad} & & s_0\otimes R_\triangleleft(s_1) \\
{\scriptstyle\nabla}\downarrow & & & & \downarrow{\scriptstyle\alpha_r} \\
R_\triangleleft(s_0)R_\triangleleft(s_1) & = & R_\triangleleft(s_0\triangleleft s_1) & \xleftarrow[\pi]{} & s_0\triangleleft s_1
\end{array}$$

for $s_0, s_1 \in S$, which incorporates the associativity of $(S, \triangleleft)$.

The commuting of the lower right square is verified by the chase

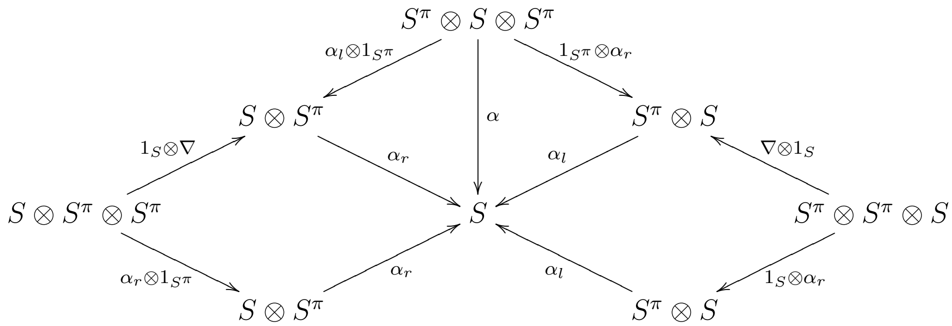
$$\begin{array}{ccc}
R_\triangleleft(s_0) & \xleftarrow{\qquad\pi\qquad} & s_0 \\
{\scriptstyle\nabla}\downarrow & & \downarrow{\scriptstyle\beta_r} \\
R_\triangleleft(s_0)\otimes R_\triangleleft(s_0) & \xleftarrow[\pi\otimes 1_{S^\pi}]{} & s_0\otimes R_\triangleleft(s_0)
\end{array}$$

for $s_0 \in S$. Commutativity of the left hand squares is verified in similar fashion. □

Reflection of the equivariance diagram in the vertical axis through its center interchanges "left" and "right". Bialgebra structure on S^π forms the unique invariant part of the diagram under this symmetry.

4.5.2. *The associativity diagram.*

Proposition 4.20. *Suppose that* $(S, E, \lhd, \rhd)$ *is a pure pregrue. Then by Lemmas 4.11(a),(b) and 4.12(a) the diagram*

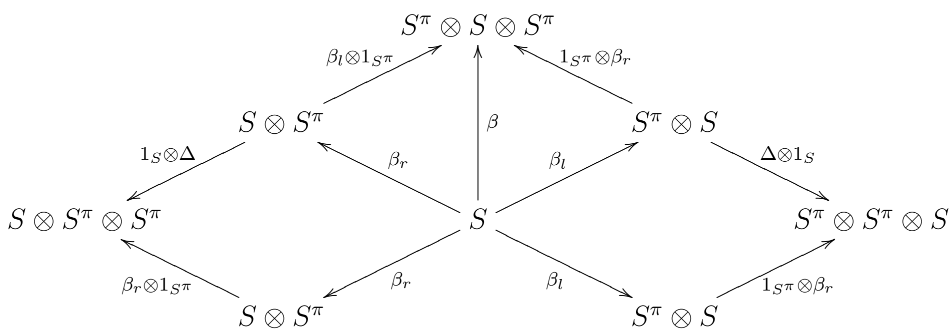


commutes.

Reflection of the associativity diagram in the vertical axis through its center interchanges "left" and "right". The unique invariant part of the diagram under this symmetry is the Cayley action of Definition 4.13.

4.5.3. *The coassociativity diagram.*

Proposition 4.21. *Suppose that* $(S, E, \lhd, \rhd)$ *is a pure pregrue. Then by Lemmas 4.11(c),(d) and 4.12(b), the diagram*

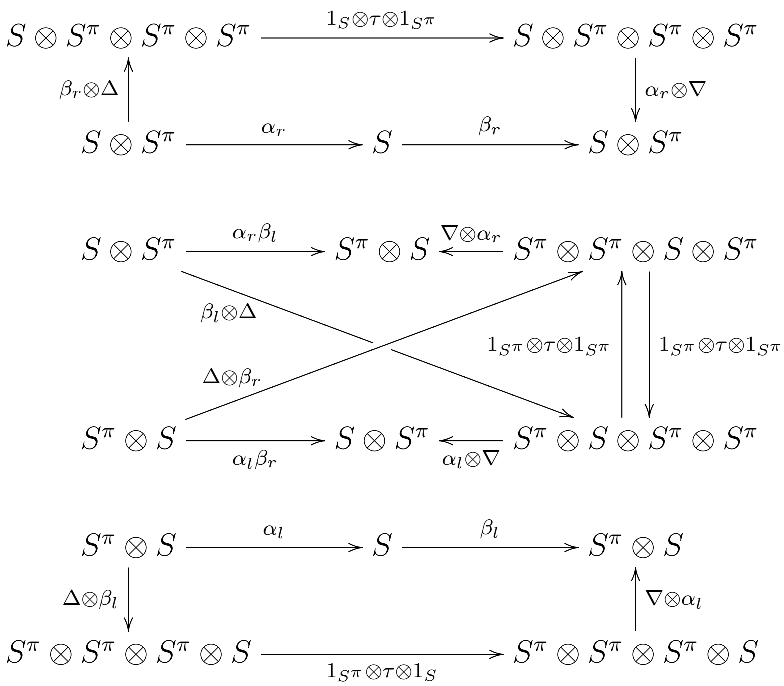


commutes.

Reflection of the coassociativity diagram in the vertical axis through its center interchanges "left" and "right". The unique invariant part of the diagram under this symmetry is the embedding (4.3) of the Cayley Theorem 4.4.

4.5.4. *The bimagma diagram.*

Proposition 4.22. *Suppose that $(S, E, \lhd, \rhd)$ is a pure pregrue. Then by Theorem 4.17 and Definition 4.15(H3), the diagram*



commutes.

Reflection of the bimagma diagram in the horizontal axis through its center interchanges "left" and "right". No part of the diagram is invariant.

5. Invertibility structure in pregrues

This section begins a study of inversion in pregrues.

5.1. **Left and right inverses.**

Definition 5.1. Let x be an element of a pregrue $(S, E, \lhd, \rhd)$.

(a) The element x is *left invertible* if there is an element x^l of S such that $x^l \lhd x$ is a bar unit. In this case, such elements x^l of S are known as *left inverses* of x.

(b) The element x is *right invertible* if there is an element x^r of S such that $x \rhd x^r$ is a bar unit. In this case, such elements x^r of S are known as *right inverses* of x.

(c) The element x is (*bilaterally*) *invertible* if there is an element y of S that is both a left and a right inverse of x. In this case, such elements y of S are known as *inverses* of x.

Remark 5.2. If $E = \emptyset$, there are no left, right, or bilaterally invertible elements.

Lemma 5.3. *Bar units of a pregrue* $(S, E, \triangleleft, \triangleright)$ *are invertible. More specifically, if* $e, f \in E$*, then* $f \triangleleft e = f$ *and* $e \triangleright f = f$*, so each bar unit* f *is an inverse for every bar unit* e*.*

Lemma 5.4. *Let* x *be an element of a pregrue* $(S, E, \triangleleft, \triangleright)$*.*

(a) *If* x *is right invertible, then* x^r *is left invertible, with* $x^{rl} = x$*.*
(b) *If* x *is left invertible, then* x^l *is right invertible, with* $x^{lr} = x$*.*
(c) *If* x *is invertible, with inverse* y*, then* y *is also invertible, with inverse* x*.*

Proof. For each element y of S, one has

$$y \triangleleft (x \triangleleft x^r) = y \triangleleft (x \triangleright x^r) = y = (x \triangleright x^r) \triangleright y = (x \triangleleft x^r) \triangleright y$$

by bar-side irrelevance, verifying (a). The proof of (b) is similar. Then (c) follows on setting $x^r = y$ in (a) and $x^l = y$ in (b). □

Lemma 5.5. *Let* x_1, x_2 *be elements of a pregrue* $(S, E, \triangleleft, \triangleright)$*.*

