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Commutative Moufang Loops and Bessel Functions

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Abstract. Let F be a field. For each $k>1$, let G be a finite group containing $\{x_1,\ldots,x_k\}!\times\{y_1,\ldots,y_k\}!$. Then in the group algebra FG,

$$\operatorname{codim}_F \sum_{j=1}^{k-1} (1+(x_j x_{j+1}))(1+(y_j y_{j+1}))\,FG = \frac{|G|}{2\pi i} \oint_{|z|=1} \frac{dz}{J_0(2\sqrt{z})\,z^{k+1}}.$$

Connections with the theory of commutative Moufang loops are discussed, including a conjectured answer to Manin's problem of specifying the 3-rank of a finitely generated free commutative Moufang loop.

Contents

1. Introduction

This paper has two aims: to present a conjectured solution (Conjecture 8.1) to Manin's problem ([7], Vopros 10.3; [8], Problem 10.2) on the 3-rank of the free commutative Moufang loop on a given finite number of generators, and, as a first stage in the proof of this conjecture, to give an algebraic interpretation (Theorem 4.1) to the coefficients of the power series expansion of the reciprocal of the Bessel function $J_0(x)$ in a neighbourhood of 0. These coefficients were studied earlier by Carlitz [2], Forsyth [4], and Riordan [10, § 5].

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The proof of the main theorem involves two matters which may be of independent interest. The first, in Sects. 4–6, might be called "combinatorial topology in the strong sense": the homology of a complex with an elaborate system of subcomplexes, where the combinatorics of this system plays an important rôle. Generalised "boundary mappings" of degrees lower than -1 arise - the ε_k of Sect. 5. The second matter, in Sect. 7, is the theory of compositions or ordered partitions of positive integers.

In order to make the material more accessible the relevant facts about commutative Moufang loops are mentioned briefly. Further details, which often turn out to be extremely involved, may be found in [1, 11], and [12].

2. Basic Facts about Commutative Moufang Loops

A *commutative Moufang loop* or CML for short $(L, ., 1)$ is a set L with a binary operation (*multiplication*) denoted by . or juxtaposition and a (nullary operation selecting an) *identity* element 1 such that the following axioms are satisfied:

(2.1) In the equation $x \cdot y = z$ knowledge of any two of x, y, z in L specifies the third uniquely, i.e. $(L, .)$ is a *quasigroup.*

(2.2) For x in L, $x \cdot 1 = x = 1 \cdot x$, i.e. $(L, ., 1)$ becomes a *loop.*

(2.3) For x, y, z in L, $x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z$, i.e. $(L, ., 1)$ becomes a *Moufang* loop.

(2.4) For x, y in L, $x \cdot y = y \cdot x$ (*commutativity*).

As a consequence of (2.1) to (2.3), each two-element subset of L generates an associative subloop of $(L, ., 1)$ ([1], VIII.4). One may thus apply the usual group-theoretic notions of inverse and exponent to (commutative) Moufang loops.

Groups are associative but not necessarily commutative. CMLs are commutative but not necessarily associative ([1], VIII.1). To a certain extent CML theory is like group theory with the rôles of commutativity and associativity exchanged. Just as the deviation of a group from commutativity is measured by the commutator $[x, y] = (yx)^{-1}xy$ of two elements x, y, the deviation of a CML from associativity is measured by the *associator* $(x, y, z) = (x(yz))^{-1}((xy)z)$ of three elements x, y, z. A group commutator is skew-symmetric in its arguments: $[y, x] = [x, y]^{-1}$. Similarly, a CML associator is skew-symmetric in its arguments: $(y, z, x) = (x, y, z) = (y, x, z)^{-1}$ ([1], VIII(2.3)). A group G has a lower central series $G_0 = G, \ldots, G_i = [G_{i-1}, G], \ldots$, where $[G_{i-1}, G]$ is the subgroup of G generated by $\{[x, y] | x \in G_{i-1}, y \in G\}$. Similarly, a CML L has a *lower central series* $L_0 = L, \ldots, L_i = (L_{i-1}, L, L), \ldots$, where (L_{i-1}, L, L) is the subloop of L generated by $\{(x, y, z) | x \in L_{i-1}, y, z \in L\}$. As in the group case, a CML L is said to be *nilpotent of class* c if $L_{c-1} > L_c = \{1\}$. The terms of the lower central series of a loop are *normal subloops*, i.e. equivalence classes of congruences on the loop, and the corresponding quotients may be realised by cosets under multiplication, just as for groups. In group theory the notation

$[a, b, c, \ldots, u, v]$ is useful for the left-normalised repeated commutator $[[[\ldots[[a, b], c], \ldots], u], v]$. In CMLs $(a, b, c!\, d, e!\ldots!\, u, v!\, w, x)$ is equally useful as an abbreviation of $(((\ldots((a, b, c), d, e), \ldots), u, v), w, x)$.

Despite these and other analogies, however, the theory of CMLs displays far more special features than the theory of groups. The first sign of this is the stronger form of skew-symmetry enjoyed by associators: $(x^{-1}, y, z) = (x, y, z)^{-1}$ ([1], Lemma VII.5.5). Only a narrow class of groups satisfies the analogous $[x^{-1}, y] = [x, y]^{-1}$ ([1], Lemma VII.5.3; [14], Proposition 2). The prime 3 also plays a distinguished rôle in CML theory, far more marked than the special rôle of the prime 2 in group theory. For example, the *derived loop* L_1 of a CML L has exponent 3 ([1], Lemma VII.5.7). Above all, there is the deep Bruck-Slaby Theorem ([1], Theorem VIII.10.1) stating that a CML generated by n elements is nilpotent of class less than n.

Let L now denote the free CML on an n-element set X of generators, $n > 2$. From [6] or [12] it follows that L has the lower central series $L = L_0 > L_1 > L_2 > \ldots > L_{n-1} = \{1\}$. The successive factors $L_1/L_2, \ldots, L_{n-2}/L_{n-1}$ are elementary abelian groups of exponent 3, having the permutation group $X!$ of X acting as a group of automorphisms. One may thus consider $L_1/L_2 \oplus \ldots \oplus L_{n-2}/L_{n-1}$ as a right module for the group algebra $GF(3)\,X!$ of $X!$ over the Galois field $GF(3)$ of three elements. Manin's problem is to specify the dimension $\delta(n)$ of this module.

