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COMMUNICATING PROCESSES AND ENTROPIC ALGEBRAS

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A significant contribution to the analysis of certain aspects of the communicating processes model was made by D. Benson's proposal to view incompletely specified nondeterministic processes as modules over certain semirings, and dually as comodules over corresponding coalgebras. The effectiveness of the proposal in treating the synthesis of such processes under mutual communication depended on the good behaviour of these algebraic systems with respect to tensor products. The aim of the paper is to draw attension to the algebraic theory underlying Benson's proposal, the theory of entropic algebras. Working with entropic algebras guarantees that tensor products are sufficiently well-behaved to make Benson's theory work.

1. INTRODUCTION

The communicating processes model is a fundamental object of study in many areas of application of modern mathematics. Beyond its familiar use in the analysis of distributed computation [Bp], [M1], [M2], it appears for example in theoretical biology in the guise of *neural nets* [AA] and *genetic nets* [Wa, pp. 18 ff.]. There are also less obvious applications. In a reversal of the usual technique of hiding internal events to merge communicating processes into a single whole, F. ROBERT [Ro, § 1.4] uses the model to study a single iterative process by viewing it as a network of communicating subprocesses.

A significant contribution to the analysis of certain aspects of the model was made by D. BENSON's stimulating and elegant proposal [Be], [BM] to view incompletely specitied nondeterministic processes as modules over certain semirings, and dually as comodules over corresponding coalgebras. The effectiveness of the proposal in treating the synthesis of such processes under mutual communication depended on the good behaviour of these algebraic systems with respect to tensor products. Part of BENSON's motivation was the desire to translate the tensor product formalism from the real vector spaces used in topological dynamics to the semiring modules of his approach to communicating processes.

The aim of the current paper, inspired by [Be], is to draw attention to the algebraic theory underlying BENSON's proposal. This is the theory of *entropic algebras* - algebras in which each operation is a homomorphism. Examples of entropic algebras are provided by the semilattices of SCCS [Bp, A1-A3],[M2, Prop. 5.1(2)(4)], the modes of [RS], [Sm], and vector spaces. Working with entropic algebras guarantees that tensor products are sufficiently well-behaved[DD] to make BENSON's theory work. As an indication of the need for some circumspection when dealing with tensor products, note that the tensor product of semigroups is not associative [Gr, p. 271].

The basic algebraic theory is presented in the second and third sections. It is then used in the fourth section to give a general formulation of BENSON's proposal. The general formulation avoids the use of duality made in [Be, pp. 11-12]. The string-reversing anti-isomorphism turns out to be adequate to change a right comodule into a left comodule. This removes the apparent dependence on the synchronisation algebra for CCS (in the sense of [Wi, 4.4]), enabling other synchronisation algebras (such as the parallel compositions "||" and "|||" of HOARE-BROOKES-ROSCOE [Wi, 4.5-6]) to be given the elegant description of [Be] that avoids the spurious elements * and O of [Wi]. Besides giving a theoretical underpinning to the Boolean semiring-module treatments of [Be], [BM], the general entropic algebra formulation offers two other advantages. Firstly, it makes it easy to attempt a translation of known results from topological dynamics expressed in the language of vector spaces (cf. Example 2.1 below) - simply rewrite them in terms of entropic algebras, and interpret them in other varieties (such as those in further examples of § 2). Secondly, it provides a ready-made framework within which to study extensions of the techniques of [Be], [BM] from nondeterministic choices to probabilistic choices (cf. [Ko], [Mn, 2.2], [Ms, 4.3.10]) and other contexts, making it potentially available for a wide variety of applications to biology and other fields. In this way, the formulation suggests a procedure "to establish bridges between the continuous methods, the stochastic analysis and the boolean analysis" [Th, p. IX] of communicating processes. Of course, there are even more general approaches available, such as that of bi-

triples (cf. [Ba], etc.) or commutative theories [Ms], but entropic algebras have the advantage of providing sufficient generality, while at the same time remaining concrete enough for the usual algebraic intuitions to act as reliable guides.

