What is a CSP?

Informally, a **C**onstraint **S**atisfaction **P**roblem consists of

- a list of variables ranging over a finite domain and
- a set of constraints on those variables.

Problem: can we assign values to all the variables so that all of the constraints are satisfied?

Examples

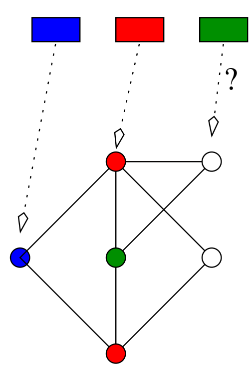
A system of linear equations is a CSP

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Also, a system of nonlinear equations is a CSP

$$\begin{array}{llll}
a_{11}x_1^2x_3 + a_{12}x_2x_3x_7 & + \cdots + a_{1n}x_4x_n^3 & = b_1 \\
a_{21}x_2x_5 + a_{22}x_2 & + \cdots + a_{2n}x_4^3 & = b_2 \\
& \vdots & \\
a_{m1}x_3x_5x_8 + a_{m2}x_2 & + \cdots + a_{mn}x_n & = b_m
\end{array}$$

For a fixed k, determining whether a graph is k-colorable is a CSP



Algorithms

There is an efficient algorithm (Gaussian elimination) for solving any linear system. That is

There is an algorithm that accepts as **input** a linear system and decides whether that system has a solution.

The running time of the algorithm is bounded above by $f(s)$ where f is a polynomial and s is the size of the system.

The particular system is an **instance** of the **problem** LINEAR SYSTEM

Similarly

There is an algorithm that accepts as input a graph and decides whether the graph is 2-colorable.

The running time is bounded by $f(s)$ where f is a polynomial and s is the size of the graph.

The graph is an instance of the problem 2-COLORABILITY.

We say these algorithms run in *polynomial time.*

No polynomial-time algorithm is known for either NONLINEAR SYSTEM or 3-COLORABILITY.

However, any candidate solution to either of these problems can be checked in polynomial-time.

Thus these problems are solvable in *nondeterministic polynomial time.*

Let X and Y be two problems. We write $X \leq_{\mathrm{p}} Y$ to indicate that Y is at least as hard as X.

Somewhat more precisely: any algorithm for solving Y can be transformed into an algorithm for X without drastically increasing its running time.

It is possible for $X \leq_{\mathrm{p}} Y \leq_{\mathrm{p}} X$. In that case, write $X \equiv_{\mathrm{p}} Y$.

$\mathbb{P}$ is the class of all problems solvable in polynomial time. Its members are called *tractable.*

$\mathbb{NP}$ is the class of problems solvable in nondeterministic polynomial time.

- $\mathbb{P} \subseteq \mathbb{NP}$
- Both $\mathbb{P}$ and $\mathbb{NP}$ are downsets, i.e., $Y \in \mathbb{P}$ & $X \leq_{\mathrm{p}} Y \implies X \in \mathbb{P}$

The maximal members of $\mathbb{NP}$ are called $\mathbb{NP}$-*complete.*

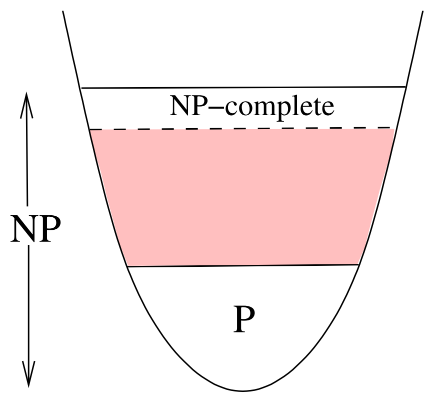
Both 3-COLORABILITY and NONLINEAR SYSTEM are known to be $\mathbb{NP}$-complete.

\$1,000,000 question: $\mathbb{P} \stackrel{?}{=} \mathbb{NP}$**.**

If $\mathbb{P} = \mathbb{NP}$ then all of the above distinctions go away. Almost every problem that mathematicians actually care about can be solved efficiently. Just build bigger computers.

In particular, this talk becomes pointless. So assume $\mathbb{P} \neq \mathbb{NP}$.

Theorem 1 (Ladner, 1975 [9])**.** *If* $\mathbb{P} \neq \mathbb{NP}$ *then there are problems in* $\mathbb{NP} - \mathbb{P}$ *that are not* $\mathbb{NP}$-*complete.*



If $\mathbb{P} \neq \mathbb{NP}$ then the pink area is nonempty.