3. Some Critical Identities for Repeated Associators

In a CML one has the identities

$$(a, b, c!\, d, e)(b, c, d!\, e, a)(c, d, e!\, a, b)(d, e, a!\, b, c)(e, a, b!\, c, d) = 1 \tag{3.1}$$

([11], §4) and

$$(e, a, b!\, c, d) = (e, c, d!\, a, b)(a, c, d!\, b, e)(b, c, d!\, e, a) \tag{3.2}$$

([11], 2.3) ([1], Lemma VIII.6.4). Note that all the factors here lie in an abelian group A, so the products are unambiguous. Since derived loops have exponent 3, (3.1) may be rewritten as

$$\begin{gathered}(a, b, c!\, d, e)(c, d, e!\, a, b)(e, a, b!\, c, d)^{-1}(b, c, d!\, e, a)\\ (e, a, b!\, c, d)^{-1}(d, e, a!\, b, c) = 1.\end{gathered} \tag{3.3}$$

By (3.2) and the strong skew-symmetry of associators, $(c, d, a!\, e, b) = (c, d, e!\, a, b)(e, a, b!\, c, d)^{-1}(b, c, d!\, e, a)$. Using this to rewrite the second through fourth factors of (3.3) gives

$$(a, b, c!\, d, e)(c, d, a!\, e, b)(e, a, b!\, c, d)^{-1}(d, e, a!\, b, c) = 1,$$

or, applying the strong skew-symmetry again,

$$(a, b, c!\, d, e)(a, d, c!\, b, e)(a, b, e!\, d, c)(a, d, e!\, b, c) = 1.$$

On setting $z=a$, $x_1=b$, $y_1=c$, $x_2=d$, $y_2=e$, and passing to right $GF(3)\{x_1,y_1,x_2,y_2\}!$-module notation, this identity takes the suggestive form

$$(z,x_1,y_1!x_2,y_2)(1+(x_1x_2)+(y_1y_2)+(x_1x_2)(y_1y_2))=0. \tag{3.4}$$

More generally, consider the repeated associator $(z,x_1,y_1!\dots!x_k,y_k)$ and its images under permutations of $\{x_1,y_1,\dots,x_k,y_k\}$. By the Bruck-Slaby Theorem these images all lie in an abelian group. Let ξ_i, η_i respectively denote the transpositions (x_ix_{i+1}), (y_iy_{i+1}) for $1\leqq i<k$. Using right $GF(3)\{x_1,y_1,\dots,x_k,y_k\}!$-module and conventional CML notations,

$$\begin{aligned}
&(z,x_1,y_1!\dots!x_k,y_k)(1+\xi_i+\eta_i+\xi_i\eta_i)\\
&=(z,x_1,y_1!\dots!x_i,y_i!x_{i+1},y_{i+1}!\dots)(z,x_1,y_1!\dots!x_{i+1},y_i!x_i,y_{i+1}!\dots),\\
&\quad(z,x_1,y_1!\dots!x_i,y_{i+1}!x_{i+1},y_i!\dots)(z,x_1,y_1!\dots!x_{i+1},y_{i+1}!x_i,y_i!\dots)\\
&=(((z,x_1,y_1!\dots!x_{i-1},y_{i-1}),x_i,y_i!x_{i+1},y_{i+1})\\
&\quad\cdot(1+\xi_i+\eta_i+\xi_i\eta_i)!x_{i+2},y_{i+2}!\dots)\\
&=(1,x_{i+2},y_{i+2}!\dots)=0.
\end{aligned}$$

The second equality here follows from the general multiplication formula ([11], Proposition 10.3(i)) and ([11], Lemma 6.1). The third follows from (3.4) with z replaced by $(z,x_1,y_1!\dots!x_{i-1},y_{i-1})$ and x_1, y_1, x_2, y_2 by x_i, y_i, x_{i+1}, y_{i+1} respectively. Summarising, for $1\leqq i<k$,

$$(z,x_1,y_1!\dots!x_k,y_k)(1+\xi_i)(1+\eta_i)=0. \tag{3.5}$$

A key step in the solution of Manin's problem on the 3-rank is the determination of the dimension of the $GF(3)\{x_1,y_1,\dots,x_k,y_k\}!$-module generated by $(z,x_1,y_1!\dots!x_k,y_k)$, i.e. the codimension of the annihilator of $(z,x_1,y_1!\dots!x_k,y_k)$. The identity (3.5) shows that this annihilator contains all the $(1+\xi_i)(1+\eta_i)$ for $1\leqq i<k$. One is thus led to study the right ideal of $GF(3)\{x_1,y_1,\dots,x_k,y_k\}!$ generated by $\{(1+\xi_i)(1+\eta_i)|1\leqq i<k\}$. It is this right ideal which yields the algebraic interpretation of the coefficients of the reciprocal of the Bessel function $J_0(x)$.

4. An Algebraic Interpretation of the Coefficients of $1/J_0$

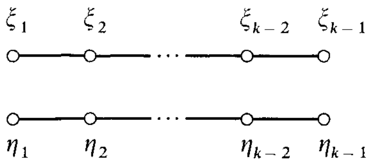
The next four sections, which are self-contained, are concerned with the proof of the following theorem:

Theorem 4.1. *Let G be a finite group containing the product $\{x_1,\dots,x_k\}!\times\{y_1,\dots,y_k\}!$ of the symmetric groups on the sets $\{x_1,\dots,x_k\}$, $\{y_1,\dots,y_k\}$ respectively, $k\geqq 2$. Let F be a field, and $R=FG$ the group algebra of G over F. For $1\leqq i<k$, let $\xi_i=(x_ix_{i+1})$, $\eta_i=(y_iy_{i+1})$. Then*

$$\operatorname{codim}_F\sum_{j=1}^{k-1}(1+\xi_j)(1+\eta_j)R=\frac{|G|}{2\pi i}\oint_{|z|=1}\frac{dz}{J_0(2\sqrt{z})\,z^{k+1}}. \tag{4.2}$$

Let S denote the group algebra of the subgroup $H=\{x_1,\ldots,x_k\}! \times\{y_1,\ldots,y_k\}!$ of G, considered as a subring of R. The group H is freely generated by the set $\{\xi_1,\ldots,\xi_{k-1},\eta_1,\ldots,\eta_{k-1}\}$ subject to the relations specified in the graph $2A_{k-1}$

(4.3)



that two generators commute if the vertices they label are not joined by an edge, that their product has order 3 if the corresponding vertices are joined, and that each generator has order 2. (Cf. [3], particularly §6.2, for this and some subsequent notions.) One may form the Cayley diagram of H with respect to this presentation, and embed it in Euclidean space of dimension at least $2k-2$ as the vertices and edges of the generalised prism $\Pi_{k-1}\times\Pi_{k-1}$. The prism is subdivided into cells consisting of the convex hulls of the right cosets of the subgroups of H generated by the various subsets of $\{\xi_1,\ldots,\xi_{k-1},\eta_1,\ldots,\eta_{k-1}\}$. These cells may be formed into an abstract cell complex K^k (as in [5], §2.12), taking the (abstract) dimension of a cell to be its geometric dimension as a convex subset of the Euclidean space, and deriving incidence numbers from an orientation of $\Pi_{k-1}\times\Pi_{k-1}$. The complex K^k is contractible, since the convex figure $\Pi_{k-1}\times\Pi_{k-1}$ is. Positive integers will be thought of as colours, and edges of the Cayley diagram corresponding to ξ_i or η_i will be said to be *coloured* with the integer i. Note that the dual of the boundary of K^k is Tits' *Coxeter complex* for $2A_{k-1}$.