(a) *If* x_1, x_2 *are right invertible, then so are* $x_1 \triangleleft x_2$ *and* $x_1 \triangleright x_2$*, with* $(x_1 \triangleleft x_2)^r = (x_1 \triangleright x_2)^r = x_2^r \triangleright x_1^r$*.*
(b) *If* x_1, x_2 *are left invertible, then so are* $x_1 \triangleleft x_2$ *and* $x_1 \triangleright x_2$*, with* $(x_1 \triangleleft x_2)^l = (x_1 \triangleright x_2)^l = x_2^l \triangleleft x_1^l$*.*
(c) *If* x_1, x_2 *are invertible, then so are* $x_1 \triangleleft x_2$ *and* $x_1 \triangleright x_2$*.*

Proof. (a) For $i = 1, 2$, suppose that $x_i \triangleright x_i^r$ ia a bar unit. Then

$$\begin{aligned}(x_1 \triangleleft x_2) \triangleright (x_2^r \triangleright x_1^r) &= (x_1 \triangleright x_2) \triangleright (x_2^r \triangleright x_1^r) \\ &= x_1 \triangleright [(x_2 \triangleright x_2^r) \triangleright x_1^r] = x_1 \triangleright x_1^r \in E\,.\end{aligned}$$

The proof of (b) is similar, and (c) then follows. □

Lemma 5.6. *Let* s *be an invertible element of a pregrue* $(S, E, \triangleleft, \triangleright)$*, with inverse* t*. Then* $R_\triangleleft(s) \colon S \to S$ *and* $L_\triangleright(s) \colon S \to S$ *are invertible, with respective inverses* $R_\triangleleft(t)$ *and* $L_\triangleright(t)$*.*

Proof. Since $s \triangleright t$ is a bar unit, one has

$$s \triangleright (t \triangleright x) = (s \triangleright t) \triangleright x = x$$

and

$$x = x \triangleleft (s \triangleright t) = x \triangleleft (s \triangleleft t) = (x \triangleleft s) \triangleleft t$$

for all x in S. Thus $L_\triangleright(s) \circ L_\triangleright(t) = 1_X$ and $R_\triangleleft(s) \cdot R_\triangleleft(t) = 1_X$. Now by Lemma 5.4(c), t is invertible, with inverse s, so it follows that $L_\triangleright(t) \circ L_\triangleright(s) = 1_X$ and $R_\triangleleft(t) \cdot R_\triangleleft(s) = 1_X$. □

5.2. **Invertibility in transformation pregrues.** At the outset, it is worth recalling Proposition 3.27: the set $E_X = \{1_X\} \times X \times \{1_X\}$ is the full set of bar units of the transformation pregrue $(T''XT', E_X, \triangleleft, \triangleright)$ on a set (X, T) with unary operations from T. The following proposition uses the notation of (3.10).

Proposition 5.7. *Let $(XG, E_X, \triangleleft, \triangleright)$ be a transformation pregrue on a structure (X, T), for a submonoid G of $(T'', \cdot, 1_X) \times (T', \circ, 1_X)$. Write G^* for the groups of units of the monoid G. Consider an element (g, x, h) of XG.*

(a) *If the element (g, x, h) is right invertible, then $g \colon X \to X$ is injective, with a retract in U_G, and $h \colon X \to X$ is surjective, with a section in ${}_GV$.*
(b) *If the element (g, x, h) is left invertible, then $g \colon X \to X$ is surjective with a section in U_G, and $h \colon X \to X$ is injective with a retract in ${}_GV$.*
(c) *Suppose that the element (g, x, h) is invertible. Then $g \colon X \to X$ and $h \colon X \to X$ are bijective, with respective inverses g^{-1} in $U_G{}^*$ and h^{-1} in ${}_GV^*$. Thus $(g, h) \in G^*$, with inverse (g^{-1}, h^{-1}).*

Proof. Suppose that (g, x, h) has a right inverse (g^r, x^r, h^r) in XG, so that

$$(g, x, h) \triangleright (g^r, x^r, h^r) = (g \cdot g^r, hx^r, h \circ h^r) = (1_X, hx^r, 1_X)\,. \tag{5.1}$$

Then the element g of U_G is injective, with retract g^r in U. Similarly, the element h of ${}_GV$ is surjective, with section h^r in V. This verifies (a). The proof of (b) is dual.

Now if (g, x, h) is invertible, it is both left and right invertible. The bijectivity of g and h then follow by (a) and (b). □

Corollary 5.8. *Suppose that (g, x, h) is an invertible element of a transformation pregrue $(XG, E_X, \triangleleft, \triangleright)$ on a structure (X, T).*

(a) *The element (g, x, h) lies in XG^*.*
(b) *Each element of the set*

$$\{(g^{-1}, y, h^{-1}) \mid (g, h) \in G\}$$

may serve as an inverse of (g, x, h) in either transformation pregrue $(XG, E_X, \triangleleft, \triangleright)$ or $(XG^*, E_X, \triangleleft, \triangleright)$.

Definition 5.9. Let $(XG, E_X, \lhd, \rhd)$ be a transformation pregrue on a structure (X, T). Consider an invertible element (g, x, h) of XG. Define

$$I_e\colon XG \to XG; (g,x,h) \mapsto (g^{-1}, yg^{-1}, h^{-1}) \tag{5.2}$$

and

$$J_e\colon XG \to XG; (g,x,h) \mapsto (g^{-1}, h^{-1}y, h^{-1}) \tag{5.3}$$

for each element $e = (1_X, y, 1_X)$ of E_X.

Proposition 5.10. *Let $(XG, E_X, \lhd, \rhd)$ be a transformation pregrue on a structure (X, T). Consider an invertible element (g, x, h) of XG. Then in* (5.2),

$$I_e\colon (g,x,h) \mapsto (g^{-1}, yg^{-1}, h^{-1}) = (1_X, y, 1_X) \lhd (g^{-1}, x, h^{-1}),$$

for each element $e = (1_X, y, 1_X)$ of E_X, so that the image of (g, x, h) under I_e lies in the right orbitoid $XGL_{\lhd}(e)$. Similarly,

$$J_e\colon (g,x,h) \mapsto (g^{-1}, h^{-1}y, h^{-1}) = (g^{-1}, x, h^{-1}) \rhd (1_X, y, 1_X)$$

in (5.3) *for each element $e = (1_X, y, 1_X)$ of E_X, so that the image of (g, x, h) under J_e lies in the left orbitoid $XGR_{\rhd}(e)$.*

Remark 5.11. Proposition 5.10 motivates the orbitoid terminology of §2.2. The central component of $(g^{-1}, yg^{-1}, h^{-1})$ involves elements from the right orbit of y under T'', while the central component of $(g^{-1}, h^{-1}y, h^{-1})$ involves elements from the left orbit of y under T'.

Proposition 5.12. *Let $(XG, E_X, \lhd, \rhd)$ be a transformation pregrue on a structure (X, T). Consider an invertible element (g, x, h) of XG.*

(a) *The equations*

$$(g,x,h)^{I_e} \lhd (g,x,h) = (g^{-1}, yg^{-1}, h^{-1}) \lhd (g,x,h) = (1_X, y, 1_X)$$

and

$$(g,x,h) \rhd (g,x,h)^{J_e} = (g,x,h) \rhd (g^{-1}, h^{-1}y, h^{-1}) = (1_X, y, 1_X)$$

hold for each element $e = (1_X, y, 1_X)$ of E_X.

(b) *There is a unique bar unit e such that $(g, x, h) \in XGI_e$. Then*

$$I_{(1_X, xg^{-1}, 1_X)}\colon (g^{-1}, z, h^{-1}) \mapsto (g, x, h)$$

for any $z \in X$.

(c) *There is a unique bar unit e such that $(g, x, h) \in XGJ_e$. Then*

$$J_{(1_X, h^{-1}x, 1_X)}\colon (g^{-1}, z, h^{-1}) \mapsto (g, x, h)$$

for any $z \in X$.

Proof. Part (a) is an immediate calculation. For (b), note that

$$I_{(1_X,y,1_X)}\colon (g^{-1},z,h^{-1})\mapsto (g,yg,h)=(g,x,h) \tag{5.4}$$

implies $y=xg^{-1}$. The proof of (c) is similar. □

Proposition 5.12(b),(c) may be compared with [31, Lemma 3], which was presented in the context of grues.

6. Grues

In this section, the pregrue Cayley Theorem 4.4 is applied to the study of inversion in pregrues, ultimately providing motivation for the abstract definition of a grue that was presented in §2.3. As an analogue of the group of units of a monoid, Theorem 6.9 constructs a grue $\Im S$ of invertible elements of a pregrue S. Theorem 6.11 shows that each pure grue is its own grue of invertible elements. The Cayley Theorem for grues is obtained as an immediate corollary. To begin, we consider permutation grues on a set with unary operations.