2. ENTROPIC ALGEBRAS

This section and the next give illustrations and a brief summary of those aspects of the theory of entropic algebras that are used here for studying communicating processes. Most of the results are well-known "folk theorems" in universal algebra and category theory. The present applications may prove interesting to specialists in those disciplines. Further details of the algebraic background are given in [Ms] and [RS, Chapter 1].

An *algebra* (A,Ω) is a set A equipped with an operator domain Ω and a *type* or *arity function* $\tau:\Omega \to \mathbb{N}$ such that each ω in Ω determines a map

$$\omega:A^{\omega\tau} \to A \; ; \; (a_1,\ldots,a_n) \mapsto a_1\ldots a_n\omega \; ,$$

where $n = \omega\tau$. A *homomorphism* $f:(A,\Omega) \to (B,\Omega)$ of algebras of the same type $\tau:\Omega \to \mathbb{N}$ is a set mapping $f:A \to B$ with $a_1\ldots a_n\omega f = a_1f\ldots a_nf\omega$ for all ω in Ω and $a_1, \ldots, a_n$ in A . The direct product (A^n,Ω) has componentwise operations, making it an algebra of the same type as (A,Ω) . Then (A,Ω) is said to be *entropic* if each $\omega:(A^n,\Omega) \to (A,\Omega)$ in Ω is a homomorphism. *Varieties* $\underline{V}$ are classes of algebras of the same type closed under the taking of subalgebras, arbitrary direct products, and homomorphic images.

<u>Example 2.1</u> (Vector spaces). For a field F , vector spaces A over F form a variety of entropic algebras of type $\{(+,2)\} \cup (F\times\{1\})$, where for λ in F ,

$$\lambda:A \to A \; ; \; a \mapsto \lambda a$$

is the unary operation of scalar multiplication. The effect $(x_1,\ldots,x_n)w$ of a derived operation w on a set $\{x_1,\ldots,x_n\}$ of arguments from A is a linear combination of the arguments. □

<u>Example 2.2</u> (Modules over a commutative ring). Example 2.1 may be generalised by relaxing the requirements on F , so that it is merely a commutative ring. One obtains the variety of modules over the ring F . The commutativity of F corresponds to

$\mu:(A,\lambda) \to (A,\lambda)$ being a homomorphism for each pair λ, μ of elements of F . □

Example 2.3 (Semilattices). The variety of semilattices is the variety of algebras $(A,.)$ with a single binary idempotent commutative associative operation. The effect $(x_1,\ldots,x_n)w$ of a derived operation w on a set $\{x_1,\ldots,x_n\}$ of arguments from A represents a non-empty subset of the arguments. Indeed $(x_1,\ldots,x_n)w = y_1\cdot\ldots\cdot y_r$ for $\{y_1,\ldots,y_r\} \subseteq \{x_1,\ldots,x_n\}$ represents $\{y_1,\ldots,y_r\}$. □

Example 2.4 (Barycentric algebras [RS, 2.1]). Let $I^\circ = \{p \in R \mid 0 < p < 1\}$. For p in I°, set $p' = 1-p$. A *barycentric algebra* is an algebra (A,I°) of type $I^\circ \times \{2\}$, with $\underline{p}:A^2 \to A$; $(x,y) \mapsto xy\underline{p}$ for p in I°, satisfying the identities $xx\underline{p} = x$ of idempotence, $xy\underline{p} = yx\underline{p'}$ of *skew-commutativity*, and $xy\underline{p}z\underline{q} = xyz\underline{(q/(p'q')')}\,\underline{(p'q')'}$ of *skew-associativity*. Semilattices form a class of barycentric algebras with $xy\underline{p} = x\cdot y$ for all p in I° . Another important class of barycentric algebras consists of the *convex sets* as identified in [RS, Ch. 2]. Free barycentric algebras are convex sets. The effect $(x_1,\ldots,x_n)w$ of a derived operation w on a set $\{x_1,\ldots,x_n\}$ of arguments from a convex set A represents a probability distribution on the arguments. In particular, $xy\underline{p}$ represents the distribution with probabilities $P(x) = 1-p$ and $P(y) = p$. □