For each pair of subsets α, β of $\{1,\ldots,k-1\}$, K^k has a subcomplex $K^k(\alpha\times\beta)$ whose cells are precisely the convex hulls of the right cosets of the subgroups of H generated by the various subsets of $\{\xi_i|i\in\alpha\}\cup\{\eta_i|i\in\beta\}$. Note that the sets of O-cells of K^k and of each $K^k(\alpha\times\beta)$ are identical – namely the set H of vertices of $\Pi_{k-1}\times\Pi_{k-1}$. The subcomplex $K^k(\{1,\ldots,k-2\}^2)$ for $k>2$ may be regarded as a disjoint union of k^2 copies of K^{k-1}. For induction it is convenient to denote $K^k(\{1,\ldots,k-2\}^2)$ by $\bar{K}^{k-1}$, and to extend the $\bar{\ }$-notation to other features of K^{k-1} correspondingly reproduced k^2 times within K^k. This is known as the $\bar{\ }$*-convention*.

Since the complex K^k is finite, the sets K_p^k of F-linear combinations of p-cells of K^k and $C^p(K^k;F)$ of functions from the set of p-cells of K^k to F may be considered as F-vector spaces. There are linear boundary mappings ∂: $K_p^k\to K_{p-1}^k$ (cf. [5], 2.12.4); the boundary of the cell consisting of the convex hull of a right coset of $\langle\xi_i,\eta_j|i\in\alpha,j\in\beta\rangle$ (for subsets α, β of $\{1,\ldots,k-1\}$) is a linear combination of right cosets of $\langle\xi_i,\eta_j|i\in\alpha-\{i_0\},\ j\in\beta\rangle$ and of $\langle\xi_i,\eta_j|i\in\alpha,\ j\in\beta-\{j_0\}\rangle$ as i_0 and j_0 range over α, β respectively.

The space $C^0(K^k;F)$ may be identified with the dual S' of the F-vector space S. Under the isomorphism $'$ from S to S' sending H to a dual basis, $(1+\xi_i)(1+\eta_i)S$ corresponds to the set of functions taking equal values at every vertex of each of the cells of $K_2^k(\{i\}^2)$. Identifying $K_0^k=K_0^k(\{i\}^2)$ with the dual of S', the annihilator of this set of functions is just the boundary $K_1^k(\{i\}^2)\partial$ of the set of linear combinations of i-coloured edges. Thus the annihilator of the subspace of S' corresponding to the subspace $\sum_{i=1}^{k-1}(1+\xi_i)(1+\eta_i)S$ of S under $'$: $S\to S'$ is $A_k=\bigcap_{i=1}^{k-1}K_1^k\partial$. The proof of Theorem 4.1 reduces to showing that the dimension of A_k is $\frac{(k!)^2}{2\pi i}\oint_{|z|=1}\frac{dz}{J_0(2\sqrt{z})\,z^{k+1}}$. To this end a complex over A_k is constructed in the next section, and shown to be a resolution of A_k in § 6. This resolution is then used to count dimensions in § 7.

5. A Complex Over A_k

Call a cell of K^k *full* if it contains an edge of each colour, i.e. if it is the convex hull of a right coset of a subgroup of H that contains either ξ_i or η_i for each $i=1,\dots,k-1$. A bounding cell of a non-full cell cannot be full, so the non-full cells form a subcomplex W^k of K^k, namely

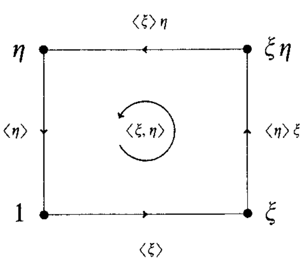
$$W^k=K^k(\{2,3,\dots,k-1\}^2)\cup K^k(\{1,3,\dots,k-1\}^2)\cup\dots$$
$$\cup K^k(\{1,2,\dots,k-2\}^2).$$

Let X^k be the direct complement of W^k in K^k spanned by the full cells, with projections α_W: $K^k\to X^k$ and β_W: $K^k\to W^k$. Let $\hat{\partial}=\partial\alpha_W$ denote the boundary map in the complex X^k.

A linear map ε_k: $K^k_{k-1}\to A_k$ will be defined so that

$$(5.1)\qquad 0\to X^k_{2k-2}\xrightarrow{\hat{\partial}}X^k_{2k-3}\xrightarrow{\hat{\partial}}\dots\xrightarrow{\hat{\partial}}X^k_k\xrightarrow{\hat{\partial}}X^k_{k-1}\xrightarrow{\varepsilon_k}A_k\to 0$$

is a complex of F-vector spaces. For each $j=1,2,\dots,k-1$, define a j-dimensional subcell of a full cell to be *initial* if it contains edges of each of the colours $1,2,\dots,j$ (and thus is itself a full cell in one of the subcomplexes $\bar{K}^{k-1}$, $\bar{\bar{K}}^{k-2}$, etc.). For example, the initial 1-cells are just the 1-coloured edges. The desired effect of ε_k on a full cell c with its orientation may then be described as follows. Take the initial cells in the boundary of c with the orientation induced in the boundary, then the initial cells in their boundary with the induced orientation, and so on down to the 1-coloured edges with the induced orientation. Now define a linear mapping e_k: $X^k_{k-1}\to K^k_0$ by taking ce_k, the desired $c\varepsilon_k$, to be the boundary of these oriented 1-coloured edges. For $k=2$ one may realise $\Pi_{k-1}\times\Pi_{k-1}$ as the unit square $[0,1]\times[0,1]$ in $\mathbb{R}^2$, oriented as shown in the figure:



Here the cells have been assigned their "group-theoretical" names with $\xi=\xi_1$, $\eta=\eta_1$, e.g. $[0,1]\times\{1\}$ is called $\langle\xi\rangle\eta$, the point $(1,0)$ is called ξ, and $[0,1]\times[0,1]$ is called $\langle\xi,\eta\rangle$. The space X_2^2 is spanned by $\{\langle\xi,\eta\rangle\}$, X_0^2 by $\{\langle\xi\rangle,\langle\eta\rangle,\langle\xi\rangle\eta,\langle\eta\rangle\xi\}$, and K_0^2 by $\{1,\xi,\eta,\xi\eta\}$. Note that $\langle\xi,\eta\rangle\partial=\langle\xi\rangle+\langle\eta\rangle\xi+\langle\xi\rangle\eta+\langle\eta\rangle$. The space A_2 is just the (double dual of the) augmentation ideal of $F\langle\xi,\eta\rangle$. The space X_1^2 is all of K_1^2 (i.e. $W_1^2=0$), and ε_2: $X_1^2\to A_2$ is defined to be the boundary map ∂ of K^2, e.g. $\langle\eta\rangle\xi\varepsilon_2=\langle\eta\rangle\xi\partial=\xi\eta-\xi$. Since $\partial\varepsilon_2=\partial^2$ is the zero mapping on X_2^2, (5.1) becomes a complex in this case.