Formal Definition of CSP

Let D be a set, n a positive integer An *n-ary relation on D* is a subset of D^n

$\mathrm{Rel}_n(D)$ denotes the set of all n-ary relations on D

$\mathrm{Rel}(D) = \bigcup_{n>0} \mathrm{Rel}_n(D)$

Let D be a finite set and $\Delta \subseteq \mathrm{Rel}(D)$

$\mathrm{CSP}(\langle D, \Delta \rangle)$ is the problem: **instance:** A finite set $V = \{ v_1, \dots, v_n \}$ of *variables* and a finite set $\{ C_1, \dots, C_m \}$ of *constraints*

Each constraint C_i is a pair $(\langle x_{i1}, \dots, x_{i\,p_i} \rangle, \delta_i)$ in which $x_{i1}, \dots, x_{i\,p_i} \in V$ and $\delta_i \in \Delta$

Question: Does there exist a mapping $f \colon V \to D$ such that for all $i \leq m$, $\langle f(x_{i1}), \dots, f(x_{i\,p}) \rangle \in \delta_i$?

$\mathrm{CSP}(\langle D, \Delta \rangle)$ always lies in $\mathbb{NP}$.

$\mathrm{CSP}\langle D, \Delta \rangle$ is *finitary* if Δ is finite.

Example: Linear Equations over $\mathbb{F}_2$

$D = \{0,1\}$ Δ consists of all relations

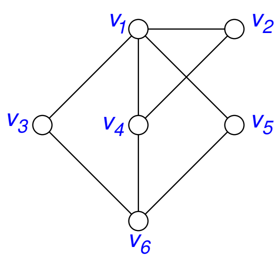
$$\delta^b_{n,\mathbf{a}} = \{\, \langle x_1,\dots,x_n\rangle \in D^n : a_1x_1 + \cdots a_nx_n = b\}$$

Then $\mathrm{CSP}(\langle D,\Delta\rangle)$ is the linear equations problem

Example: 3-colorability

$D = \{r, g, b\}$, $\Delta = \{\kappa_3\}$ $\kappa_3 = \{\, (x,y) \in D : x \neq y\,\}$

Then $\mathrm{CSP}(\langle D,\Delta\rangle)$ is the 3-colorability problem



$V = \{v_1,\dots,v_6\}$ $\langle v_1,v_2\rangle \in \kappa$ $\langle v_1,v_3\rangle \in \kappa$ $\langle v_1,v_4\rangle \in \kappa$ $\langle v_2,v_4\rangle \in \kappa$ $\vdots$ $\langle v_5,v_6\rangle \in \kappa$

Schaefer's Dichotomy

Theorem 2 (Schaefer, 1978 [12])**.** *Let $D = \{0,1\}$. There are six families $\Delta_1,\dots,\Delta_6$ such that*

$$\mathrm{CSP}(\langle D,\Delta\rangle) \in \mathbb{P} \iff \Delta \subseteq \Delta_i, \textit{some } i \leq 6$$

Otherwise $\mathrm{CSP}(\langle D,\Delta\rangle)$ *is $\mathbb{NP}$-complete.*

Two Motivating Questions

1. *Dichotomy Conjecture* Every $\mathrm{CSP}(\langle D,\Delta\rangle)$ either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.
2. *Tractability Problem* Characterize those CSPs that lie in $\mathbb{P}$.

Graph Homomorphisms

Well-known fact: A graph G is 3-colorable iff there is a graph homomorphism from G to K_3.

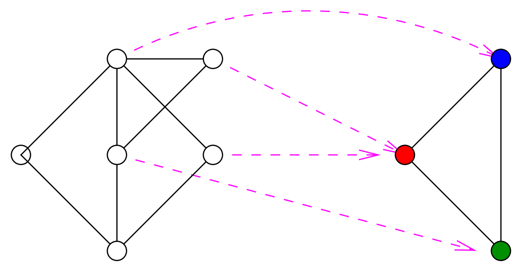
Definition 3. Let $\langle G,E\rangle$ and $\langle H,F\rangle$ be (di)graphs. A *homomorphism* is a function $f\colon G \to H$ such that

$$(x,y) \in E \implies \big(f(x),\, f(y)\big) \in F.$$

Remark. This definition makes sense

- for both undirected and directed graphs;
- for graphs with all/some/no loops.

3-Coloring G as a homomorphism to K_3



Definition 4. Let H be a digraph. $\mathrm{CSP}(H)$ is the problem: *Instance:* A digraph G *Question:* Is there a homomorphism from G to H?

Note that $\mathrm{CSP}(H)$ is a constraint satisfaction problem. (The "H-coloring" problem.)