Example 2.5 (Semimodules over semirings). A *semiring* [RS, 265] is an algebra $(R,+,.)$ of type $\{+,.\} \times \{2\}$, where $(R,+)$ and $(R,.)$ are semigroups connected by the distributive laws $x.(y+z) = x.y + x.z$ and $(x+y).z = x.z + y.z$. The semiring R is "with 1" if $(R,.,1)$ is a monoid, and "with 0" if $(R,+,0)$ is a monoid. Rings are semirings with 0 . The set $\mathbb{Z}^+$ of positive integers forms a semiring with 1 under the usual operations. Distributive lattices, and indeed dissemilattices [RS, 326], are semirings. Main and Benson [BM] reserve the term "semiring" for semirings with 0 and 1 in which $(R,+)$ is commutative. Thus the set $\mathbb{N}$ of natural numbers under the usual operations is a semiring even in this more restrictive sense.

For an entropic semigroup $(S,+)$, the *endomorphism semiring* $(\mathrm{End}(S,+),+,.)$ or just $\mathrm{End}S$ is the set of semigroup homomorphisms $\theta:(S,+) \to (S,+)$ with semiring operations defined by $s(\theta+\phi) = s\theta + s\phi$ and $s(\theta.\phi) = s\theta\phi$ for θ, ϕ in $\mathrm{End}S$ and s in S . A commutative semigroup $(S,+)$ is said to be a *semimodule* over a semiring $(R,+,.)$ [RS, 326] if there is a semiring homomorphism $(R,+,.) \to (\mathrm{End}(S,+),+,.)$. A semimodule over a semiring with 1 is *unital* if this semiring homomorphism is also a monoid homomorphism $(R,.,1) \to (\mathrm{End}S,.,1)$. Semimodules over a semiring $(F,+,.)$ with $(F,.)$ commutative form a variety of entropic algebras of type $\{(+,2)\} \cup (F \times \{1\})$, generalizing Examples 2.1 and 2.2 above. If A is a free algebra in such a variety, then the effect $(x_1,\ldots,x_n)w$ of a derived operation w on a set $X = \{x_1,\ldots,x_n\}$ of arguments from A may be interpreted as a non-deterministic distribution of the arguments [BM, §2.1]. If F is $\mathbb{N}$ or $\mathbb{Z}^+$, the distribution is a multiset from X , while if F is a distributive lattice of "conditions" (in Dijkstra's sense [Di], cf. [BM, §3]), then the distribution records the conditions attached to the appearance of each argument. □

3. TENSOR PRODUCTS, COALGEBRAS, AND COMODULES

Given algebras $M_1, \ldots, M_r, N$ in any variety $\underline{V}$, a mapping $f:M_1\times\ldots\times M_r \to N$ is an *r-homomorphism* if the mapping

$$M_i \to N \ ; \ m_i \mapsto (m_1,\ldots,m_i,\ldots,m_r)f$$

is a homomorphism for each choice of $1 \leq i \leq r$ and m_j in M_j for $j \neq i$. (This generalizes the concept of an r-multilinear mapping of vector spaces.) Then the *tensor product* $M_1\otimes M_2\otimes\ldots\otimes M_r$ of the ordered list of algebras $(M_1, M_2,\ldots, M_r)$ is an algebra in $\underline{V}$ with r-homomorphism

$$\eta:M_1\times\ldots\times M_r \to M_1\otimes\ldots\otimes M_r \ ; \ (m_1,\ldots,m_r) \mapsto m_1\otimes\ldots\otimes m_r$$