Now consider the case $k>2$. Since $K^{k-1}=X^{k-1}\oplus W^{k-1}$, one also has $\bar{K}^{k-1}=\bar{X}^{k-1}\oplus\bar{W}^{k-1}$, in accordance with the $^-$-convention. Further, there is a direct sum decomposition $K^k=\bar{K}^{k-1}\oplus(K^k/\bar{K}^{k-1})$, where $K^k/\bar{K}^{k-1}$ is realised on the subspaces of each K_i^k spanned by the i-dimensional cells of K_i^k not in $\bar{K}_i^{k-1}$. Let the corresponding projections be $\alpha_{\bar{K}}$: $K^k\to K^k/\bar{K}^{k-1}$ and $\beta_{\bar{K}}$: $K^k\to\bar{K}^{k-1}$. By induction on k a map ε_k: $K_{k-1}^k\to A_k$ will be defined, with the properties:

(5.2) (i) ε_k restricts to the zero mapping on W_{k-1}^k;

(ii) $\partial\varepsilon_k=0$: $K_k^k\to A_k$;

(iii) for a full $(k-1)$-dimensional cell c, $c\varepsilon_k=ce_k$.

This has already been done for $k=2$. Suppose it done for $k-1$; then according to the $^-$-convention there is a map $\bar{\varepsilon}_{k-1}$: $\bar{K}_{k-2}^{k-1}\to\bar{A}_{k-1}$ restricting to zero on $\bar{W}_{k-1}^{k-1}$ and satisfying $\partial\beta_{\bar{K}}\bar{\varepsilon}_{k-1}=0$: $\bar{K}_{k-1}^{k-1}\to\bar{A}_{k-1}$. The map ε_k is then defined to be the composite

$$K_{k-1}^k\xrightarrow{\partial}K_{k-2}^k\xrightarrow{\beta_{\bar{K}}}\bar{K}_{k-2}^{k-1}\xrightarrow{\bar{\varepsilon}_{k-1}}\bar{A}_{k-1}.$$

To verify property (i) of ε_k, consider cells in W_{k-1}^k. These cells fall into two classes: those in $\bar{K}_{k-1}^{k-1}$, and those not. Now $\bar{K}_{k-1}^{k-1}\partial$ lies in $\bar{K}_{k-2}^{k-1}$, on which $\beta_{\bar{K}}$: $K_{k-2}^k\to\bar{K}_{k-2}^{k-1}$ acts as the identity. By property (ii) of ε_{k-1}, and correspondingly of $\bar{\varepsilon}_{k-1}$, it follows that $\varepsilon_k=\partial\beta_{\bar{K}}\bar{\varepsilon}_{k-1}=0$: $\bar{K}_{k-1}^{k-1}\to\bar{A}_{k-1}$. Now a cell of W_{k-1}^k not in $\bar{K}_{k-1}^{k-1}$ is in the convex hull of a right coset of $\langle\xi_i,\eta_j|i\in\alpha,j\in\beta\rangle$ for some $\alpha,\beta\subseteq\{1,\dots,k-1\}$ with $k-1\in\alpha\cup\beta$, $|\alpha|+|\beta|=k-1$, and $|\alpha\cap\beta|>0$. Bounding cells of such cells either lie in $\bar{W}_{k-1}^{k-1}$ or off $\bar{K}^{k-1}$. Cells of the first kind are preserved by $\beta_{\bar{K}}$, but are then mapped to zero by $\bar{\varepsilon}_{k-1}$, while cells of the second kind are immediately killed off by $\beta_{\bar{K}}$. The upshot is that a cell of W_{k-1}^k not lying in $\bar{K}_{k-1}^{k-1}$ is also mapped to zero by $\partial\beta_{\bar{K}}\bar{\varepsilon}_{k-1}$. Thus property (i) for ε_k is verified. Property (ii) is immediate: $\partial\varepsilon_k=\partial\partial\beta_{\bar{K}}\bar{\varepsilon}_{k-1}=0$. Property (iii) follows from the definition of ε_k and the corresponding property of $\bar{\varepsilon}_{k-1}$.

It remains to show that $K_{k-1}^k\varepsilon_k$ – or, what is now seen to be equivalent, $X_{k-1}^k\varepsilon_k$ – is a subspace of A_k. A priori $X_{k-1}^k\varepsilon_k$ is a subspace of $\bar{A}_{k-1}$, i.e. of $\bigcap_{i=1}^{k-2}K_1^k(\{i\}^2)\partial$. Thus it suffices to check that $X_{k-1}^k\varepsilon_k\leq K_1^k(\{k-1\}^2)\partial$. Consider a $(k-1)$-coloured edge of a full $(k-1)$-dimensional cell b. Each of the endpoints x_0, x_1 of this edge lies on precisely one edge of each colour $1,2,\dots,k-2$, these $k-2$ edges determining respective initial cells c_0, c_1. The cells c_0 and c_1 are oriented within the boundary $b\partial$ of b. It must be shown that the points x_0 and

x_1 appear with opposite signs on applying ε_k to b, and thus on applying $\bar{\varepsilon}_{k-1}$ to c_0 and c_1. Let c_2 denote the $(k-2)$-dimensional cell of b containing x_0, x_1 and having edges coloured $1, 2, \ldots, k-3$, $k-1$. Let d_0, d_1 be the initial cells of dimension $k-3$ at the points x_0, x_1 respectively determined by the edges there of colours $1, 2, \ldots, k-3$. If d_0 and d_1 are given their orientations within the boundaries of c_0 and c_1, it must be shown that the points x_0 and x_1 appear with opposite signs on applying $\bar{\bar{\varepsilon}}_{k-2}$ to d_0 and d_1. But d_0 and d_1 also appear within the boundary of c_2, their orientations there being opposite to their orientations within the respective boundaries of c_0 and c_1 in order that $b\partial\partial$ be zero. It thus suffices to note that x_0 and x_1 appear with opposite signs on applying $\bar{\bar{\varepsilon}}_{k-2}$ to d_0 and d_1 with their orientations in the boundary of c_2, this following by the induction hypothesis, the rôles of the colours $k-2$ and $k-1$ being interchanged.