Theorem 5 (Feder and Vardi, 1998 [5])**.** *For every finitary CSP X there is a digraph H such that $X \equiv_{\mathrm{p}} \mathrm{CSP}(H)$.*

The CSP for Graphs

Definition 6. Let H be a digraph.

1. An induced subgraph H' is a *retract* of H if there is $r\colon H \to H'$ with $r \circ r = r$.
2. A *core* of H is a minimal retract.

Easy fact: any two cores of H are isomorphic.

H is called a *core* if $H = \mathrm{core}(H)$.

Lemma 7. *For any digraph H, $\mathrm{CSP}(H) \equiv_{\mathrm{p}} \mathrm{CSP}(\mathrm{core}(H))$.*

Theorem 8 (Hell & Nešetřil, 1990 [6])**.** *Let H be an undirected, loopless graph. Then $\mathrm{CSP}(H)$ lies in $\mathbb{P}$ if and only if H is bipartite, otherwise it is $\mathbb{NP}$-complete.*

Corollary 9. *The dichotomy conjecture holds for undirected graphs.*

Theorem 10 (Barto, Kozik and Niven, 2008 [1])**.** *Let H be a smooth digraph. If each component of $\mathrm{core}(H)$ is a circle, then $\mathrm{CSP}(H) \in \mathbb{P}$. Otherwise $\mathrm{CSP}(H)$ is $\mathbb{NP}$-complete.*

H is *smooth* if each vertex has an incoming and an outgoing edge.

A *circle* is a directed cycle with no chords.

Corollary 11. *The dichotomy conjecture holds for smooth digraphs.*

Polymorphisms

Definition 12. Let $\delta \in \mathrm{Rel}_k(D)$ and $f\colon D^n \to D$. We say f *preserves* δ if

$$(a_{11}, \ldots, a_{1k}), \ldots, (a_{n1}, \ldots, a_{nk}) \in \delta \implies \\ \big(f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1k}, \ldots, a_{nk})\big) \in \delta$$

f is an *n-ary operation* on D.

$$\begin{array}{cccccc} a_{11} & a_{12} & \ldots & a_{1k} & \in & \delta \\ a_{21} & a_{22} & \ldots & a_{2k} & \in & \delta \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} & \in & \delta \\ \downarrow_f & \downarrow_f & & \downarrow_f & & \\ \star & \star & \ldots & \star & \in & \delta \end{array}$$

Definition 13. Let Δ be a set of relations on D. Then $\mathrm{Pol}(\Delta)$ denotes the set of all operations preserving all members of Δ. These are the *polymorphisms* of Δ.

Let F be a set of operations on D. Then $\mathrm{Inv}(F)$ denotes the set of all relations preserved by all operations in F.

Theorem 14. *Let* $\Gamma, \Delta \subseteq \mathrm{Rel}(D)$. *Then*

$$\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta) \implies \mathrm{CSP}(\Delta) \leq_{\mathrm{p}} \mathrm{CSP}(\Gamma).$$

Note: Δ is a core iff $\mathrm{Pol}_1(\Delta)$ is a set of permutations of D. $\mathrm{Pol}_1(\Delta)$ is the set of unary polymorphisms

One can go back and forth between relational and algebraic structures

$$\begin{array}{ccc} \textbf{Relational} & & \textbf{Algebraic} \\ \langle D, \Delta \rangle & \longrightarrow & \langle D, \mathrm{Pol}(\Delta) \rangle \\ \langle D, \mathrm{Inv}(F) \rangle & \longleftarrow & \langle D, F \rangle \end{array}$$

$\mathrm{CSP}\langle D, \Delta \rangle \equiv_{\mathrm{p}} \mathrm{CSP}\langle D, \mathrm{Inv}(\mathrm{Pol}(\Delta)) \rangle$

Perhaps the expressive power of algebra can be used to classify CSPs.

Notation: $\mathrm{CSP}(F) = \mathrm{CSP}(\mathrm{Inv}(F))$

Algebraic Facts

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras

$\mathbf{B}$ a subalgebra of $\mathbf{A} \implies \mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$.

$\mathbf{B}$ a homomorphic image of $\mathbf{A} \implies \mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$.

Let $\Delta \subseteq \mathrm{Rel}(D)$. For $a \in D$ write $\mu_a = \{a\} \in \mathrm{Rel}_1(D)$ $\Delta^* = \Delta \cup \{\, \mu_a : a \in D \,\}$.

Every member of $\mathrm{Pol}(\Delta^*)$ is *idempotent,* that is, $f(x, x, \ldots, x) = x$.

Theorem 15 (Bulatov, Jeavons, Krokhin, 2000 [4])**.** *If Δ is a core then* $\mathrm{CSP}(\Delta) \equiv_{\mathrm{p}} \mathrm{CSP}(\Delta^*)$.