6.1. **Permutation grues.** Let $(XG,E_X,\lhd,\rhd)$ be a transformation pregrue on a structure (X,T). Consider the group G^* of units of the monoid G.

Proposition 6.1. *The set XG^* comprises the invertible elements of the transformation pregrue $(XG,E_X,\lhd,\rhd)$.*

Proof. Suppose that (g,x,h) is an invertible element of $(XG,\lhd,\rhd)$. Then Corollary 5.8 shows that (g,x,h) lies in XG^*.

Conversely, suppose that (g,x,h) lies in XG^*. Thus $g^{-1}\in {U_G}^*$ and $h^{-1}\in {}_GV^*$. The element (g^{-1},x,h^{-1}) then serves as an inverse of (g,x,h) in XG. □

Definition 5.9 may be used to place an inversion structure on the subpregrue $(XG^*,E_X,\lhd,\rhd)$ of $(XG,E_X,\lhd,\rhd)$.

Theorem 6.2. *Let $(XG,E_X,\lhd,\rhd)$ be a transformation pregrue on a structure (X,T). Then the transfomation pregrue $(XG^*,E_X,\lhd,\rhd)$, which consists of the invertible elements of the transformation pregrue $(XG,E_X,\lhd,\rhd)$, forms a grue $(XG^*,E_X,\lhd,\rhd,I,J)$ equipped with the left inversion*

$$I\colon E_X\times XG^*\to XG^*; \big((1_X,y,1_X),(g,x,h)\big)\mapsto (g^{-1},yg^{-1},h^{-1})$$

and right inversion

$$J\colon XG^*\times E_X\to XG^*; \big((g,x,h),(1_X,y,1_X)\big)\mapsto (g^{-1},h^{-1}y,h^{-1})$$

given in parametrized form by (5.2) and (5.3).

Proof. The grue structure on the pregrue $(XG^*, E_X, \lhd, \rhd)$ is confirmed by Proposition 5.12(a). □

Definition 6.3. Let (X, T) be a set X equipped with a set T of unary operations.

(a) Write $(T'' \times T')^* = T''^* \times {}^*T'$ and $X(T'' \times T')^* = T''^* X^* T'$. Then the grue $(T''^* X^* T', E_X, \lhd, \rhd, I, J)$ presented by Theorem 6.2 is described as **the** *permutation grue* of, or the *symmetric grue* on, the set (X, T).

(b) Let U be a subgroup of T''^*, and let V be a subgroup of ${}^*T'$. Then the grue $(UXV, E_X, \lhd, \rhd, I, J)$, presented by Theorem 6.2 as $X(U \times V)$, is described as **a** (*balanced*) *permutation grue* on (X, T).

(c) Suppose that G is a subgroup of $T'' \times T'$. Then the grue $(XG, E_X, \lhd, \rhd, I, J)$ presented by Theorem 6.2 is described as **a** *transformation grue* on the structure (X, T).

Remark 6.4. (a) If T consists entirely of bijections, then $T \subseteq T''^*$, and ${}^*T'$ is the centralizer of T in $X!$. Two extreme cases are worthy of note:

(i) If T is empty, or if $T = \{1_X\}$, then ${}^*T' = X!$ and $T''^* = \{1_X\}$. Then the permutation grue $(T''^* X^* T', E_X, \lhd, \rhd, I, J)$ gives the left action of $\mathbf{Set}(X, X)^* = {!X}$ on X.

(ii) If $T = X!$, then ${}^*T' = \{1_X\}$ and $T''^* = X!$. In this case, the permutation grue $(T''^* X^* T', E_X, \lhd, \rhd, I, J)$ gives the right action of $\mathbf{Set}(X, X)^* = X!$ on X.

(b) The unique permutation grue on $(\varnothing, \varnothing)$ or $(\varnothing, 1_\varnothing)$ is the pure empty grue $(\varnothing, \varnothing, \lhd, \rhd, I^\varnothing, I^\varnothing)$ with vacuous inversions $I^\varnothing = 1_\varnothing$.

Example 6.5. (a) Let ${}_GX_H$ be a *bitorsor* in the sense of [5, Def'n. 1.1]: respective commuting left and right regular actions of groups G, H on a set X. Then GXH is a balanced permutation grue on (X, G).

(b) Let ${}_GX$ be a *torsor* in the sense of [4, Def'n. 2.1]: a set X with a regular left action of a group G. Then, in the notation of Remark 6.4, ${}_GX_{{}^*G'}$ is a bitorsor [5, §1.2].

(c) Each bitorsor ${}_GX_H$ is naturally isomorphic to ${}_GX_{{}^*G'}$ [5, §1.2].

6.2. **The grue of invertible elements.** Let $(S, E. \lhd, \rhd)$ be a pregrue. Recall the embedding

$$\beta\colon (S, E, \lhd, \rhd) \to (R_\lhd S L_\rhd, E_S, \lhd, \rhd); s \mapsto \big(R_\lhd(s), s, L_\rhd(s)\big) \tag{6.1}$$

that is provided by the pregrue Cayley Theorem 4.4. Then the image $(S\beta, E\beta, \lhd, \rhd)$ forms a subpregrue of $R_\lhd S L_\rhd$ that is isomorphic to

S via β. Consider the subpregrue $\big(R_\triangleleft{}^*SL_\triangleright{}^*, E_S, \triangleleft, \triangleright,\big)$ of $R_\triangleleft SL_\triangleright$ provided by Theorem 6.2. The intersection $S\beta \cap R_\triangleleft{}^*S^*L_\triangleright$ then forms a subpregrue

$$(S\beta \cap R_\triangleleft{}^*S^*L_\triangleright, E\beta, \triangleleft, \triangleright) \tag{6.2}$$

of $(R_\triangleleft SL_\triangleright, E_S, \triangleleft, \triangleright)$. Let $\Im S$ or $(\Im S, E, \triangleleft, \triangleright)$ be the preimage of (6.2) under the pregrue-homomorphic embedding β.

Example 6.6. Consider the iterated semigroup $S = (\mathbb{Z}^+, \varnothing, +, +)$ of positive integers under addition, as discussed in §4.2. In this case, (6.2) reduces to the pure empty pregrue $(\varnothing, \varnothing, \triangleleft, \triangleright)$.

Proposition 6.7. *Let s be an element of S. Then s is invertible in $(S, E, \triangleleft, \triangleright)$ if and only if $s\beta$ is an invertible element of $\Im S\beta$.*

Proof. If s is invertible, say with inverse t, then $s\beta = \big(R_\triangleleft(s), s, L_\triangleright(s)\big)$ lies in $R_\triangleleft{}^*S^*L_\triangleright$ by Lemma 5.6. Thus $s\beta$ is an invertible element of $\Im S\beta$, with inverse $t\beta$.

Conversely, suppose s is an element of S for which $s\beta$ is an invertible element of $\Im S\beta$. Let $t\beta = \big(R_\triangleleft(t), t, L_\triangleright(t)\big)$ be an inverse of $s\beta$ in $\Im S\beta$. Then $s\beta \triangleright t\beta = (1_S, s \triangleright t, 1_S) \in E\beta$. The central component of this inclusion implies that $s \triangleright t \in E$. Similarly, $t \triangleleft s \in E$. Thus s is invertible in $(S, E.\triangleleft, \triangleright)$. □

By Theorem 6.2, the set $R_\triangleleft{}^*S^*L_\triangleright$ is the carrier of a grue structure $\big(R_\triangleleft{}^*S^*L_\triangleright, E_S, \triangleleft, \triangleright, I, J\big)$ with

$$I\colon E_S \times R_\triangleleft{}^*S^*L_\triangleright \to R_\triangleleft{}^*S^*L_\triangleright;$$
$$\big((1_S, y, 1_S), (g, x, h)\big) \mapsto (g^{-1}, yg^{-1}, h^{-1})$$

and

$$J\colon R_\triangleleft{}^*S^*L_\triangleright \times E_S \to R_\triangleleft{}^*S^*L_\triangleright;$$
$$\big((g, x, h), (1_S, y, 1_S)\big) \mapsto (g^{-1}, h^{-1}y, h^{-1})$$

as the left and right inversions.