6. The Complex is a Resolution

This section comprises the proof that the complex (5.1) over A_k is actually a resolution of A_k. For exactness of (5.1) at X^k_{i+k-1}, $i=0, \ldots, k-1$, one needs the following

Lemma 6.1. *Let $(a_1, \ldots, a_s)$ be a sequence of subsets of $\{1, \ldots, k\}$. Then for $r \geqq s$, $H_r(K^k(a_1^2) \cup \ldots \cup K^k(a_s^2)) = 0$.*

Proof. For $s=1$, $K^k(a_1^2)$ is homotopic to a set of isolated points, so $H_r(K^k(a_1^2)) = 0$ for $r \geqq 1$. For $s>1$, suppose $r \geqq s$. By an induction hypothesis on s, $H_r(K^k(a_1^2) \cup \ldots \cup K^k(a_{s-1}^2)) = 0$ and

$$\begin{aligned} &H_{r-1}((K^k(a_1^2) \cup \ldots \cup K^k(a_{s-1}^2)) \cap K^k(a_s)) \\ &= H_{r-1}((K^k(a_1^2) \cap K^k(a_s^2)) \cup \ldots \cup (K^k(a_{s-1}^2) \cap K^k(a_s^2))) \\ &= H_{r-1}(K^k((a_1 \cap a_s)^2) \cup \ldots \cup K^k((a_{s-1} \cap a_s)^2)) = 0. \end{aligned}$$

Exactness of the piece of Mayer-Vietoris sequence ([5], p. 93)

$$\begin{aligned} &H_r(K^k(a_1^2) \cup \ldots \cup K^k(a_{s-1}^2)) \oplus H_r(K^k(a_s^2)) \\ &\to H_r(K^k(a_1^2) \cup \ldots \cup K^k(a_{s-1}^2) \cup K^k(a_s^2)) \\ &\quad \to H_{r-1}((K^k(a_1^2) \cup \ldots \cup K^k(a_{s-1}^2)) \cap K^k(a_s^2)) \end{aligned}$$

then implies that $H_r(K^k(a_1^2) \cup \ldots \cup K^k(a_s^2)) = 0$.

Remark. The requirement that $r \geqq s$ in Lemma 6.1 is essential, e.g. $H_0(K^2(\{1\}^2)) \neq 0$, $H_1(K^3(\{1\}^2) \cup K^3(\{2\}^2)) \neq 0$, etc.

Exactness of (5.1) at X^k_{i+k-1} for $i>0$ is just the vanishing of $H_{i+k-1}(X^k)$. This follows from the exact homology sequence [5, 2.9.1]

$$H_{i+k-1}(K^k) \to H_{i+k-1}(X^k) \to H_{i+k-2}(W^k).$$

Here, $H_{i+k-1}(K^k) = 0$ since K^k is contractible, while $H_{i+k-2}(W^k) = 0$ is an application of Lemma 6.1 with $r = i+k-2$, $s = k-1$, $a_j = \{1, \ldots, k-1\} - \{j\}$ for j

$=1,\dots,k-1$. Exactness of (5.1) at X_{k-1}^k is more complicated, following from a proof by induction over k that $\ker \varepsilon_k = X_k^k \partial \alpha_W + W_{k-1}^k$. This is certainly true for $k=2$. For $k>2$, recall first that by (5.2)(i), (ii), $\ker \varepsilon_k \geqq X_k^k \partial \alpha_W + W_{k-1}^k$. Now suppose x in X_{k-1}^k satisfies $x\varepsilon_k=0$, i.e. by induction $x\partial\beta_{\bar{K}} \in \bar{X}_{k-1}^{k-1}\partial\beta_{\bar{K}}\alpha_{\bar{W}} + \bar{W}_{k-2}^{k-1}$, say via $\bar{y}\in\bar{X}_{k-1}^{k-1}$ and $\bar{w}\in\bar{W}_{k-2}^{k-1}$ such that $x\partial\beta_{\bar{K}} = \bar{y}\partial\beta_{\bar{K}}\alpha_{\bar{W}}+\bar{w}$. The aim is to find a y in X_k^k with $y\hat{\partial}=x$. Consider $\bar{y}\partial\beta_{\bar{K}} = \bar{y}\partial\beta_{\bar{K}}\alpha_{\bar{W}} + \bar{y}\partial\beta_{\bar{K}}\beta_{\bar{W}}$. Now $\alpha_{\bar{K}} + \beta_{\bar{K}}=1$ and $\alpha_{\bar{W}}+\beta_{\bar{W}}=1$, so

$$
\begin{aligned}
(x-\bar{y})\partial &= (x-\bar{y})\partial\alpha_{\bar{K}} + (x-\bar{y})\partial\beta_{\bar{K}} \\
&= (x-\bar{y})\partial\alpha_{\bar{K}} + \bar{y}\partial\beta_{\bar{K}}\alpha_{\bar{W}} + \bar{w} - \bar{y}\partial\beta_{\bar{K}} = (x-\bar{y})\partial\alpha_{\bar{K}} + (\bar{w} - \bar{y}\partial\beta_{\bar{K}}\beta_{\bar{W}}) = z,
\end{aligned}
$$

say. Since $z\partial = (x-\bar{y})\partial\partial=0$, $z\in Z_{k-2}(K^k)$. However, z also lies in $(K^k/\bar{K}^{k-1})_{k-2} + \bar{W}_{k-2}^{k-1}$.

Let $\tilde{K} = K^k(a_1^2)\cup\dots\cup K^k(a_{k-2}^2)$, where $a_i=\{1,\dots,k-1\}-\{i\}$. $\tilde{K}$ is the subcomplex of W consisting of those cells for which at least one of $1,\dots,k-2$ does not appear as an edge colour. As usual, there is a direct sum decomposition $W^k=\tilde{K}\oplus(W^k/\tilde{K})$ with projections $\alpha_{\tilde{K}}$: $W^k\to W^k/\tilde{K}$ and $\beta_{\tilde{K}}$: $W^k\to\tilde{K}$. Now $(K^k/\bar{K}^{k-1})_{k-2}\leqq\tilde{K}_{k-2}$, since a $(k-2)$-dimensional cell in $K^k/\bar{K}^{k-1}$, having $(k-1)$ as an edge colour, but at most $k-2$ edge colours, must lack one of $1,\dots,k-2$. Similarly $\bar{W}_{k-2}^{k-1}\leqq\tilde{K}_{k-2}$, since a cell of $\bar{W}^{k-1}$ does not have a full set $\{1,\dots,k-2\}$ of edge colours. Thus z lies in $Z_{k-2}(\tilde{K})$. By Lemma 6.1, $H_{k-2}(\tilde{K})=0$, so z lies in $B_{k-2}(\tilde{K})$, i.e. there is a t in $\tilde{K}_{k-1}$ such that $t\partial=z=(x-\bar{y})\partial$. Then $(x-\bar{y}-t)\partial=0$. Since K^k is contractible, there is a y' in K_k^k such that $y'\partial=x-\bar{y}-t$. Take y to be $y'\alpha_W$ in X_k^k, so that $y=y'-y'\beta_W$. Then $y\hat{\partial}=y\partial\alpha_W=(y'-y'\beta_W)\partial\alpha_W = y'\partial\alpha_W = x\alpha_W - \bar{y}\alpha_W - t\alpha_W = x$ as required.