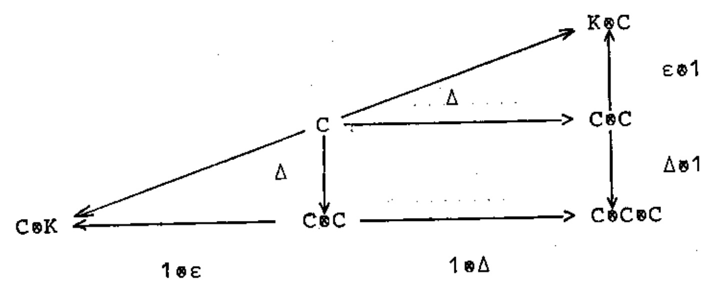
such that for any r-homomorphism $f:M_1\times\ldots\times M_r \to N$, there is a unique homomorphism $\overline{f}:M_1\otimes\ldots\otimes M_r \to N$ such that $\eta\overline{f} = f$. ([DD, §1], cf. [Ms, Ex. 3.6.7]).

If $\underline{V}$ is a variety of entropic algebras, then tensor products are very well-behaved. The technical formulation of this good behaviour is that the class $\underline{V}$ of objects together with homomorphisms as morphisms forms a "closed category" [DD, Corollary 3.4], [ML, VII. 7], [Ms, Ex. 3.6.7(c)] under the binary tensor product $\otimes$ (as bifunctor) and the free algebra K in $\underline{V}$ on the singleton $\{1\}$. This means in particular that $\otimes$ is commutative and associative, with K as a unit, up to unique natural isomorphisms. Given $\underline{V}$-algebras A and B , the natural isomorphism

$A \otimes B \cong B \otimes A$ is the "twisting" $\tau : a\otimes b \to b\otimes a$.

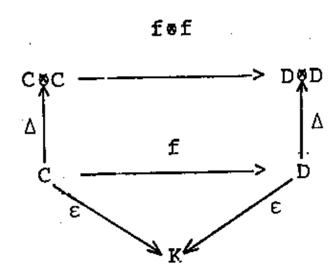
In a variety $\underline{\underline{V}}$ of entropic algebras, a *coalgebra* C (cf. [Be, pp. 6-7], [Sw, pp. 4-5]) is a $\underline{\underline{V}}$-algebra C equipped with homomorphisms $\Delta : C \to C\otimes C$ and $\varepsilon : C \to K$ such that the diagram

(3.1)



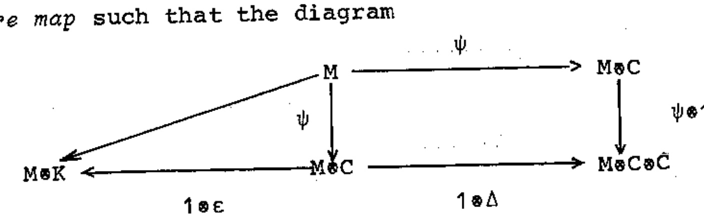
commutes, where the sloping maps are the natural isomorphisms $K \otimes C \cong C$ and $C \cong C \otimes K$. Given coalgebras C and D in $\underline{\underline{V}}$, a $\underline{\underline{V}}$-homomorphism $f : C \to D$ is a *coalgebra homomorphism* (cf. [Sw, pp. 13-14]) if the diagram

(3.2)



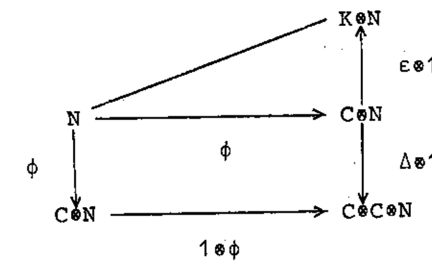
commutes. A *right C-comodule* (cf. [Be, pp. 8-9], [Sw, p. 30]) is a $\underline{\underline{V}}$-algebra M with a homomorphism $\psi : M \to M \otimes C$ called the *structure map* such that the diagram

(3.3)



commutes. A *left C-comodule* is a $\underline{\underline{V}}$-algebra N with a homomorphism $\phi : N \to C \otimes N$ (again called the *structure map*) such that the diagram

(3.4)



commutes. As in (3.1), the sloping maps in (3.3) and (3.4) denote the natural isomorphisms $M \otimes K \cong M$ and $N \cong K \otimes N$.