The inversions restrict respectively to

$$E\beta \times \Im S\beta \to R_\triangleleft{}^*S^*L_\triangleright;$$
$$\big((1_S, e, 1_S), (R_\triangleleft(s), s, L_\triangleright(s))\big) \mapsto (R_\triangleleft(s)^{-1}, eR_\triangleleft(s)^{-1}, L_\triangleright(s)^{-1}) \tag{6.3}$$

and

$$\Im S\beta \times E\beta \to R_\triangleleft{}^*S^*L_\triangleright;$$
$$\big((R_\triangleleft(s), s, L_\triangleright(s)), (1_S, e, 1_S)\big) \mapsto (R_\triangleleft(s)^{-1}, L_\triangleright(s)^{-1}e, L_\triangleright(s)^{-1}) \tag{6.4}$$

on $\Im S\beta$.

Lemma 6.8. *The images of* (6.3) *and* (6.4) *lie in* $\Im S\beta$.

Proof. Consider (6.3). Since s is invertible, it has some inverse t in S, with $R_\triangleleft(s)^{-1} = R_\triangleleft(t)$ and $L_\triangleright(s)^{-1} = L_\triangleright(t)$ by Lemma 5.6. Then:

$$\begin{aligned} R_\triangleleft(s)^{-1} &= R_\triangleleft(t) = R_\triangleleft(e)R_\triangleleft(t) = R_\triangleleft(e \triangleleft t); \\ eR_\triangleleft(s)^{-1} &= eR_\triangleleft(t) = e \triangleleft t; \text{ and} \\ L_\triangleright(s)^{-1} &= L_\triangleright(t) = L_\triangleright(e)L_\triangleright(t) = L_\triangleright(e \triangleleft t)\,, \end{aligned}$$

so

$$(R_\triangleleft(s)^{-1}, eR_\triangleleft(s)^{-1}, L_\triangleright(s)^{-1}) = \big(R_\triangleleft(e \triangleleft t), e \triangleleft t, L_\triangleright(e \triangleleft t)\big) = (e \triangleleft t)\beta$$

as required. The treatment of (6.4) is dual. □

Now (6.1) is injective. Thus, for each pair

$$\big((R_\triangleleft(s), s, L_\triangleright(s)), (1_S, e, 1_S)\big)$$

in (6.3) and (6.4), there are unique elements $sI_e^{\Im S}$ and $sJ_e^{\Im S}$ of S such that

$$\beta\colon sI_e^{\Im S} \mapsto (R_\triangleleft(s)^{-1}, eR_\triangleleft(s)^{-1}, L_\triangleright(s)^{-1}) \tag{6.5}$$

and

$$\beta\colon sJ_e^{\Im S} \mapsto (R_\triangleleft(s)^{-1}, L_\triangleright(s)^{-1}e, L_\triangleright(s)^{-1})\,. \tag{6.6}$$

Pulling (6.3) and (6.4) back along β, inversions

$$I^{\Im S}\colon E \times \Im S \to \Im S; (e, s) \mapsto sI_e^{\Im S} \tag{6.7}$$

and

$$J^{\Im S}\colon \Im S \times E \to \Im S; (s, e) \mapsto sJ_e^{\Im S} \tag{6.8}$$

are defined such that (6.5) and (6.6) hold for each pair $(s, e) \in \Im S \times E$. In summary, one has the following result.

Theorem 6.9. *If* $(S, E.\triangleleft, \triangleright)$ *is a pregrue, its invertible elements form a grue* $(\Im S, E, \triangleleft, \triangleright, I^{\Im S}, J^{\Im S})$.

Example 6.10. Consider the impure pregrue $(\mathbb{Z}^+, \varnothing, +, +)$ discussed in Example 6.6. Its (non-existent) invertible elements form the empty grue $(\varnothing, \varnothing, \triangleleft, \triangleright, 1_\varnothing, 1_\varnothing)$. By Remark 2.13(d),

$$\big(\mathbb{Z}^+, \varnothing, +, +, \varnothing \hookrightarrow \mathbb{Z}^+, \varnothing \hookrightarrow \mathbb{Z}^+\big) \quad \text{or} \quad \big(\mathbb{Z}^+, \varnothing, +, +, \varnothing, \varnothing\big)$$

is a grue in its own right.

6.3. **Cayley's theorem for grues.** Example 6.10 shows the necessity of the purity assumption in the following.

Theorem 6.11. *Suppose that* $(S, E, \lhd, \rhd, I, J)$ *is a pure grue.*

(a) *The equality*

$$(\Im S, E, \lhd, \rhd, I^{\Im S}, J^{\Im S}) = (S, E, \lhd, \rhd, I, J) \tag{6.9}$$

holds.

(b) *Let* e *be a bar unit. Then*

$$s^{I_e} = eR_\lhd(s)^{-1} \quad \text{and} \quad s^{J_e} = L_\rhd(s)^{-1}e \tag{6.10}$$

for each element s *of* S*.*

Proof. If $(S, E, \lhd, \rhd, I, J)$ is a pure grue, each element is invertible, so $S \subseteq \Im S \subseteq S$. The pregrue structures $(S, E, \lhd, \rhd)$ and $(\Im S, E, \lhd, \rhd)$ thus coincide. In particular, if $(S, E, \lhd, \rhd, I, J)$ is the empty grue, each side of (6.9) is the empty grue, with vacuous inversions.

Now consider an element s of S and an element e of E. Then

$$I_e^{\Im S}\beta \colon s \mapsto \big(R_\lhd(s)^{-1}, eR_\lhd(s)^{-1}, L_\rhd(s)^{-1}\big) \tag{6.11}$$

by (6.5). On the other hand, the equation $s^{I_e} \lhd s = e$ holds in $(S, E, \lhd, \rhd, I, J)$ by (2.9). Under β, this equation maps to

$$\begin{aligned} &\big(R_\lhd(s^{I_e}), s^{I_e}, L_\rhd(s^{I_e})\big) \lhd \big(R_\lhd(s), s, L_\rhd(s)\big) \\ &= \big(R_\lhd(s^{I_e}) \cdot R_\lhd(s), s^{I_e}R_\lhd(s), L_\rhd(s^{I_e}) \circ L_\rhd(s)\big) = (1_S, e, 1_S)\,, \end{aligned} \tag{6.12}$$

so

$$I_e\beta \colon s \mapsto \big(R_\lhd(s)^{-1}, eR_\lhd(s)^{-1}, L_\rhd(s)^{-1}\big) \tag{6.13}$$

by (6.12). Recalling the injectivity of β, a comparison of (6.11) with (6.13) shows that the inversions $I^{\Im S}$ and I coincide. Moreover, the first equation of (6.10) is confirmed. Dually, the maps $J^{\Im S}$ and J agree, completing the verification of the equality (6.9) and the second equation of (6.10). □

Lemma 6.12. *Suppose that* $(S, E, \lhd, \rhd, I, J)$ *is a pure grue. Then* $(R_\lhd S L_\rhd, E_S, \lhd, \rhd, I, J)$ *is a permutation grue on* $\big(S, R_\lhd(S)\big)$*.*

Proof. The case of the empty grue is described by Remark 6.4(c). If S and E are nonempty, so each element of S is invertible, then Lemma 5.6 shows that $R_\lhd(S) = R_\lhd = R_\lhd{}^*$ and $L_\rhd(S) = L_\rhd = {}^*L_\rhd$. In this case, Theorem 6.2 shows that the transformation pregrue $(R_\lhd S L_\rhd, E_S, \lhd, \rhd)$ on $\big(S, R_\lhd(S)\big)$ becomes a permutation grue $(R_\lhd S L_\rhd, E_S, \lhd, \rhd, I, J)$ on $\big(S, R_\lhd(S)\big)$. □

Theorem 6.13 (Grue Cayley Theorem)**.** *Suppose that* $(S, E, \lhd, \rhd, I, J)$ *is a pure grue. Then the injective pregrue homomorphism*

$$\beta\colon S \to R_\lhd S L_\rhd \,;\, s \mapsto (R_\lhd(s), s, L_\rhd(s)) \tag{6.14}$$

of Theorem 4.4 becomes an injective grue homomorphism when its codomain is taken to be the permutation grue $(R_\lhd S L_\rhd, E_S, \lhd, \rhd, I, J)$ *on* $\big(S, R_\lhd(S)\big)$ *provided by Lemma 6.12.*

The Grue Cayley Theorem 6.13 compares with the structure theorem for digroups that appeared as [12, Th. 4.8], and the Cayley Theorem for generalized digroups by Rodríguez-Neto, Salazar-Díaz, and Velásquez [32, Th. 13]..

