CHARACTERS OF CENTRAL PIQUES

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Abstract. As a first step towards a duality theory for central quasigroups, the paper presents an explicit computation of the characters of a central pique (quasigroup with pointed idempotent) using Wigner's "little groups" method. The characters of a central pique's cloop (principally isotopic abelian group) form a dual pique. The conjugacy classes of the dual correspond to the characters of the primal; indeed the unitary character table of the dual is the inverse of the unitary character table of the primal. Together with its dual, a central pique forms a structure known as the double. The double satisfies identities indexed by loops of 2-power order. These identities project onto the unit circle to yield identities involving character values.

1. Introduction

Character theory for finite quasigroups was introduced in [6]-[7], [8]-[13], [19] as an extension of group character theory. Within the latter theory, the character theory of abelian groups exhibits special features, generally arising from the duality theory for abelian groups that finds its fullest and most satisfactory formulation in the topological context of Pontryagin duality for locally compact abelian groups. The quasigroup analogues of abelian groups are *central* quasigroups or $\mathfrak{Z}$-quasigroups, those quasigroups for which the diagonal subquasigroup of the direct square is a normal subquasigroup, i.e. an equivalence class of a congruence relation on the direct square. One thus expects the character theory of central quasigroups to exhibit special features extending those exhibited by abelian groups. The current paper represents a first step towards the investigation of these special features. To simplify the presentation, it focusses on central piques. Recall that a pique is a quasigroup having a pointed idempotent element. Thus one of the main goals is to provide an explicit determination of the character table of a finite central pique. In fact each central quasigroup is centrally isotopic to a central pique [3, Th. III.5.6], and central isotopy

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preserves character tables (cf. [10, §3] and [3, Prop. III.4.6]), so the work of this paper might be viewed as giving recipes for the character tables of general finite central quasigroups. Nevertheless, essential differences (such as the fact that pique characters take one argument, while quasigroup characters take two) mean that a satisfactory treatment of the character theory of central quasigroups must wait for a future paper.

Character tables of central piques have already appeared at various times in the literature, and there are several possible approaches to their computation. For example, one may use the fusion geometry of [10], or Schur rings of abelian groups [1, §II.2.6]. However, these methods can be somewhat *ad hoc* in their application. The approach presented in this paper uses the Wigner-Mackey "little groups" technique (Section 4) to analyse the permutation character of the natural action of the multiplication group of a finite central pique (Theorem 5.1). Reliance on the group theory rapidly produces the required results, but the greatest advantages of this approach are the explicit form in which the character table is presented (Theorem 7.1), and the way in which the formulation leads naturally to the concept of the dual of a finite central pique (Section 6). The duality appears in its most striking form when the character tables are normalized as unitary matrices (3.4). The unitary character table of the dual of a finite central pique is then just the inverse of the unitary character table of the primal (Corollary 7.4). Incidentally, the existence of an association scheme dual to the association scheme of a central pique is an immediate consequence of Schur ring theory [1, Th. 2.6.4]. The dual pique construction shows that this dual scheme is actually a quasigroup scheme. (Compare [13] for a discussion of association schemes which may or may not be quasigroup schemes.)

Together with its dual, a finite central pique forms an object known as the *double* (Section 8), consisting of a tensor product of the pique algebras of the primal and the dual. The double satisfies curious identities corresponding to each loop of 2-power order (Section 10). The existence of these identities is significant as the first application of one area of quasigroup theory to another (as opposed to the involvement of other parts of mathematics within quasigroup theory, or the application of quasigroup theory to an external area such as coding theory or mathematical physics).

As indicated earlier, the current paper is only intended to be a first step towards a duality theory for central quasigroups. Among the many open problems remaining, the following are worthy of mention:

(1) investigation of the dependencies between the loop-indexed identities on the double of a finite central pique;
(2) extension of the current theory to the general unpointed case of finite central quasigroups that do not necessarily contain an idempotent element;
(3) extension of the current theory from finite (discrete) central piques to general locally compact central piques or quasigroups.