The final task of this section is to show that ε_k: $X_{k-1}^k\to A_k$ surjects, or, in view of (5.2)(i), to show that ε_k: $K_{k-1}^k\to A_k$ surjects. Let $a\in A_k$. For $i=1,\dots,k-1$, there is a b_i in $K_1^k(\{i\}^2)$ such that $b_i\partial=a$. Let $B=b_1F+\dots+b_{k-1}F\leqq K_1^k$. The exterior algebra ΛB is a complex under the homogeneous linear mapping d of degree -1 defined by

$$
(b_{i_1}\wedge\dots\wedge b_{i_r})d = \sum_{j=1}^{r}(-1)^{r-j} b_{i_1}\wedge\dots\wedge\widehat{b_{i_j}}\wedge\dots\wedge b_{i_r}, \qquad b_i d = 1
$$

(cf. the argument of [5, 2.2.5]). By induction over r, a linear zero-degree chain mapping h: $\bigoplus_{i\leqq r}\Lambda^i B\to\bigoplus_{i\leqq r}K_i^k$ will be defined so that, for $r\geqq 1$, $(b_{i_1}\wedge\dots\wedge b_{i_r})h\in K_r^k(\{i_1,\dots,i_r\}^2)$ and $(b_1\wedge\dots\wedge b_r)h\bar{\varepsilon}_{r+1}=a$. For $r=0$, set h: $\Lambda^0 B=F\to K_0^k$; $1\mapsto a$. For $r=1$, set $b_ih=b_i$. Then $b_ih\partial=b_i\partial=a=b_idh$, in particular $b_1h\bar{\varepsilon}_2=b_1h\partial=a$. Suppose h has been defined appropriately for degrees up to $r-1\geqq 0$. Then for each r-tuple $i_1<\dots<i_r$, $(b_{i_1}\wedge\dots\wedge b_{i_r})dh\partial=(b_{i_1}\wedge\dots\wedge b_{i_r})ddh=0$, i.e.

$$
(b_{i_1}\wedge\dots\wedge b_{i_r})dh\in Z_{r-1}(K^k(\{i_1,\dots,i_r\}^2))=B_{r-1}(K^k(\{i_1,\dots,i_r\}^2))
$$

using Lemma 6.1, i.e. there is a $c_{i_1\dots i_r}$ in $K_r^k(\{i_1,\dots,i_r\}^2)$ such that $c_{i_1\dots i_r}\partial=(b_{i_1}\wedge\dots\wedge b_{i_r})dh$. Defining $(b_{i_1}\wedge\dots\wedge b_{i_r})h=c_{i_1\dots i_r}$ then ensures $(b_{i_1}\wedge\dots\wedge b_{i_r})h\partial=(b_{i_1}\wedge\dots\wedge b_{i_r})dh$. Finally,

$$\begin{aligned}(b_1 \wedge \ldots \wedge b_r)\, h\bar{\varepsilon}_{r+1} &= c_{1\ldots r}\, \partial \bar{\beta}_{\bar{K}^r} \bar{\bar{\varepsilon}}_r \\ &= (b_1 \wedge \ldots \wedge b_r)\, dh \bar{\beta}_{\bar{K}^r} \bar{\bar{\varepsilon}}_r = \sum_{j=1}^{r} (-1)^{r-j} c_{1\ldots\hat{j}\ldots r} \bar{\beta}_{\bar{K}^r} \bar{\bar{\varepsilon}}_r \\ &= c_{1\ldots(r-1)} \bar{\varepsilon}_r = (b_1 \wedge \ldots \wedge b_{r-1})\, h\bar{\varepsilon}_r = a.\end{aligned}$$

Thus ε_k: $c_{1\ldots(k-1)} \mapsto a$, showing ε_k surjects as required.

7. Counting Dimensions - Compositions

This section completes the proof of Theorem 4.1. Let

$$\beta_k = \dim_F A_k = \operatorname{codim}_F \sum_{i=1}^{k-1} (1+\xi_i)(1+\eta_i) S, \qquad k \geqq 2,$$

and $\beta_1 = \beta_0 = 1$. The aim is to demonstrate the

Proposition 7.1. $\sum_{k=0}^{\infty} \beta_k x^k / k!\, k! = 1/J_0(2\sqrt{x})$.

Then for $k > 1$,

$$\dim A_k = \beta_k = \frac{(k!)^2}{2\pi i} \oint_{|z|=1} \frac{dz}{J_0(2\sqrt{z})\, z^{k+1}},$$

as required from §4. The current section involves the combinatorics of MacMahon's notion of a *composition* [9, Chap. 6], an ordered partition $i_1 + \ldots + i_m = k$ of a positive integer k into positive integers $i_1, \ldots, i_m$. For semigroups M "written additively" with operation $+$, let M^+ denote the free semigroup on the underlying set of M. Let N be the semigroup with unique element 1. Then N^+ is the semigroup of positive integers under $+$: write $1+1+\ldots+1$ with r summands as r. The semigroup N^{++} is the semigroup of compositions of integers. The mappings λ: $N^{++} \to N^+$; $i_1 + \ldots + i_m \mapsto m$ and σ: $N^{++} \to N^+$; $i_1 + \ldots + i_m \mapsto k$ (assigning k to each composition $i_1 + \ldots + i_m = k$ of k) are semigroup homomorphisms; in the monadic cohomology notation of [13, p. 119] they are the morphisms ε_2^0 and ε_2^1 respectively. There are also mappings δ: $N^+ \to N^{++}$; $m \mapsto 1 + \ldots + 1$ (m summands) and γ: $N^+ \to N^{++}$; $k \mapsto k$; the mapping δ is the morphism δ_1^0 of [13, p. 119] and γ is the non-homomorphic mapping h_1 of [13, p. 120], the embedding of N^+ in N^{++} as a free generating set of N^{++}.