Example 6.14. The pregrue reduct $(\mathbb{Z}^+, \varnothing, \lhd, \rhd)$ of the impure grue $(\mathbb{Z}^+, \varnothing, +, +, \varnothing, \varnothing)$ of Example 6.10 embeds (according to Theorem 4.4) into the transformation pregrue $\big(R_+(\mathbb{N})\mathbb{Z}^+ L_+(\mathbb{N}), E_{\mathbb{Z}^+}, \lhd, \rhd\big)$, which in turn has $\big(\{1_{\mathbb{Z}^+}\} \times \mathbb{Z}^+ \times \{1_{\mathbb{Z}^+}\}, E_{\mathbb{Z}^+}, \lhd, \rhd, I, J\big)$ as its grue of invertible elements, according to Theorem 6.2. This grue cannot accommodate the image of the embedding β. Such behavior demonstrates the need for the purity assumption in the Cayley Theorem for grues.

6.4. **The orbitoid groups of a pure nonempty grue.** For a pure grue $(S, E, \lhd, \rhd, I, J)$, the pregrue properties of §4.3 may be refined. By making use of the Cayley theorem for grues, the results here offer semantic counterparts to results that were derived in syntactic fashion in [31].

Proposition 6.15. *Let* $(S, E, \lhd, \rhd, I, J)$ *be a non-empty pure grue. Consider* $e \in E$.

(a) *The right orbitoid* $(e \lhd S, \lhd, e, I_e)$ *of* e *in* S *forms a group.*
(b) *The left orbitoid* $(S \rhd e, \rhd, e, J_e)$ *of* e *in* S *forms a group.*

Proof. Note $e\beta = (1_S, e, 1_S)$. Then for each element s of S, one has $s^\beta = \big(R_\lhd(s), s, L_\rhd(s)\big)$ and $sL_\lhd(e)\beta = (1_S, e, 1_S) \lhd \big(R_\lhd(s), s, L_\rhd(s)\big) = \big(R_\lhd(s), eR_\lhd(s), L_\rhd(s)\big)$. In other words,

$$(e \lhd S)\beta = \left\{ \big(R_\lhd(s), eR_\lhd(s), L_\rhd(s)\big) \,\middle|\, s \in S \right\}. \tag{6.15}$$

Then

$$\begin{aligned} &\big(R_\lhd(s), eR_\lhd(s), L_\rhd(s)\big) \lhd \big(R_\lhd(s), eR_\lhd(s), L_\rhd(s)\big) I_{(1_S, e, 1_S)} \\ &\qquad \overset{(5.2)}{=} \big(R_\lhd(s), eR_\lhd(s), L_\rhd(s)\big) \lhd \big(R_\lhd(s)^{-1}, eR_\lhd(s)^{-1}, L_\rhd(s)^{-1}\big) \\ &\qquad = (1_S, e, 1_S)\,, \end{aligned}$$

along with the usual inversion property of Proposition 5.12(a), shows that $(e \lhd S)\beta$ is a group. Since β is an injective homomorphism, it follows that $e \lhd S$ is a group. The proof of (b) is similar, noting the dual version

$$(S \rhd e)\beta = \left\{ \left(R_{\lhd}(s), L_{\rhd}(s)e, L_{\rhd}(s) \right) \,\middle|\, s \in S \right\}. \tag{6.16}$$

of (6.15). ☐

In a nonempty pure grue $(S, E, \lhd, \rhd, I, J)$, the pregrue properties from Proposition 4.9 may be strengthened by noting that the diagram is now located in the category of groups. Proposition 4.8 is strengthened as follows.

Corollary 6.16. *Let e be a bar unit of a grue $(S, E, \lhd, \rhd, I, J)$.*

(a) *The right orbitoid of e is $e \lhd S = eR_{\lhd} = SI_e$.*
(b) *The left orbitoid of e is $S \rhd e = L_{\rhd}e = SJ_e$.*

Proof. The respective right equalities follow by (6.13) and its dual, together with the injectivity of β. ☐

A further distinction with the general pregrue situation is highlighted by the following.

Proposition 6.17. *Let s be an element of a pure grue $(S, E, \lhd, \rhd, I, J)$.*

(a) *There is a unique right orbitoid containing s: the right orbitoid of $e = sR_{\lhd}(s)^{-1}$.*
(b) *There is a unique left orbitoid containing s: the left orbitoid of $f = L_{\rhd}(s)^{-1}s$.*
(c) *Suppose that the element s lies in the intersection of the right orbitoid of a bar unit e with the left orbitoid of a bar unit f. Then $L_{\rhd}(s)^{-1}eR_{\lhd}(s) = f$.*

Proof. Using the final orbitoid characterizations from Corollary 6.16, the existence and uniqueness claims made in (a) and (b) follow from the Cayley map and Proposition 5.12(b)(c). In the Cayley-embedded version of S, the latter proposition gives $e\beta = (1_S, sR_{\lhd}(s)^{-1}, 1_S)$ and $f\beta = (1_S, L_{\rhd}(s)^{-1}s, 1_S)$. For (c), one has $s = eR_{\lhd}(s) = L_{\rhd}(s)f$. ☐

Definition 6.18. Let $(S, E, \lhd, \rhd, I, J)$ be a pure grue. Define maps

$$\varepsilon_r \colon S \to E \quad \text{and} \quad \varepsilon_l \colon S \to E \tag{6.17}$$

where, for an element s of S, the bar units $s\varepsilon_r = e$ and $s\varepsilon_l = f$ are such that $s \in (e \lhd S)$ and $s \in (S \rhd f)$.

Example 6.19. (a) Let E be a set. In the grue $(E, E, \lhd, \rhd, \lhd, \rhd)$ of Example 2.11, $\varepsilon_l = 1_E = \varepsilon_r$.

(b) Let G be a group. In the grue $(G, \{e_G\}, \cdot, \cdot, I, J)$ of Example 2.12, ε_l and ε_r are the constant map e_G.

Proposition 6.17 stands in sharp contrast with Proposition 4.9 as applied to the infinity pregrues of Definition 3.22. There, the only orbitoid elements are the unique bar unit $(1_X, \infty, 1_X)$, and $(c_\infty, \infty, c_\infty)$; the latter is not invertible. Thus if x is finite, the pregrue element (c_∞, x, c_∞) does not lie in an orbitoid. In particular, the maps (6.17) are not available in general pregrues.

Theorem 6.20. *The replica congruence of a pure grue is the kernel of its adjoint map.*

Proof. If the grue is empty, the result is trivial. Thus, suppose now that $(S, E, \lhd, \rhd, I, J)$ is a nonempty pure grue. Since the codomain of the adjoint map $\pi\colon S \to S^\pi$ is an iterated group, Lemma 2.15 implies that the replica congruence υ is a subset of $\ker\pi$.

Conversely, Proposition 6.17(a) shows that

$$S = \{\, e \lhd s \mid e \in E,\ s \in S \,\} \, .$$

Note that for $e, f \in E$ and $s, t \in S$, one has

$$\begin{aligned}(e \lhd s, f \lhd t) \in \ker\pi \ &\Rightarrow\ R_\lhd(e \lhd s) = R_\lhd(f \lhd t)\\ &\Rightarrow\ R_\lhd(s) = R_\lhd(t) \ \Rightarrow\ f \lhd t = fR_\lhd(t) = fR_\lhd(s) = f \lhd s \, .\end{aligned}$$

In other words, each element of $\ker\pi$ has the form $(e \lhd s, f \lhd s)$ for some $e, f \in E$ and $s \in S$. By Lemma 2.16(a), one has $(e, f) \in \upsilon$. Then $(e \lhd s, f \lhd s) = (e, f) \lhd (s, s) \in \upsilon$, so that $\ker\pi$ is a subset of υ. □

6.5. **Bar unitors.**

Definition 6.21. Let $(S, E, \lhd, \rhd)$ be a pure pregrue.

(a) The composite

$$E \otimes S^\pi \xrightarrow{\eta_r \otimes 1_{S^\pi}} S \otimes S^\pi \xrightarrow{\alpha_r} S$$

is the *left bar-unitor* $\overline{\lambda}_S\colon E \otimes S^\pi \to S; e \otimes R_\lhd(s) \mapsto e \lhd s$ of $(S, E, \lhd, \rhd)$.

(b) The composite

$$S^\pi \otimes E \xrightarrow{1_{S^\pi} \otimes \eta_l} S^\pi \otimes S \xrightarrow{\alpha_l} S$$

is the *right bar-unitor* $\overline{\rho}_S\colon S^\pi \otimes E \to S; L_\rhd(s) \otimes e \mapsto s \rhd e$ of $(S, E, \lhd, \rhd)$.