Concerning Problem 3, recall that Suvorov exhibited a duality for idempotent, entropic, locally compact topological quasigroups on the basis of Pontryagin duality [22].

2. Central piques

This section and the following present a quick summary of the requisite aspects of the theory of quasigroups and their characters. For concepts and notational conventions not otherwise defined explicitly in this paper, see [21].

A *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. In particular, the body of the multiplication table of a (finite) quasigroup is a Latin square, while each Latin square may be bordered to yield the multiplication table of a quasigroup. Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{ll} \text{(IL)} & y \backslash (y \cdot x) = x; \\ \text{(IR)} & x = (x \cdot y)/y; \\ \text{(SL)} & y \cdot (y \backslash x) = x; \\ \text{(SR)} & x = (x/y) \cdot y. \end{array}$$

(Note that one often suppresses explicit mention of the division operations of a quasigroup, denoting it merely as $(Q, \cdot)$ instead.) The equational definition of quasigroups means that they form a variety in the sense of universal algebra, and are thus susceptible to study by the concepts and methods of universal algebra [21]. An element e of a quasigroup Q is said to be *idempotent* if $\{e\}$ forms a singleton subquasigroup of Q. A *pique* or *pointed idempotent quasigroup* [3, §III.5] is a quasigroup P, containing an idempotent element 0, that has its quasigroup structure of multiplication and the divisions enriched by a nullary operation selecting the idempotent element 0. Note that piques also form a variety.

For each element q of a quasigroup $(Q, *)$, the *right multiplication* $R_*(q)$ or

$$R(q) : Q \to Q; x \mapsto x * q$$

and *left multiplication* $L_*(q)$ or

$$L(q) : Q \to Q; x \mapsto q * x$$

are elements of the group $Q!$ of bijections from the set Q to itself. The subgroup of $Q!$ generated by all the right and left multiplications is called the *multiplication group* $\mathrm{Mlt}\, Q$ of Q. For a pique P with pointed idempotent 0, it is conventional to set $R = R(0)$ and $L = L(0)$. The stabilizer of 0 in the permutation group $\mathrm{Mlt}\, P$ is called the *inner multiplication group* $\mathrm{Inn}\, P$ of P. For example, if P is a group, then the inner multiplication group of the pique P is just the inner automorphism group of the group P. One thus defines the (*pique*) *conjugacy classes* of a pique to be the orbits of its inner multiplication group.

A *loop* L is a pique in which the pointed idempotent element 1 acts as an identity, so that $1x = x = x1$ for all elements x of L. For a general pique $(P, \cdot, 0)$, the *cloop* or *corresponding loop* is the loop $B(P)$ or $(P, +, 0)$ in which the "multiplication" operation $+$ is defined by

$$x + y = xR^{-1} \cdot yL^{-1}. \tag{2.1}$$

Inverting (2.1), the multiplication of a pique is recovered from the cloop by

$$x \cdot y = xR + yL. \tag{2.2}$$

Definition 2.1. A pique $(P, \cdot, 0)$ is said to be *central*, or to lie in the class $\mathfrak{Z}$, if:

(1) $B(P)$ is an abelian group, and
(2) $\mathrm{Inn}\, P$ is a group of automorphisms of $B(P)$.

Remark 2.2. The syntactical Definition 2.1 of pique centrality is chosen for its concreteness, and because it is well suited to the purposes of the current paper. The equivalence of this definition with the structural characterization given by normality of the diagonal in the direct square is discussed in [3, §III.5].

Let P be a central pique. Note that $\mathrm{Inn}\, P$ is generated by R and L. For an element x of P, (2.2) yields $L(x) = LL_+(xR)$ and $R(x) = RR_+(yL)$. Identifying $\mathrm{Mlt}\, B(P)$ with $B(P)$ by regularity, one obtains the following

Theorem 2.3. *The multiplication group* $\operatorname{Mlt} P$ *of a central pique* P *is the split extension of the abelian group* $B(P)$ *by the inner multiplication group* $\operatorname{Inn} P$.