Corollary 16. *For every algebra $\mathbf{A}$, there is an idempotent algebra $\mathbf{B}$ such that* $\mathrm{CSP}(\mathbf{A}) \equiv_{\mathrm{p}} \mathrm{CSP}(\mathbf{B})$.

Corollary 17. *For every algebra $\mathbf{A}$, there is an idempotent algebra $\mathbf{B}$ such that* $\mathrm{CSP}(\mathbf{A}) \equiv_{\mathrm{p}} \mathrm{CSP}(\mathbf{B})$.

$\mathbf{B}$ is efficiently computable from $\mathbf{A}$.

Thus for our two questions, we can restrict our attention to idempotent algebras.

CSP results for Algebras

Theorem 18 (Jeavons, Cohen, Gyssens, 1997 [7])**.** *If* $\mathrm{Pol}(\Delta)$ *contains a semilattice operation, then* $\mathrm{CSP}(\Delta) \in \mathbb{P}$.

semilattice: $x \cdot (y \cdot z) = (x \cdot y) \cdot z, \quad x \cdot y = y \cdot x, \quad x \cdot x = x$

Examples: logical '$\wedge$', '$\vee$'; '$\cap$', '$\cup$', 'gcd', 'lcm', $\langle H, K \rangle \in \mathrm{Sub}(G)$,...

Theorem 19 (Bulatov, 2002 [2])**.** *If* $\mathrm{Pol}(\Delta)$ *contains a Maltsev operation, then* $\mathrm{CSP}(\Delta) \in \mathbb{P}$.

Maltsev: $q(x, x, y) = q(y, x, x) = y$

Examples: groups, quasigroups, Boolean algebras, etc.

Corollary 20. *Both 2-COLORABILITY and LINEAR SYSTEM are tractable.*

Theorem 21 (Jeavons, Cohen, Gyssens, 1997 (?) [7])**.** *If* $\mathrm{Pol}(\Delta)$ *contains a majority operation, then* $\mathrm{CSP}(\Delta) \in \mathbb{P}$.

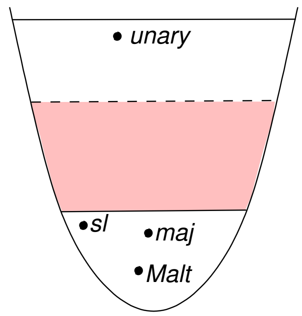
majority: $m(x, y, y) = m(y, x, y) = m(y, y, x) = y$

This gives another proof that 2-COLORABILITY is tractable

Theorem 22 (Bulatov, Jeavons, Krokhin, 2000 [4])**.** *If Δ is a core and every polymorphism is essentially unary, then* $\mathrm{CSP}(\Delta)$ *is* $\mathbb{NP}$*-complete.*

f is *essentially unary* if $f(x_1, \ldots, x_n) = g(x_j)$ for some unary g and some $j \leq n$.

Corollary 23. *Both 3-COLORABILITY and NONLINEAR SYSTEM are* $\mathbb{NP}$*-complete.*



Informal reformulation of the dichotomy conjecture If $\mathbf{A}$ has some kind of decent algebraic structure then $\mathrm{CSP}(\mathbf{A}) \in \mathbb{P}$ otherwise $\mathrm{CSP}(\mathbf{A})$ is $\mathbb{NP}$-complete.

Definition 24. Let $n > 1$. An n-ary operation f is called a *weak near-unanimity operation* if it is idempotent and satisfies

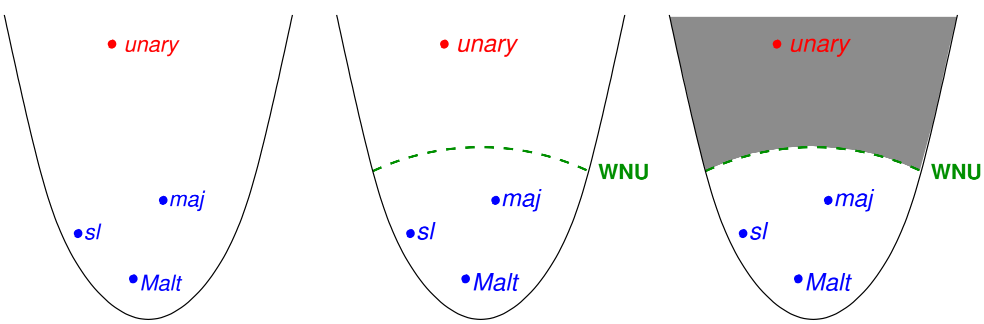
$$f(y, x, x, x, \ldots, x) = f(x, y, x, x, \ldots, x) = \cdots = f(x, x, \ldots, x, y)$$

Note that an essentially unary operation (on a nontrivial set) can not be a WNU operation.