Remark 6.22. If $(S, E, \lhd, \rhd)$ is an infinity pregrue as in Definition 3.22, there are elements of S that are not of the form $e \lhd s$ for any $e \in E$ and $s \in S$. Thus the left bar-unitor is not surjective in this case.

Proposition 6.23. *Let $(S, E, \lhd, \rhd, I, J)$ be a pure grue.*

(a) *The left bar-unitor has its dual*

$$S \xrightarrow{\beta_r} S \otimes S^\pi \xrightarrow{\varepsilon_r \otimes 1_{S^\pi}} E \otimes S^\pi$$

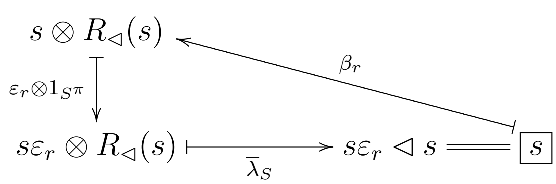
as a two-sided inverse.

(b) *The right bar-unitor has its dual*

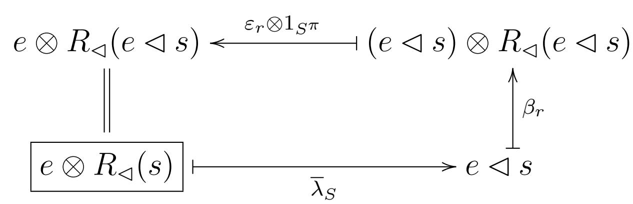
$$S \xrightarrow{\beta_l} S^\pi \otimes S \xrightarrow{1_{S^\pi} \otimes \varepsilon_l} S^\pi \otimes E$$

as a two-sided inverse.

Proof. (a): At the elementary level, one has the diagram



for $s \in S$, in which the equality follows by Proposition 6.15(a) and Definition 6.18. In the other direction, one has



for $e \otimes R_\triangleleft(s) \in E \otimes S^\pi$. Here, the equality holds by $R_\triangleleft(e \triangleleft s) = R_\triangleleft(e)R_\triangleleft(s) = R_\triangleleft(s)$. In each diagram, the starting point of the chase is boxed for easy location. The proof of (b) is similar. □

6.6. **Grue diagrams.** This section builds on the foundation of §4.5, pictorially formulating those additional structural features of a pure grue $(S, E, \triangleleft, \triangleright, I, J)$ that are not observed in pregrues. Overall, grue diagrams split the diagrams of a Hopf algebra structure $(H, \nabla, \Delta, \eta, \varepsilon, \nu)$ into two halves. The diagrams are again formulated in a locally small symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, currently interpreted as the Cartesian $(\mathbf{Set}, \otimes, \top)$. The left/right symmetry axis is noted for each diagram.

The notations of §4.5 are used, along with the Hopf algebra structure $(S^\pi, \nabla, \Delta, \eta, \varepsilon, \nu)$ of S^π in $(\mathbf{Set}, \otimes, \top)$. Table 1 summarizes how the grue notation splits the Hopf algebra notation. Replacing the left- and right-handed structure in the grue diagrams here for S recovers the usual Hopf algebra diagrams for S^π (as displayed, for example, in [39, §2.3]). Given that grue features oftem mix left and right, their assignment to a left or right side in the table is sometimes arbitrary.

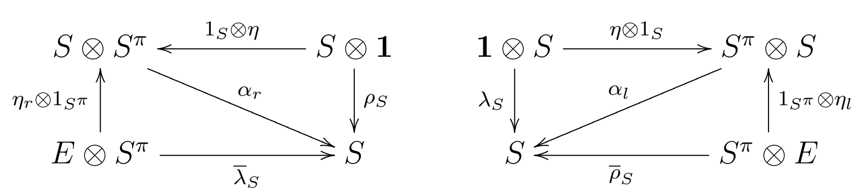
In relation to the issue discussed at the end of §4.4, the bimonoid diagrams of counitality and biunitality require grue structure for their splitting, since the split versions (6.17) of the bimonoid counit ε are only available in grues. The unitality diagram of Proposition 6.24 works for pregrues, since it does not require invertibility of the bar-unitors.

Left	Hopf	Right
α_l	∇	α_r
β_l	Δ	β_r
η_l	η	η_r
ε_l	ε	ε_r
J	ν	I
S	S^π	S
E	$\mathbf{1}$	E
$\overline{\lambda}_S$	λ_{S^π}	ρ_S
λ_S	ρ_{S^π}	$\overline{\rho}_S$

TABLE 1. Splitting Hopf structure to grue structure

6.6.1. *The unitality diagram.*

Proposition 6.24. *Suppose that* $(S, E, \lhd, \rhd)$ *is a pure pregrue. Then the diagram*



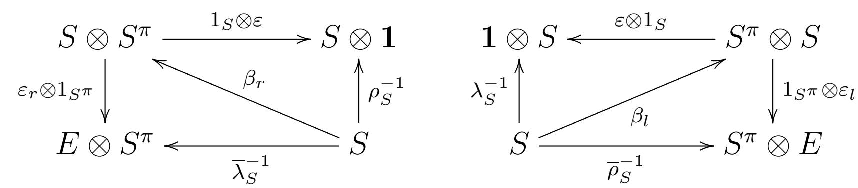
commutes.

Proof. The lower triangles just reflect the definitions of the bar-unitors, while the upper triangles form part of the tetraset structure. □

Reflection of the unitality diagram in the vertical axis through its center interchanges "left" and "right". No part of the diagram remains invariant.

6.6.2. *The counitality diagram.*

Proposition 6.25. *Suppose that* $(S, E, \lhd, \rhd, I, J)$ *is a pure grue. Then the diagram*



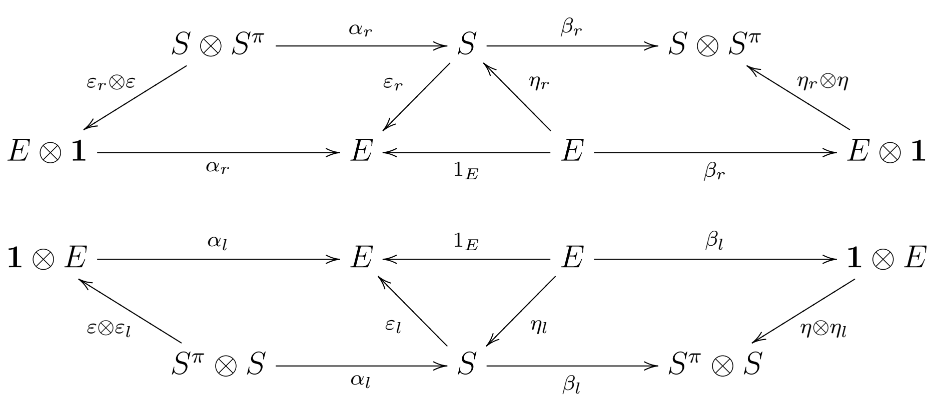
commutes.

Proof. The definition and commutativity of the lower triangles is given by Proposition 6.23, while the upper triangles are part of the tetraset structure of any pure pregrue. $\square$

Reflection of the counitality diagram in the vertical axis through its center interchanges "left" and "right". No part of the diagram remains invariant.

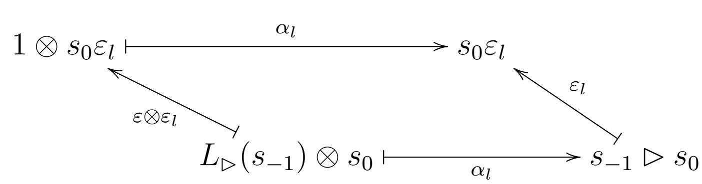
6.6.3. *The biunitality diagram.*

Proposition 6.26. *Suppose that* $(S, E, \lhd, \rhd, I, J)$ *is a pure grue. Then the diagram*



commutes.

Proof. The commuting of the lower left parallelogram traces to



at the elementary level. For the correctness of the action of ε_l on the right, note that $s_0 \in S \triangleright e$ implies $s_{-1} \triangleright s_0 \in S \triangleright e$ since $\triangleright$ is associative. Other verifications are similar or "trivial". $\square$

Reflection of the biunitality diagram in the horizontal axis through its center interchanges "left" and "right". Reflection in the vertical axis through its center dualizes.