Example 2.4. The simplest algebraic specification of the dihedral group D_n is as the multiplication group of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ considered as a central pique under the operation of subtraction [21, Ch. I, Ex. 2.1.2].

3. Quasigroup characters

If A is an abelian group, then the *dual* or *character* group is the set $\widehat{A}$ of abelian group homomorphisms from A into the circle group S^1 of unit modulus complex numbers under multiplication. The dual group carries the abelian group operation defined by $(\alpha+\beta)(a) = \alpha(a)\beta(a)$ for $\alpha, \beta \in \widehat{A}$ and $a \in A$. Recall that each finite abelian group is isomorphic to its dual [5, V.6.4(b)].

Following the classical extension of character theory by Frobenius, Schur, et al. from abelian to finite non-commutative groups, a further extension to finite quasigroups was obtained using the theories of association schemes and S-rings [6], [8]-[13], [19]. (Note that character theory and ordinary representation theory, which are virtually indistinguishable in the usual treatments given for finite groups, diverge for finite quasigroups.)

Let G be the multiplication group of a quasigroup Q of finite order n. Let $\{C_1, C_2, \ldots, C_s\}$ be the orbits of G in its diagonal action on Q^2, numbered so that C_1 is the diagonal orbit or equality relation. These orbits are known as the (*quasigroup*) *conjugacy classes* of Q. Let $\{A_1 = I_n, A_2, \ldots, A_s\}$ be the incidence matrices of the quasigroup conjugacy classes. Then the linear span of these matrices in the algebra of $n \times n$ complex matrices is a commutative algebra, the *centralizer ring* or *Vertauschungsring* $V(G, Q)$ of G in its multiplicity-free action on Q. Diagonalizing the algebra $V(G, Q)$, one obtains a basis $\{E_1 = J_n/n, E_2, \ldots, E_s\}$ for $V(G, Q)$ consisting entirely of idempotent matrices, with J_n as the $n \times n$ all-ones matrix.

Set $|C_i| = nn_i$ and $\operatorname{tr} E_i = f_i$ for $1 \leq i \leq s$. Suppose

$$A_i = \sum_{j=1}^{s} \xi_{ij} E_j \tag{3.1}$$

and

$$E_i = \sum_{j=1}^{s} \eta_{ij} A_j \tag{3.2}$$

for $1 \leq i \leq s$. Then $\xi_{ji} f_i = \overline{\eta}_{ij} n n_j$ for $1 \leq i, j \leq s$. The *character table* of Q is defined to be the $s \times s$ matrix Ψ with entries

$$\psi_{ij} = \frac{\sqrt{f_i}}{n_j}\, \xi_{ji} = \frac{n}{\sqrt{f_i}}\, \overline{\eta}_{ij} \tag{3.3}$$

for $1 \leq i, j \leq s$. For a pique P, the i-th *irreducible character* is the function $\psi_i : P \to \mathbb{C}$ with $\psi_i(p) = \psi_{ij}$ for $(0, p) \in C_j$. If P is a group, then this definition agrees with the usual group-theoretic concept, ψ_1 being the trivial character [1, Th. 2.7.2]. In fact, the most natural normalization is obtained by considering the $s \times s$-matrix U whose i, j-entry is

$$U_{ij} = \sqrt{\frac{f_i}{n n_j}}\xi_{ji} = \sqrt{\frac{n n_j}{f_i}}\overline{\eta}_{ij} \tag{3.4}$$

[4, (A.8)][20, (4.7)]. This unitary matrix, known as the *unitary character table* of P, has an interpretation in terms of quantum mechanics [20].