Let N_k^{++} denote the subset $\sigma^{-1}(k)$ of N^{++} for each positive integer k. An element $\mathbf{i} = i_1 + \ldots + i_m$ of N_k^{++} is specified uniquely by the set of partial sums $\{\sigma(i_1), \sigma(i_1 + i_2), \ldots, \sigma(i_1 + \ldots + i_{m-1})\}$, and conversely each subset of $\{1, 2, \ldots, k-1\}$ is the set of partial sums of an element of N_k^{++} (cf. [0, III.1.G]). In particular N_k^{++} is finite, having 2^{k-1} elements [9, p. 124]. The set of subsets of $\{1, 2, \ldots, k-1\}$ forms a Boolean algebra B_{k-1} under intersection, union, and complementation. Define a bijection ϕ: $N_k^{++} \to B_{k-1}$ by setting $\phi(\mathbf{i})$ for $\mathbf{i}$ in N_k^{++} to be the complement of the set of partial sums of $\mathbf{i}$. The elements of $\phi(\mathbf{i})$ are referred to as the *colours* of the composition $\mathbf{i}$. One may obtain an intuitive picture of ϕ as follows. Consider the elements $x_1, \ldots, x_k$ taken in this order.

For a given composition $i_1+\ldots+i_m=k$, link consecutive elements amongst the first i_1, the next i_2, the next i_3, and so on. Colour the link between x_i and x_{i+1} with the colour i. Then $\phi(i_1+\ldots+i_m)$ is the set of colours used in the linking determined by $i_1+\ldots+i_m$. For example, $\phi(2+3+1+2)=\{1,3,4,7\}$ may be displayed as:

$$x_1 \underset{1}{\text{—}} x_2 \quad x_3 \underset{3}{\text{—}} x_4 \underset{4}{\text{—}} x_5 \quad x_6 \quad x_7 \underset{7}{\text{—}} x_8.$$

The bijection ϕ may be used to induce a Boolean algebra structure on N_k^{++} from B_{k-1}. In particular $\delta(k)$ is the least element of N_k^{++} and $\gamma(k)$ the greatest. Note, too, that $|\phi(\mathbf{i})|=\sigma(\mathbf{i})-\lambda(\mathbf{i})$ for $\mathbf{i}\in N^{++}$.

For a function $f\colon N^+\to\mathbb{R}$, one may form the formal power series $F(x)=\sum_{k=1}^{\infty} f(k)x^k$. Now $\mathbb{R}$ is a semigroup under multiplication, so there is a unique semigroup homomorphism $f^+\colon N^{++}\to(\mathbb{R},.)$; $\mathbf{i}\to f(i_1)f(i_2)\ldots f(i_{\lambda(\mathbf{i})})$ such that $\gamma\circ f^+=f$.

Lemma 7.2. $F(x)/(1-F(x))=\sum_{k=1}^{\infty} x^k \sum_{\sigma(\mathbf{i})=k} f^+(\mathbf{i})$.

Proof.

$$F(x)/(1-F(x))=\sum_{l=1}^{\infty}(F(x))^l=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}x^k\sum_{(\sigma,\lambda)(\mathbf{i})=(k,l)} f^+(\mathbf{i})=\sum_{k=1}^{\infty}x^k\sum_{\sigma(\mathbf{i})=k}f^+(\mathbf{i}).$$

Note. Lemma 7.2 may be interpreted purely formally, or one may think of the expansion of $F(x)$ as being valid for x in a domain D over which F is analytic. The validity of the expansion of $F(x)/(1-F(x))$ then depends on $F(D)$ lying within the open unit disc.

Taking $f\colon N^+\to\mathbb{R}$ to be the constant function with value 1, i.e. $F(x)=x/(1-x)$, Lemma 7.2 says that the coefficient of x^k in the expansion of $F(x)/(1-F(x))=x/(1-2x)$, namely 2^{k-1}, is just $\sum_{\sigma(\mathbf{i})=k} 1=|N_k^{++}|$, as concluded earlier from the existence of the bijection ϕ of N_k^{++} with the Boolean algebra B_{k-1}.

For $f\colon N^+\to\mathbb{R}$; $k\mapsto k!$, the integer $f^+(\mathbf{i})$, namely $i_1!\ldots i_{\lambda(\mathbf{i})}!$, will be denoted by $\mathbf{i}!$. Note that $\mathbf{i}!=|\langle\xi_j|j\in\phi(\mathbf{i})\rangle|$, a relation that will be needed later.

Corollary 7.3. $(-1)^k/k!=\sum_{\sigma(\mathbf{i})=k}(-1)^{\lambda(\mathbf{i})}/\mathbf{i}!$.

Proof. Take $F(x)=1-e^x$ in Lemma 7.2.

From now on, take $f\colon N^+\to\mathbb{R}$; $k\mapsto -\beta_k/k!\,k!$, with the corresponding F and f^+.

Proposition 7.4. $\sum_{\sigma(\mathbf{i})=k} f^+(\mathbf{i})=(-1)^k/k!\,k!$.

Proof. By Corollary 7.3,

$$\begin{aligned}(-1)^k/k!\,k! &= (-1)^k\Big(\sum_{\sigma(\mathbf{i})=k}(-1)^{\lambda(\mathbf{i})}/\mathbf{i}!\Big)\Big(\sum_{\sigma(\mathbf{j})=k}(-1)^{\lambda(\mathbf{j})}/\mathbf{j}!\Big)\\ &= (-1)^k \sum_{\sigma(\mathbf{i})=k,\,\sigma(\mathbf{j})=k}(-1)^{\lambda(\mathbf{i})+\lambda(\mathbf{j})}/\mathbf{i}!\,\mathbf{j}!\\ &= (-1)^k \sum_{\sigma(\mathbf{q})=k}\ \sum_{\mathbf{i}\cup\mathbf{j}=\mathbf{q}}(-1)^{\lambda(\mathbf{i})+\lambda(\mathbf{j})}/\mathbf{i}!\,\mathbf{j}!\\ &= (-1)^k \sum_{\sigma(\mathbf{q})=k}\ \prod_{1\leqq h\leqq\lambda(\mathbf{q})}\ \sum_{\mathbf{i}\cup\mathbf{j}=\gamma(q_h)}(-1)^{\lambda(\mathbf{i})+\lambda(\mathbf{j})}/\mathbf{i}!\,\mathbf{j}!\end{aligned}$$

by distributivity

$$= \sum_{\sigma(\mathbf{q})=k}\ \prod_{1\leqq h\leqq\lambda(\mathbf{q})}(q_h!^{-2}\sum_{\mathbf{i}\cup\mathbf{j}=\gamma(q_h)}(-1)^{q_h+\lambda(\mathbf{i})+\lambda(\mathbf{j})}q_h!^2/\mathbf{i}!\,\mathbf{j}!).$$