6.6.4. *The antipode diagram.*

Proposition 6.27. *Suppose that* $(S, E, \triangleleft, \triangleright, I, J)$ *is a pure grue. Then the diagram*

$$\begin{array}{ccccccc} & & S\otimes S^\pi & \xrightarrow{1_S\otimes\nu} & S\otimes S^\pi & & \\ & \nearrow^{\beta_r} & & & & \searrow^{\alpha_r} & \\ S & & \xrightarrow{\quad\varepsilon_r\quad} & E & \xrightarrow{\quad\eta_r\quad} & & S\,. \end{array}$$

$$\begin{array}{ccccccc} S & & \xrightarrow{\quad\varepsilon_l\quad} & E & \xrightarrow{\quad\eta_l\quad} & & S\,. \\ & \searrow_{\beta_l} & & & & \nearrow_{\alpha_l} & \\ & & S^\pi\otimes S & \xrightarrow[\nu\otimes 1_S]{} & S^\pi\otimes S & & \end{array}$$

commutes.

Proof. Note

$$\begin{array}{ccccccc} & & s\otimes R_\triangleleft(s) & \stackrel{1_S\otimes\nu}{\longmapsto} & s\otimes R_\triangleleft(s)^{-1} & & \\ & \nearrow^{\beta_r} & & & & \searrow^{\alpha_r} & \\ s & & \stackrel{\varepsilon_r}{\longmapsto} & e & \stackrel{\eta_r}{\longmapsto} & & sR_\triangleleft(s)^{-1} \end{array}$$

$$\begin{array}{ccccccc} s & & \stackrel{\varepsilon_l}{\longmapsto} & f & \stackrel{\eta_l}{\longmapsto} & & L_\triangleright(s)^{-1}s\,. \\ & \searrow_{\beta_l} & & & & \nearrow_{\alpha_l} & \\ & & L_\triangleright(s)\otimes s & \stackrel{\nu\otimes 1_S}{\longmapsto} & L_\triangleright(s)^{-1}\otimes s & & \end{array}$$

at the elementary level, by Proposition 6.17 and Definition 6.18. $\square$

Reflection of the antipode diagram in the horizontal axis through its center interchanges "left" and "right". Reflection in the vertical axis through its center dualizes.

6.6.5. *Recovering grue structure from diagram data.*

Proposition 6.28. *Let $(S, E, \lhd, \rhd, I, J)$ be a pure grue. Then the diagram*

$$\begin{array}{ccc} S\otimes S^\pi & \xleftarrow{1_S\otimes\pi} & S\otimes S \\ {\scriptstyle \eta_r\otimes\pi\nu}\uparrow & \searrow^{\alpha_r} & \downarrow{\scriptstyle \lhd} \\ E\otimes S & \xrightarrow[I]{} & S \end{array} \qquad \begin{array}{ccc} S\otimes S & \xrightarrow{\pi\otimes 1_S} & S^\pi\otimes S \\ {\scriptstyle \rhd}\downarrow & \swarrow^{\alpha_l} & \uparrow{\scriptstyle \pi\nu\otimes\eta_l} \\ S & \xleftarrow[J]{} & S\otimes E \end{array}$$

commutes.

Proof. For the upper left triangle, note

$$s_0\otimes s_1 \xmapsto{1_S\otimes\pi} s_0\otimes R_\lhd(s_1) \xmapsto{\alpha_r} s_0 \lhd s_1$$

for $s_0, s_1 \in S$. For the lower left triangle, note

$$\begin{array}{ccc} e\otimes R_\lhd(s)^{-1} & & \\ {\scriptstyle \eta_r\otimes\pi\nu}\uparrow & \searrow^{\alpha_r} & \\ e\otimes s & \xmapsto[I]{} & e\lhd R_\lhd(s)^{-1} \end{array}$$

for $e \in E$ and $s \in S$, using (6.10). The right-hand side triangles are similar. $\square$

Reflection of the recovery diagram in the vertical axis through its center interchanges "left" and "right".

6.7. **Convolutions.** This section uses the notation and conventions of the previous section.

Definition 6.29. Suppose that $(S, E, \lhd, \rhd)$ is a pure pregrue. Take endomorphisms $f \in \mathbf{V}(S, S)$ and $g \in \mathbf{V}(S^\pi, S^\pi)$.

(a) The composite

$$S \xrightarrow{\beta_r} S\otimes S^\pi \xrightarrow{f\otimes g} S\otimes S^\pi \xrightarrow{\alpha_r} S$$

is the *right convolution* $f \dashv g \colon S \to S$.

(b) The composite

$$S \xrightarrow{\beta_l} S^\pi\otimes S \xrightarrow{g\otimes f} S^\pi\otimes S \xrightarrow{\alpha_l} S$$

is the *left convolution* $g \vdash f \colon S \to S$.

If S is undirected, so that $S = S^\pi$, then the right and left convolution reduce to the usual convolution

$$S^\pi \xrightarrow{\Delta} S^\pi\otimes S^\pi \xrightarrow{f\otimes g} S^\pi\otimes S^\pi \xrightarrow{\nabla} S^\pi$$

forming the multiplication of the convolution monoid $\big(\mathbf{V}(S^\pi, S^\pi), *, \varepsilon\eta\big)$ of the bimonoid $(S^\pi, \nabla, \Delta, \eta, \varepsilon)$ [25, Ch. 6].

Proposition 6.30. *Suppose that $(S, E, \triangleleft, \triangleright)$ is a pure pregrue. Then the endomorphism set $\mathbf{V}(S, S)$ is a $\big(\mathbf{V}(S^\pi, S^\pi), *, \varepsilon\eta\big)$-bimodule under the left and right convolution actions.*

Proof. Note

$$\begin{array}{ccc} s \otimes R_\triangleleft(s) & \xmapsto{f\otimes g} & s^f \otimes R_\triangleleft(s)^g \\ {\scriptstyle \beta_r}\uparrow & & \downarrow{\scriptstyle \alpha_r} \\ s & \xmapsto[f\dashv g]{} & s^f R_\triangleleft(s)^g \end{array} \quad \text{and} \quad \begin{array}{ccc} s \otimes R_\triangleleft(s) & \xmapsto{g_1\otimes g_2} & R_\triangleleft(s)^{g_1} \otimes R_\triangleleft(s)^{g_2} \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle \nabla} \\ R_\triangleleft(s) & \xmapsto[g_1 * g_2]{} & R_\triangleleft(s)^{g_1} R_\triangleleft(s)^{g_2} \end{array}$$

for $s \in S$, $f \in \mathbf{V}(S, S)$, and $g, g_1, g_2 \in \mathbf{V}(S^\pi, S^\pi)$, along with the mirror-image counterparts for the left action. Now, since $R_\triangleleft(s)\varepsilon y = 1 \in S^\pi$, the equations

$$f = f \dashv \varepsilon\eta \quad \text{and} \quad f = \varepsilon\eta \vdash f \tag{6.18}$$

hold for any endomorphism $f \colon S \to S$, so $V(S^\pi, S^\pi)$ acts unitally. Then, $s^{(f\dashv g_1)\dashv g_2} = s^{f\dashv g_1} R_\triangleleft(s)^{g_2}$

$$= s^f R_\triangleleft(s)^{g_1} R_\triangleleft(s)^{g_2} = s^f R_\triangleleft(s)^{g_1 * g_2} = s^{f\dashv(g_1 * g_2)}\,,$$

verifying the mixed associativity of the right action. The left action is treated similarly. Finally, $s^{(g_1\vdash f)\dashv g_2} = s^{g_1\vdash f} R_\triangleleft(s)^{g_2}$

$$= \big(L_\triangleright(s)^{g_1} s^f\big) R_\triangleleft(s)^{g_2} = L_\triangleright(s)^{g_1}\big(s^f R_\triangleleft(s)^{g_2}\big) = s^{g_1\vdash(f\dashv g_2)}$$

verifies the internal associativity, the commuting of the left and right actions. □

Definition 6.31. In the context of (6.18), the map $\varepsilon\eta \colon S^\pi \to S^\pi$ is called the *convolution action bar-unit.*

Proposition 6.32. *Let $(S, E, \triangleleft, \triangleright, I, J)$ be a pure grue.*

(a) *The unique solution $\xi \colon S \to S$ to the convolutional equation*

$$1_S = \xi \dashv 1_{S^\pi}$$

is $\xi = \varepsilon_r \eta_r$.

(b) *The unique solution $\xi \colon S \to S$ to the convolutional equation*

$$1_S = 1_{S^\pi} \vdash \xi$$

is $\xi = \varepsilon_l \eta_l$.