Example 3.1. Continuing Example 2.4, the character table of the central pique $(\mathbb{Z}/4\mathbb{Z}, -)$ is

$$\Psi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 \end{bmatrix}.$$

Consider the conformal field theory describing the scaling limit of the Ising model at the critical point (cf. [2, Ex. 5.2.12] or [14]). This theory has three physical representations $\rho_0, \rho_1, \rho_{1/2}$, with respective statistical dimensions $1, 1, \sqrt{2}$ ([2, Ex. 11.3.22] or [14, (1.57)]). These statistical dimensions are the dimensions of the irreducible characters ψ_1, ψ_2, and ψ_3 of $(\mathbb{Z}/4\mathbb{Z}, -)$. The centraliser ring of $(\mathbb{Z}/4\mathbb{Z}, -)$ yields the fusion rules of the conformal field theory under the assignments $\rho_0 \mapsto \mathbf{A}_1$, $\rho_1 \mapsto \mathbf{A}_2$, $\rho_{1/2} \mapsto \mathbf{A}_3/\sqrt{2}$.

4. The little groups method

This section recalls the details and conventions of the Mackey/Wigner "little groups method" [17, §8.2]. Suppose that a group G is a semidirect product of a normal abelian subgroup A with a subgroup H. (In the application of Section 5 below, guaranteed by Theorem 2.3, G will be the multiplication group of a central pique with cloop A and inner multiplication group H.) The group G has a right action on A by conjugation, and a left action on the dual group $\widehat{A}$ of A by

$$G \times \widehat{A} \to \widehat{A}; (g, \alpha) \mapsto ({}^{g}\alpha : a \mapsto \alpha(a^g)). \tag{4.1}$$

Consider the restriction of the action (4.1) to the subgroup H. Let

$$\{\alpha_i \mid i \in H \setminus \widehat{A}\} \tag{4.2}$$

be a set of orbit representatives for the restricted action. For $i \in H \setminus \widehat{A}$, let H_i be the stabilizer of α_i in H, and let $G_i = AH_i$. Extend the domain of $\alpha_i : A \to S^1$ to G_i by

$$\alpha_i(ah) = \alpha_i(a) \tag{4.3}$$

for $a \in A$ and $h \in H_i$, thereby obtaining a linear character of G_i. For each irreducible character $\rho : H_i \to \mathbb{C}$ of H_i, let $\widetilde{\rho}$ be the irreducible character of G_i obtained by composition with the natural projection $G_i \to H_i$. Finally, let $\theta_{i,\rho}$ be the (irreducible) character of G obtained by induction from the irreducible character $\alpha_i \otimes \widetilde{\rho}$ of G_i. One then has the following

Theorem 4.1. *Each irreducible character of G is obtained in unique fashion as $\theta_{i,\rho}$.*

5. Permution character of the multiplication group

This section uses the little-groups method to analyze the permutation character of the natural action of the multiplication group of a finite central pique.

Theorem 5.1. *For a finite central pique P, with multiplication group G and inner multiplication group H, let H act from the left on $\widehat{B}(P)$ by ${}^h\beta(b) = \beta(b^h)$ for $h \in H$, $\beta \in \widehat{B}(P)$, and $b \in P$. Let $\{\beta_1, \dots \beta_s\}$ be a set of representatives for the orbits of H on $\widehat{B}(P)$. For $1 \le i \le s$, let H_i be the stabilizer of β_i in H, and set $G_i = B(P)H_i$. Then the permutation character $\pi = 1 \uparrow_H^G$ of G on P decomposes into a sum of irreducible characters as*

$$\pi = \sum_{i=1}^{s} \beta_i \uparrow_{G_i}^G .$$

Proof. Use the "little groups" notation of Section 4. Then by two applications of Frobenius reciprocity, one has

$$\begin{aligned}
\langle 1_H \uparrow_H^G, \theta_{i,\rho} \rangle_G &= \langle 1_H, \beta_i \otimes \tilde{\rho} \uparrow_{G_i}^G \downarrow_H^G \rangle_H \\
&= \langle 1_H, \beta_i \otimes \tilde{\rho} \downarrow_{H_i}^{G_i} \uparrow_{H_i}^H \rangle_H \\
&= \langle 1_{H_i}, \beta_i \otimes \rho \rangle_{H_i}
\end{aligned}$$