The interpretation of these algebraic abstractions in terms of communicating processes may be summarized as follows. (More detailed examples are given in the next section.) Coalgebras C, D represent communications channels, into which messages (as distributions of strings or traces of events) may be sent or from which messages may be received. A right C-comodule M represents a transmitting machine. An element of M represents a condition of the machine. The structure map $\psi : M \to M\otimes C$ describes the transition of the transmitter from one condition to another; during the transition a message is sent into C . A left D-comodule N represents a receiving machine. The structure map $\phi : N \to D \otimes N$ describes the transition of the receiver from one condition to another; during the transition a message is received from D . A $\underline{\underline{V}}$-algebra homomorphism $e : C \otimes D \to K$ serves to synchronize communications over the channels C and D . Thus the composition

(3.5)
$$M\otimes N \xrightarrow{\psi\otimes\phi} M\otimes C\otimes D\otimes N \xrightarrow{1_M\otimes e\otimes 1_N} M\otimes K\otimes N \cong M\otimes N$$

describes the transition of the coupled machines M, N from one condition to another as a result of synchronized communications: the message sent by M into C is synchronized with the message received by N from D . In all this, "conditions" and "messages" may stand for non-deterministic or probabilistic combinations of "pure" conditions and messages, according to the variety $\underline{\underline{V}}$ of entropic algebras in which M, N, C and D lie.

4. COMMUNICATING PROCESSES

This section discusses some formulations of communicating processes (such as [Be]) in terms of entropic algebras. Throughout, let $\underline{\underline{V}}$ be a fixed non-trivial variety of entropic algebras. Let K be the free $\underline{\underline{V}}$-algebra on the singleton {1} . In the context of [Be], $\underline{\underline{V}}$ is the variety of join semilattices with zero, so that $0+x = x = x+0$. Then K is $B = \{0,1\}$. Let A be a given alphabet. The elements of A represent events. Let D be a symmetric and reflexive relation on A , known as the *dependency* relation on A . If A is finite, the pair (A,D) is a "concurrent alphabet" in the sense of [Mz]. Let A^D or $(A^D,.)$ be the monoid generated by A subject to the relations that independent events (i.e. events a, b with $(a,b) \notin D$) commute. If A is finite, then A^D is the "algebra of traces over (A,D)" in the sense of [Mz]. If $D = A \times A$, so that there are no independent events, then A^D is just the free monoid A^* on A . Let $(A^D,\sim)$ denote the *opposite* of $(A^D,.)$, with

$$a\sim b = b.a \; . \tag{4.1}$$

The identity mapping on A extends to unique mutually inverse monoid homomorphisms $r:(A^D,.) \to (A^D,\sim)$ and $r':(A^D,\sim) \to (A^D,.)$ called *reversal*, with $a_1 \ldots a_n r = a_n \ldots a_1$ for a_i in A . Let C be the free algebra in $\underline{\underline{V}}$ on A^D . Since $\underline{\underline{V}}$ is non-trivial, A^D may be identified with its image in C . The elements of C are $\underline{\underline{V}}$-words in the elements of A^D . Note that the multiplication $A^D \times A^D \to A^D$ extends to a $\underline{\underline{V}}$-homomorphism $\mu:C \otimes C \to C$.