Theorem 25 (Bulatov, Larose, Zádori, McKenzie, Maróti [3, 10, 11])**.** *If Δ is a core and* $\mathrm{Pol}(\Delta)$ *has no WNU operation then* $\mathrm{CSP}(\Delta)$ *is* $\mathbb{NP}$*-complete.*

Reformuated Dichotomy Conjecture

Let Δ be a core. Then $\mathrm{CSP}(\Delta)$ is tractable if and only if it has a WNU polymorphism. Otherwise, it is $\mathbb{NP}$-complete.



Supporting Examples

- Every semilattice and majority op is a WNU.
- Let $\mathbf{A}$ be an abelian group, $n = |A|$. Choose integers k, l with $kl \equiv 1 \pmod{n}$. Then

$$f(x_1, \ldots, x_k) = l(x_1 + \cdots + x_k)$$

  is a WNU operation.

Binary Operations

A binary operation is WNU if and only if

$$x \cdot x = x, \quad x \cdot y = y \cdot x.$$

Problem 26. *Assume* Δ *is a core and* $\mathrm{Pol}(\Delta)$ *contains a commutative, idempotent binary operation. Show* $\mathrm{CSP}(\Delta)$ *is tractable.*

Recall that a semilattice is an associative WNU.

Note that neither idempotence nor commutativity are sufficient individually

A left-zero semigroup (i.e., $x \cdot y = x$) is idempotent, but not commutative. It clearly has no WNU.

Let $\mathbf{A} = \langle \{0, 1, 2, 3\}, \cdot \rangle$ with multiplication modulo 4. This operation is commutative but not idempotent. $\mathbf{A}$ has no WNU.

Note that **"has a WNU polymorphism"** puts no bound on the number of variables. Is this even decidable?

Theorem 27 (Siggers, Kearnes, Marković, McKenzie, 2008 [13, 8])**.** $\mathrm{Pol}(\Delta)$ *has a WNU if and only if it has a 4-ary idempotent operation satisfying*

$$t(x, y, z, x) = t(y, z, x, z).$$

References

[1] Libor Barto, Marcin Kozik, and Todd Niven, *The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell)*, SIAM J. Comput. **38** (2008/09), no. 5, 1782–1802.

[2] Andrei Bulatov and Víctor Dalmau, *A simple algorithm for Mal′tsev constraints*, SIAM J. Comput. **36** (2006), no. 1, 16–27 (electronic).

[3] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin, *Classifying the complexity of constraints using finite algebras*, SIAM J. Comput. **34** (2005), no. 3, 720–742 (electronic).

[4] Andrei A. Bulatov, Andrei A. Krokhin, and Peter Jeavons, *Constraint satisfaction problems and finite algebras*, Automata, languages and programming (Geneva, 2000), Lecture Notes in Comput. Sci., vol. 1853, Springer, Berlin, 2000, pp. 272–282.

[5] Tomás Feder and Moshe Y. Vardi, *The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory*, SIAM J. Comput. **28** (1999), no. 1, 57–104 (electronic).

[6] Pavol Hell and Jaroslav Nešetřil, *On the complexity of H-coloring*, J. Combin. Theory Ser. B **48** (1990), no. 1, 92–110.

[7] Peter Jeavons, David Cohen, and Marc Gyssens, *Closure properties of constraints*, J. ACM **44** (1997), no. 4, 527–548.

[8] K. Kearnes, P. Marković, and R. McKenzie, *A note on mal'cev conditions for omitting type 1 in locally finite varieties*, (2009), preprint.

[9] Richard E. Ladner, *On the structure of polynomial time reducibility*, J. Assoc. Comput. Mach. **22** (1975), 155–171.

[10] Benoit Larose and László Zádori, *The complexity of the extendibility problem for finite posets*, SIAM J. Discrete Math. **17** (2003), no. 1, 114–121 (electronic).

[11] Miklós Maróti and Ralph McKenzie, *Existence theorems for weakly symmetric operations*, Algebra Universalis **59** (2008), no. 3-4, 463–489.

[12] Thomas J. Schaefer, *The complexity of satisfiability problems*, Conference Record of the Tenth Annual ACM Symposium on Theory of Computing (San Diego, Calif., 1978), ACM, New York, 1978, pp. 216–226.

[13] Mark H. Siggers, *A strong Mal'cev condition for locally finite varieties omitting the unary type*, Algebra Universalis **64** (2010), no. 1-2, 15–20.