for $1 \le i \le s$, the central equality holding by [5, Satz V.16.9(b)] since $G = G_iH$. By the definition (4.3) of the extension of β_i to $B(P)H_i$, the final expression reduces as

$$\frac{1}{|H_i|}\sum_{h\in H_i}(\beta_i\otimes\rho)(h)=\frac{1}{|H_i|}\sum_{h\in H_i}\rho(h)=\langle 1,\rho\rangle_{H_i},$$

which is 1 for ρ trivial, and 0 otherwise. □

6. The dual of a central pique

Let P be a finite central pique, with multiplication group G, inner multiplication group H, and pointed idempotent 0. The cloop $B(P)$ of P is a finite abelian group, with dual abelian group $\widehat{B}(P)$. Let H act from the left on $\widehat{B}(P)$ by ${}^h\beta(b)=\beta(b^h)$ for $h\in H$, $\beta\in\widehat{B}(P)$, and $b\in P$. One may then define a quasigroup operation on $\widehat{B}(P)$ by

$$\xi\cdot\eta={}^R\xi+{}^L\eta. \tag{6.1}$$

Definition 6.1. The pique $\widehat{P}$ *dual* to P consists of the set $\widehat{B}(P)$ equipped with the quasigroup operation (6.1), and pointed by the trivial character 0 of $B(P)$.

The following result is an easy consequence of duality theory for abelian groups.

Proposition 6.2. *Let P be a finite central pique, with inner multiplication group H.*

(1) *The inner multiplication group of $\widehat{P}$ is isomorphic to H.*
(2) *The double dual $\widehat{\widehat{P}}$ of P is naturally isomorphic to P.*

7. Characters of a central pique

Continuing the notation of the previous two sections, suppose that the pique conjugacy classes of a finite central pique P are $\{0\}=D_1,D_2,\dots,D_s$. Suppose that the pique conjugacy classes of $\widehat{P}$ are $\{0\}=\Delta_1,\Delta_2,\dots,\Delta_s$. In each case, these classes are the orbits of the corresponding right or left action of the group H, according to Proposition 6.2 (1). Suppose that $\beta_i\in\Delta_i$ for $1\le i\le s$. Use regularity to identify elements of central piques with the corresponding multiplications in the abelian cloops. The character table of P is then specified by the following theorem.

Theorem 7.1. *For $1\le i,j\le s$, the i,j-entry of the character table of the finite central pique P is given by*

$$\psi_{ij}=\frac{1}{n_j\sqrt{f_i}}\sum_{\beta\in\Delta_i}\sum_{b\in D_j}\beta(b). \tag{7.1}$$

Proof. By Theorem 5.1, the complex linear representation of the multiplication group G of P given by the natural permutation action of G on P decomposes as a direct sum of mutually inequivalent linear representations λ_i for $1 \leq i \leq s$, the character of each λ_i being $\beta_i \uparrow_{G_i}^{G}$. By [18, Th. 526], for $1 \leq j \leq s$ one has

$$A_j = \sum_{i=1}^{s} \sum_{b \in D_j} \lambda_i(R_+(b)). \tag{7.2}$$

For each $1 \leq i \leq s$, comparison of (7.2) with (3.1) yields $f_i \xi_{ji} = \operatorname{tr} \xi_{ji} E_i = \sum_{b \in D_j} \operatorname{tr} \lambda_i(R_+(b)) = \sum_{b \in D_j} \beta_i \uparrow_{G_i}^{G} (R_+(b))$. Now by the definition of character induction,

$$\beta_i \uparrow_{G_i}^{G} (R_+(b)) = \frac{1}{|H_i|} \sum_{h \in H} \beta_i(R_+(b)^{h^{-1}}) \tag{7.3}$$

for any b in B. Using the action of H on $\widehat{B}(P)$, (7.3) may be rewritten as

$$\beta_i \uparrow_{G_i}^{G} (R_+(b)) = \frac{1}{|H|} \sum_{h \in H} {}^{h^{-1}}\beta_i(R_+(b)) = \sum_{\beta \in \Delta_i} \beta(R_+(b)). \tag{7.4}$$