Suppose that C has a coalgebra structure (C,Δ,ε) as in (3.1) with $\Delta\mu = 1_C$. For example, suppose that $\underline{\underline{V}}$ is a variety of unital semimodules over a commutative semiring with 0 and 1 . Given a string s in A^* , the pairs (s',s'') for which $s'\otimes s''$ appears as an argument of the $\underline{\underline{V}}$-word $s\Delta$ may represent factorisations $s = s's''$. In (4.2), these pairs are the (s_i,t_i), $1 \leq i \leq n$. Take $1\varepsilon = 1$ and $(A^* - \{1\})\varepsilon = \{0\}$. Such coalgebras are called *choice coalgebras* [Be], [BM]. Given a choice coalgebra $C = (C,\Delta,\varepsilon)$ with

$$s\Delta = (s_1\otimes t_1,\ldots,s_n\otimes t_n)w_s \; , \tag{4.2}$$

where w_s is a derived operation of $\underline{\underline{V}}$ depending on s , a *reversed coalgebra* $C^r = (C,\Delta^r,\varepsilon)$ may be defined by

$$sr\Delta^r = (t_1 r\otimes s_1 r,\ldots,t_n r\otimes s_n r)w_s \; ; \tag{4.3}$$

the commuting of the diagram (3.1) for C , interpreted at the element level, gives the commuting of the corresponding diagram for C^r on reversing the strings appearing. (Note that, in general, the coalgebra C^r is not isomorphic to the coalgebra C .) Define a *C-process* to be a right C-comodule U for the coalgebra (C,Δ,ε) .

Communication between processes in [Be] depends on the assignment of left C^r-comodules to right C-comodules. This may be done (without recourse to duality as in [Be]) by using string reversal. Let

$$\psi:U \to U\otimes C \; ; \; u \mapsto (u_1\otimes c_1,\ldots,u_n\otimes c_n)w_u \tag{4.4}$$

be the structure map of the right C-process U , where w_u is a derived operation of $\underline{\underline{V}}$ depending on u . Then the structure map of a left C^r-comodule U may be defined by

$$\psi^r:U \to C^r\otimes U \; ; \; u \mapsto (c_1 r\otimes u_1,\ldots,c_n r\otimes u_n)w_u \; . \tag{4.5}$$

The commuting of the diagram (3.3) for the right C-comodule U , interpreted at the element level, gives the commuting of the corresponding diagram (3.4) for the left C^r-comodule U on reversing the strings appearing.

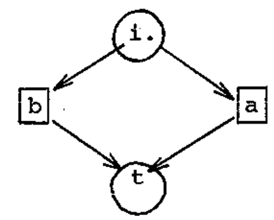
The synchronisation of events is achieved by a map $e:A^* \times A^* \to K$, extended to a $\underline{\underline{V}}$-homomorphism $e:C \otimes C^r \to K$. One defines $(s,t)e = 1$ if $s = x_n \ldots x_1$ and $t = y_1 \ldots y_n$ with x_i, y_i in $A \cup \{1\}$, such that for $i = 1, \ldots, n$, the events x_i and y_i can occur synchronously. Here the following convention applies: "event a can occur synchronously with 1 ", i.e. $(a,1)e = 1$, means precisely that the event a can occur asynchronously, independent of other events. If the pair (s,t) has no such expression $(x_n \ldots x_1, y_1 \ldots y_n)$, then $(s,t)e = 0$. Note that e need not be a coalgebra homomorphism. even under the synchronisation algebra for CCS [Wi, 4.4], since the triangle of (3.2) does not commute.

Suppose given two C-processes V, U, with respective structure maps $\psi:U \to U \otimes C$ and $\phi:V \to V \otimes C$. A run or "nondeterministic program segment" (in the sense of [BM]) for the communicating pair of processes is then obtained as the ("linear transformation" [BM] or) $\underline{\underline{V}}$-homomorphism $f:V \otimes U \to V \otimes U$ which is the composite

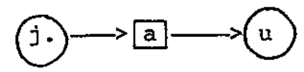
$$V\otimes U \xrightarrow{\phi\otimes\psi^r} V\otimes C\otimes C^r\otimes U \xrightarrow{1\otimes e\otimes 1} V\otimes K\otimes U \cong V\otimes U \tag{4.6}$$

as in (3.5).