Proof. It will suffice to prove (a); then (b) is similar. We require

$$\begin{array}{ccccc} s \otimes R_{\lhd}(s) & \xmapsto{\ \xi \otimes 1_{S^\pi}\ } & & & s\xi \otimes R_{\lhd}(s) \\ \uparrow{\scriptstyle \beta_r} & & & & \downarrow{\scriptstyle \alpha_r} \\ s & \xmapsto[\ 1_S\ ]{} & s & = & s\xi \lhd s \end{array}$$

for each element $s \in S$. Proposition 6.17(a) and Definition 6.18 thus yield $s\xi = sR_{\lhd}(s)^{-1} = s\varepsilon_r\eta_r$. □

The following provides an alternative formulation for establishing the antipode diagram.

Corollary 6.33. *Suppose that* $(S, E, \lhd, \rhd, I, J)$ *is a pure grue. Then the convolution equations*

$$\nu \vdash 1_S = \varepsilon_l\eta_l \quad \text{and} \quad 1_S \dashv \nu = \varepsilon_r\eta_r$$

hold.

Proof. In $\big(\mathbf{V}(S^\pi, S^\pi), *, \varepsilon\eta\big)$, the antipode ν is a two-sided inverse for 1_{S^π}. Recalling the role of $\varepsilon\eta$ as a convolution bar-unit, one then has

$$1_S = 1_S \dashv \varepsilon\eta = 1_S \dashv \big(\nu * 1_{s^\pi}\big) = \big(1_S \dashv \nu\big) \dashv 1_{s^\pi}\,,$$

whence $1_S \dashv \nu = \varepsilon_r\eta_r$ by Proposition 6.32. The other equation is obtained in similar fashion. □

Definition 6.34. Let $(S, E, \lhd, \rhd, I, J)$ be a pure grue.

(a) The map

$$\sum_{e \in E} I_e|_{e \lhd S} \colon S \to S; s \mapsto sR_{\lhd}(s)^{-2}$$

is called the (*left*) *local inversion* of $(S, E, \lhd, \rhd, I, J)$.

(b) The map

$$\sum_{e \in E} J_e|_{S \rhd e} \colon S \to S; s \mapsto L_{\rhd}(s)^{-2}s$$

is called the (*right*) *local inversion* of $(S, E, \lhd, \rhd, I, J)$.

(c) In this context, the inversions $I_e \colon S \to S$ and $J_e \colon S \to S$, for each bar unit $e \in E$, are said to be *global*.

Remark 6.35. In Definition 6.34, the action of the inversions is given by (6.10) and Proposition 6.17.

Proposition 6.36. *Let $(S, E, \lhd, \rhd, I, J)$ be a pure grue. Then the convolution equations*

$$\sum_{e\in E} I_e|_{e\lhd S} = \varepsilon_r\eta_r \dashv \nu = 1_S \dashv \nu \dashv \nu$$

and

$$\sum_{e\in E} J_e|_{S\rhd e} = \nu \vdash \varepsilon_l\eta_l = \nu \vdash \nu \vdash 1_S$$

recover the local inversions from the structure that is encoded in the grue diagrams.

Proof. It will suffice to prove the first equation; the second is similar. Note

$$\begin{array}{ccccc} s\otimes R_\lhd(s) & \xmapsto{\varepsilon_r\eta_r\otimes\nu} & & & e\otimes R_\lhd(s)^{-1} \\ \beta_r \uparrow & & & & \downarrow \alpha_r \\ s & \xmapsto[\sum_{e\in E} I_e|_{e\lhd S}]{} & sI_e & = & eR_\lhd(s)^{-1} \end{array}$$

for each $e \in E$ and $s \in (e \lhd S)$, the equality following by (6.10). □

7. Conclusion and future work

The philosophy underlying this paper has been an insistence on a complete respect for the symmetries of the objects under study. The concept of a grue as a replacement for digroups has been founded on the behavior of the more primitive concepts of disemigroup and pregrue, using a commuting pair of left and right actions to maintain the full left/right symmetry. In both pregrues and grues, all the bar units receive equal treatment. Working in consort, they support the orbitoid structure which is already present to a certain extent in pregrues. The orbitoid structure appears in full strength with grues, where it provides the bar-unitors that are needed to complete the split of the unitality and counitality diagrams of Hopf algebras.

7.1. **Actions on sets.** Concepts of permutation actions for digroups and generalized digroups have recently been presented [8, 27]. Now, transformation pregrues and permutation grues, as introduced in §3.4 and §6.1 above, may be taken as the codomains of pregrue and grue homomomorphisms to provide suitable concepts of pregrue and grue actions on sets. The Cayley Theorems 4.4 and 6.13 for pregrues and grues are prototypical examples of such actions, on the underlying sets of the domains of their embeddings β. The set action homomorphisms should appear within a diagrammatic setting, closely related to Hopf algebra actions, in an analogue to the way that the Cayley embeddings have here been identified as the symmetric part of the coassociativity diagrams for pregrues and grues.

7.2. **Cohomology of pure grues.** Loday's original motivation for the introduction of his algebras came from homological algebra [16]. Except for vestiges in the discussion of infinitesimal categories in §2.6, such considerations have been kept absent from this paper. In connection with the coquecigrue problem, Mostovoy refers to the

> "desire to find a homology theory for groups that would parallel the Leibniz homology for Lie algebras"

[22]. As a possible framework for a cohomology theory of pure grues (of course including groups according to Example 2.12), one might first homogenize them using the techniques of [23], and then consider monadic (or "triple") cohomology for the homogenized algebras [3].

7.3. **Clones.** Definition 3.26(a) established the transformation pregrue $T''XT'$ of a set (X, T) with a set T of unary operations. A particular basis for a transformation pregrue would be an *algebra* (X, Θ) of *type* $\tau\colon \Theta \to \mathbb{N}$ in the sense of universal algebra, as summarized concisely in [37, App. B], say. The elements of Θ are described as the *basic operators* of the algebra (X, Θ). Each basic operator ω determines a *basic operation*

$$\omega\colon X^{\omega\tau} \to X; (x_1, \dots, x_{\omega\tau}) \mapsto x_1 \dots x_{\omega\tau}\omega \tag{7.1}$$

of (X, Θ). Then for each basic operator with $0 < \omega\tau$, for each index choice $1 \le i \le \omega\tau$, and for each $(\omega\tau - 1)$-tuple

$$\mathbf{a} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{\omega\tau}) \in X^{\omega\tau - 1},$$

there is a curried version

$$\omega^i_{\mathbf{a}}\colon X \to X; (a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{\omega\tau}) \mapsto a_1 \dots a_{i-1} x a_{i+1} \dots a_{\omega\tau}\omega$$

of the basic operation (7.1) known as a *translation* of the universal algebra (X, Θ). The set T_Θ of all translations then constitutes a set of unary operations on X, and one may consider the transformation pregrue $T''_\Theta X T'_\Theta$ of (X, T_Θ), the *translation pregrue* of the universal algebra (X, Θ).

The commutation relation (3.6) between unary operations θ, φ on a set X extends naturally to the case of general operations $\theta\colon X^m \to X$ and $\varphi\colon X^n \to X$ for natural numbers m, n. The extended relation, described as *mutual homomorphism*, says that θ is a homomorphism $\theta\colon (X, \varphi)^m \to (X, \varphi)$ of algebras equipped with a single operator φ, or equivalently that $\varphi\colon (X, \theta)^n \to (X, \theta)$ is a homomorphism. Now, for a set X, consider the disjoint union Υ of the sets $\mathbf{Set}(X^n, X)$ for all natural numbers n, and the corresponding coproduct $\upsilon\colon \Upsilon \to \mathbb{N}$ of the functions $\mathbf{Set}(X^n, X) \to \mathbb{N}; \omega \mapsto n$. On applying the polarity notation of §3.2.1 to the relation of mutual homomorphism, each subset Σ of Υ yields a subset Σ' of Υ that is not only "closed" in the sense of Galois theory, but also closed under functional composition when interpreted as a set of operations on the set X. These closed sets of operations and their cousins (compare [2], say) are known as *clones*, a name attributed to P. Hall that predates Lazard's *analyseurs* and May's *operads*.

Starting from a universal algebra (X, Θ), it then becomes natural to move beyond its transformation pregrue $T''_\Theta X T'_\Theta$ with the commuting monoid actions T''_Θ, T'_Θ, instead considering the mutually homomorphic algebra structures Θ'', Θ' on X. These structures, interpreted in various monoidal categories, will provide a more general version of Loday's algebras.

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