Thus

$$\psi_{ij} = \frac{\sqrt{f_i}}{n_j} \xi_{ji} = \frac{1}{n_j \sqrt{f_i}} \sum_{b \in D_j} \sum_{\beta \in \Delta_i} \beta(b),$$

using the identification provided by the regularity. □

Corollary 7.2. *For $1 \leq i \leq s$, one has $f_i = |\Delta_i|$.*

Proof. Set $b = 0$ in (7.4). □

Corollary 7.3. *The unitary character table of the central pique P of finite order n is given by*

$$U_{ij} = \frac{1}{\sqrt{n}} \sqrt{\frac{|D_j|}{|\Delta_i|}} \sum_{\beta \in \Delta_i} \sum_{b \in D_j} \beta(b) \tag{7.5}$$

for $1 \leq i, j \leq s$.

Corollary 7.4. *The unitary character table of the dual of a finite central pique is the inverse of the unitary character table of the primal.*

8. The double of a central pique

Let P be a finite central pique, with inner multiplication group H and dual $\widehat{P}$. Let S be a commutative, unital ring. Let SP and $S\widehat{P}$ be free S-modules with respective bases P and $\widehat{P}$. Let SH be the group algebra of H over S. The right action of H on P extends to make SP a right SH-module. Dually, the left action of H on $\widehat{P}$ extends to make $S\widehat{P}$ a left SH module. The multiplications on P and $\widehat{P}$ extend by distributivity to respective multiplications on SP and $S\widehat{P}$, making these structures into *pique algebras* over S. Specifically, the right SH-module SP with the product given formally by (2.2) is called a *primal* pique algebra, while the left SH-module $S\widehat{P}$ with the product given formally by (6.1) is called a *dual* pique algebra. Use the tensor product terminology of [15, §5.1].

Definition 8.1. The S-*double* $D_S(P)$ of the finite central pique P is the tensor product of the right SH-module SP with the left SH-module $S\widehat{P}$. The (*integral*) *double* of P is $D(P) = D_{\mathbb{Z}}(P)$.

In the S-double, the multiplications on SP and $S\widehat{P}$ will bind more strongly than the tensor product. Note that $D_S(P)$ is again a free S-module, with basis given by

$$\{b \otimes \beta \mid b \in P, \beta \in \widehat{P}\}. \tag{8.1}$$

Proposition 8.2. *There is an abelian group homomorphism*

$$\mathrm{ev} : D(P) \to S^1 \tag{8.2}$$

from the double to the abelian group of complex numbers of unit modulus, defined on elements of the basis (8.1) *by* $b \otimes \beta \mapsto \beta(b)$.

Proof. Apply [15, Th. V.1.1]. The map

$$P \times \widehat{P} \to S^1; (b, \beta) \mapsto \beta(b)$$

is linear in its first argument since its second argument is a character. It is linear in the second argument by the definition of the character group of an abelian group. Finally, it is middle associative by the definition of the left action of H on $\widehat{P}$. □

The map (8.2) is known as *evaluation.*

9. Words in pique algebras

Let l be a natural number. Consider the set of natural numbers less than 2^l, expressed by their binary expansions of length l. (For $l = 0$, the natural number 0 is expressed by the empty expansion.)