<u>Example 4.7</u>. Let V denote the machine



and let U denote the machine

 .

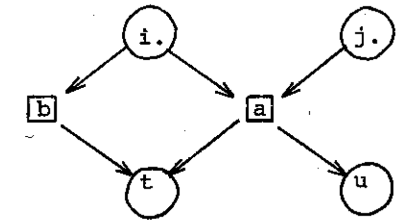
Let $A = \{a,b\}$. Let $\underline{V}$ be the variety of join semilattices. Let V be free in $\underline{V}$ on $\{i,t\}$. As a right C-comodule, suppose that V has the structure map $\phi : V \to V \otimes C$ with $\phi(i) = i\otimes 1 + t\otimes a + t\otimes b$ and $\phi(t) = t\otimes 1$. Let U be free in $\underline{V}$ on $\{j,u\}$. As a right C-comodule, suppose that U has the structure map $\psi : U \to U \otimes C$ with $\psi(j) = j\otimes 1 + u\otimes a$ and $\psi(u) = u\otimes 1$. According to (4.5), U has a left C^r-comodule structure map $\psi^r : U \to C^r \otimes U$ with $\psi^r(j) = 1\otimes j + a\otimes u$ and $\psi^r(u) = 1\otimes u$. Suppose that V and U communicate over the channel C in such a way that event b may occur asynchronously, while event a acts as a "handshake" between V and U . Thus $(a,a)e = (b,1)e = (1,b)e = (1,1)e = 1$, while $(x,y)e = 0$ for other pairs (x,y) in $(A \cup \{1\}) \times (A \cup \{1\})$. A run of the pair $V \otimes U$ of communicating machines starting from the (deterministic) initial condition $i\otimes j$ is then described by (4.6) as follows. Under $\phi \otimes \psi^r$, the element $i\otimes j$ of $V \otimes U$ is mapped to the element $(i\otimes 1 + t\otimes a + t\otimes b) \otimes (1\otimes j + a\otimes u) = i\otimes 1\otimes 1\otimes j + t\otimes a\otimes 1\otimes j + t\otimes b\otimes 1\otimes j + i\otimes 1\otimes a\otimes u + t\otimes a\otimes a\otimes u + t\otimes b\otimes a\otimes u$ of $V \otimes C \otimes C^r \otimes U$. Under $1\otimes e\otimes 1$, this element is mapped to $i\otimes 1\otimes j + t\otimes 0\otimes j + t\otimes 1\otimes j + i\otimes 0\otimes u + t\otimes 1\otimes u + t\otimes 0\otimes u = i\otimes j + t\otimes j + t\otimes u$ in $V \otimes U$. Thus there are three possible behaviours:

(i) the machines may stay in their initial conditions, without communicating;

(ii) V may change to its terminal condition t via the asynchronous event b ;

(iii) V and U may "shake hands", communicating via the event a which changes them to their respective terminal conditions t and u .

It is interesting to contrast this with the description provided by regarding V and U as (diagrams of) "elementary net systems" in the sense of [Mz, §4]. The "composition" $V + U$ [Mz, §4.4] is the net system with diagram



Under this composition, the separate identity of V and U has been lost. For example, $V + U$ may flow from the initial condition i of V , via event a , to the terminal condition u of U . □

Taking $\underline{V}$ to be a variety of finitary algebras limits the C-processes to finitely branching and finitely deep process trees (in the sense of [Bp, 1.2.1]). One may then extend the study to possibly infinitely deep trees by taking projective limits of finitely deep trees, and hence of C-processes, as in [Bp, 1.2.3]. Process graphs with cycles may be bisimulated by infinitely deep "covering" trees. This offers an alternative approach to the countable sums used in [Be], [BM].

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