This section discusses the formation of certain repeated products in pique algebras, known as *full words of depth l* [16, §8.1]. The basic definition is inductive. For $l = 0$, the full word of depth 0 is

$$f_0(a_0) = a_0 .$$

Then for each natural number l, the full word of depth $l+1$ is given as

$$f_{l+1}(a_0, \ldots, a_{2^{l+1}-1}) = f_l(a_0, \ldots, a_{2^l-1}) \cdot f_l(a_{2^l}, \ldots, a_{2^{l+1}-1}). \tag{9.1}$$

The suffix l on the symbol f already implies the number of arguments, namely 2^l. For a function

$$g : \{0, 1, \ldots, 2^l - 1\} \to \mathbb{N}$$

(which will often just be the embedding of the domain), it is convenient to introduce the abbreviated notation

$$f_l^j(a_{g(j)}) = f_l(a_{g(0)}, \ldots, a_{g(2^l-1)}).$$

Thus by convention, the dummy index j appearing in arguments of f_l is to run in order from $j = 0$ to $j = 2^l - 1$. In this notation, the inductive definition (9.1) takes the form

$$f_{l+1}^j(a_j) = f_l^j(a_j) \cdot f_l^j(a_{2^l+j}).$$

Now interpret each integer k or binary word of length l as a monoid word w_k over the alphabet $\{R, L\}$ by $0 \mapsto R$ and $1 \mapsto L$. These words may also be interpreted in the inner multiplication group H of P and $\widehat{P}$.

Proposition 9.1. *In the dual pique algebra $S\widehat{P}$,*

$$f_l^j(\chi_j) = \sum_{j=0}^{2^l-1} w_j \chi_j . \tag{9.2}$$

Proof. For $l = 0$, (9.2) is trivial. Suppose (9.2) holds for l. Then by (9.1) and (6.1),

$$\begin{aligned} f_{l+1}^j(\chi_j) &= f_l^j(\chi_j) \cdot f_l^j(\chi_{2^l+j}) \\ &= R f_l^j(\chi_j) + L f_l^j(\chi_{2^l+j}) \\ &= \sum_{j=0}^{2^l-1} R w_j \chi_j + \sum_{j=0}^{2^l-1} L w_j \chi_{2^l+j} \\ &= \sum_{j=0}^{2^l-1} w_j \chi_j + \sum_{j=2^l}^{2^{l+1}-1} w_j \chi_j = \sum_{j=0}^{2^{l+1}-1} w_j \chi_j . \end{aligned}$$

□

Corollary 9.2. *For a permutation λ of* $\{0, 1, \ldots, 2^l - 1\}$,

$$\sum_{j=0}^{2^l-1} w_{j\lambda}\chi_j = f_l^j(\chi_{j\lambda^{-1}}).$$

For a given natural number l, consider the permutation ρ or ρ^l of $\{0, 1, \ldots, 2^l - 1\}$ that reverses the length l binary expansions. The following proposition has an inductive proof analogous to that of Proposition 9.1, but the details are more subtle.

Proposition 9.3. *In the primal pique algebra SP,*

$$f_l^i(x_i) = \sum_{i=0}^{2^l-1} x_i w_{i\rho}. \tag{9.3}$$

Proof. For $l = 0$, (9.3) is trivial. Suppose (9.3) holds for l. Then by (9.1) and (2.2),

$$\begin{aligned} f_{l+1}^i(x_i) &= f_l^i(x_i) \cdot f_l^i(x_{2^l+i}) \\ &= f_l^i(x_i)R + f_l^i(x_{2^l+i})L \\ &= \sum_{i=0}^{2^l-1} x_i w_{i\rho^l} R + \sum_{i=0}^{2^l-1} x_{2^l+i} w_{i\rho^l} L \\ &= \sum_{i=0}^{2^l-1} x_i w_{i\rho^{l+1}} + \sum_{i=2^l}^{2^{l+1}-1} x_i w_{i\rho^{l+1}} = \sum_{i=0}^{2^{l+1}-1} x_i w_{i\rho}. \end{aligned}$$

□

Remark 9.4. Proposition 9.3 gives an explicit version of the addressing of arguments in repeated products discussed in [16, §8.1]. The universality of the description (9.3) is guaranteed by [3, Th. III.5.4].

Corollary 9.5. *For a permutation λ of* $\{0, 1, \ldots, 2^l - 1\}$,

$$\sum_{i=0}^{2^l-1} x_i w_{i\lambda} = f_l^i(x_{i\rho\lambda^{-1}}).$$

10. Loop-indexed identities

Let l be a natural number. Let L be a loop (with identity 0) defined on the set of natural numbers less than 2^l, expressed by their binary expansions of length l. Consider the multiplication table of the loop L, in which each column label $0 \leq c < 2^l$ is written as χ_c, and each row label $0 \leq r < 2^l$ is written as x_r. An example for $l = 2$ is given in Table 1.

L	χ_{00}	χ_{01}	χ_{10}	χ_{11}
x_{00}	00	01	10	11
x_{01}	01	11	00	10
x_{10}	10	00	11	01
x_{11}	11	10	01	00

Table 1. A loop of order 2^2.

Now let P be a central pique, with S-double $D_S(P)$ and inner multiplication group H. The variables x_i are interpreted in SP and the variables χ_j in $S\widehat{P}$, for $0 \leq i, j < 2^l$. If

$$ij = k$$

is a multiplication in the loop L, consider the corresponding element

$$x_i w_k \otimes \chi_j = x_i \otimes w_k \chi_j$$

of the S-double $D_S(P)$ of P. The entire loop L then indexes equal sums

$$\sum_{0 \leq i,j < 2^l} x_i w_{ij} \otimes \chi_j = \sum_{0 \leq i,j < 2^l} x_i \otimes w_{ij} \chi_j \tag{10.1}$$

in the S-double. By (IL) and Corollary 9.2, the right hand side of (10.1) parses as a sum

$$\sum_{0 \leq i < 2^l} x_i \otimes f_l^j(\chi_{i\backslash j}) \tag{10.2}$$

in $D_S(P)$. By (IR) and Corollary 9.5, the left hand side parses as a sum

$$\sum_{0 \leq j < 2^l} f_l^i(x_{i\rho/j}) \otimes \chi_j \tag{10.3}$$

in $D_S(P)$. In (10.2) and (10.3), the left and right divisions appearing in the suffices are taken from the loop L.

Definition 10.1. The identities

$$\sum_{0 \leq j < 2^l} f_l^i(x_{i\rho/j}) \otimes \chi_j = \sum_{0 \leq i < 2^l} x_i \otimes f_l^j(\chi_{i\backslash j}) \tag{10.4}$$

are called *loop-indexed identities*. More specifically, (10.4) is called the *loop-based identity* indexed by the loop L.

Summarizing,

Theorem 10.2. *Each loop L of 2-power order indexes a loop-based identity* (10.4) *holding in the S-double of each (finite) central pique.*

Example 10.3. The loop of Table 1 indexes the identity

$$\begin{aligned}&(x_0x_2\cdot x_1x_3)\otimes\chi_0+(x_2x_3\cdot x_0x_1)\otimes\chi_1\\&\qquad+(x_1x_0\cdot x_3x_2)\otimes\chi_2+(x_3x_1\cdot x_2x_0)\otimes\chi_3\\&=x_0\otimes(\chi_0\chi_1\cdot\chi_2\chi_3)+x_1\otimes(\chi_2\chi_0\cdot\chi_3\chi_1)\\&\qquad+x_2\otimes(\chi_1\chi_3\cdot\chi_0\chi_2)+x_3\otimes(\chi_3\chi_2\cdot\chi_1\chi_0).\end{aligned}$$

The two-element loop indexes the identity

$$(x_0x_1)\otimes\chi_0+(x_1x_0)\otimes\chi_1=x_0\otimes(\chi_0\chi_1)+x_1\otimes(\chi_1\chi_0).$$

This specializes to

$$(xx)\otimes\chi=x\otimes(\chi\chi).$$

Corollary 10.4. *Each loop L of 2-power order indexes the identity*

$$\prod_{0\le j<2^l}\chi_j(f_l^i(x_{i\rho/j}))=\prod_{0\le i<2^l}f_l^j(\chi_{i\backslash j})(x_i).$$

Proof. Apply the evaluation map (8.2) to (10.4) interpreted in the integral double of P. □

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