Proposition 7.4 follows if

$$\beta_k = \sum_{\mathbf{i}\cup\mathbf{j}=\gamma(k)}(-1)^{k+\lambda(\mathbf{i})+\lambda(\mathbf{j})+1}k!^2/\mathbf{i}!\,\mathbf{j}!.$$

Now for $\mathbf{i}\cup\mathbf{j}=\gamma(k)$,

$$\begin{aligned}|\phi(\mathbf{i})\cap\phi(\mathbf{j})| &= |\phi(\mathbf{i})|+|\phi(\mathbf{j})|-|\phi(\mathbf{i}\cup\mathbf{j})|\\ &= k-\lambda(\mathbf{i})+k-\lambda(\mathbf{j})-(k-1)=k-\lambda(\mathbf{i})-\lambda(\mathbf{j})+1.\end{aligned}$$

Then since (5.1) is a resolution,

$$\begin{aligned}\beta_k=\dim_F(A_k) &= \sum_{i=0}^{k-1}(-1)^i\dim_F X^k_{i+k-1}\\ &= \sum_{i=0}^{k-1}(-1)^i \sum_{\mathbf{i}\cup\mathbf{j}=\gamma(k),\,|\phi(\mathbf{i})\cap\phi(\mathbf{j})|=i}\dim_F K^k_{k+i-1}(\phi(\mathbf{i})\times\phi(\mathbf{j}))\\ &= \sum_{i=0}^{k-1}(-1)^i \sum_{\mathbf{i}\cup\mathbf{j}=\gamma(k),\,|\phi(\mathbf{i})\cap\phi(\mathbf{j})|=i} k!^2/\mathbf{i}!\,\mathbf{j}!\end{aligned}$$

by the remark preceding Corollary 7.3

$$= \sum_{\mathbf{i}\cup\mathbf{j}=\gamma(k)}(-1)^{k-\lambda(\mathbf{i})-\lambda(\mathbf{j})+1}k!^2/\mathbf{i}!\,\mathbf{j}!, \quad \text{as required.}$$

The proof of Proposition 7.1 now follows. By Lemma 7.2 and Proposition 7.4, $F(x)/(1-F(x))=\sum_{k=1}^{\infty}x^k(-1)^k/k!\,k!$. By definition, $J_0(z)=\sum_{k=0}^{\infty}(-1)^k(z/2)^{2k}/k!\,k!$, so that $F(x)/(1-F(X))=J_0(2\sqrt{x})-1$. Then $1-(1/J_0(2\sqrt{x}))=F(x)=-\sum_{k=1}^{\infty}\beta_k x^k/k!\,k!$ and $\beta_0=1$ yield $\sum_{k=0}^{\infty}\beta_k x^k/k!\,k!=1/J_0(2\sqrt{x})$, i.e. Proposition 7.1.

8. The Conjectured Solution to Manin's Problem

As at the end of § 2, let $3^{\delta(n)}$ denote the cardinality of the derived loop L_1 of the free CML L on n generators. Let $D(z)=\sum_{n=3}^{\infty}\delta(n)z^n/n!$.

Conjecture 8.1

$$D(z)=\int_{t=0}^{1} ze^{2z}\left(\frac{1}{J_0(2z(t(1-t))^{1/2})}-z^2t(1-t)-1\right)dt$$

$$(8.2)\qquad +\frac{1}{2\pi i}\oint_{0<|t|=\rho<1}\frac{t^2}{1-t^2}\exp\left\{z\cdot\frac{1-2t^2}{1-t^2}\right\}\sinh\left(\frac{z}{t}\right)dt.$$

The essence of this conjecture is that for given disjoint p-element subsets X and $(2k+1)$-element subsets Y of the n generators, where $p\geqq 0$, $k>0$, and $n\geqq 3$ are integers, the subspace $V(X,Y)$ of the $GF(3)$-vector space $L_1/L_2\oplus\ldots\oplus L_{n-2}/L_{n-1}$ spanned by all the associators having X as the set of symmetric arguments and Y as the set of skew arguments (see [11, § 2] for these terms) has dimension

$$(8.3)\qquad \alpha_k+\binom{p+k-1}{p},$$

where $\alpha_1=0$ and $\alpha_k=\beta_k$ for $k>1$. The contribution α_k for $k>1$ is believed to come on inducing up from a module $R/\sum_{j=1}^{k-1}(1+\xi_j)(1+\eta_j)R$ as in Theorem 4.1, this module structure occuring in $V(X,Y)$ on the span of each set of associators with fixed symmetric arguments in fixed positions and a fixed left-most skew argument. Formula (8.3) for the dimension of $V(X,Y)$ has been verified for k up to 3 and small p by extremely long calculations (in the case of $k=3$) using the method of [12].

Assuming the validity of the Triple Argument Hypothesis [11, § 1], $\delta(n)$ is then obtained as the sum of the dimensions of the various $V(X,Y)$ as (X,Y) ranges over all disjoint pairs of subsets of the n generators with Y having odd cardinality at least 3. Thus the conjecture is that

$$(8.4)\qquad \delta(n)=\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\sum_{p=0}^{n-2k-1}\frac{n!}{p!(2k+1)!(n-p-2k-1)!}\left[\alpha_k+\binom{p+k-1}{p}\right].$$

The first few values are $\delta(3)=1$, $\delta(4)=8$, $\delta(5)=44$, $\delta(6)=214$, $\delta(7)=1000$, $\delta(8)=4592$.

Rewriting (8.4) in closed generating function form then leads to Conjecture 8.1. To see this, it is convenient to divide the calculations into two parts: the *fermion* part coming from α_k and the skew arguments, and the *boson* part coming from $\binom{p+k-1}{p}$ and the symmetric arguments.

The fermion part will be dealt with first. Noting that

$$\sum_{p=0}^{n-2k-1} n!/p!(2k+1)!(n-p-2k-1)!=\binom{n}{2k+1}2^{n-2k-1},$$

this takes the form

$$\sum_{n=3}^{\infty}\frac{z^n}{n!}\sum_{k=1}^{[(n-1)/2]}\binom{n}{2k+1}2^{n-2k-1}\alpha_k.$$

Now by Theorem 4.1

$$\begin{aligned}J_0(v)^{-1}-1-v^2/4&=\sum_{k=0}^{\infty}\beta_k(v/2)^{2k}/k!^2-\beta_0-\beta_1 v^2/4\\&=\sum_{k=2}^{\infty}\beta_k(v/2)^{2k}/k!^2=\sum_{k=1}^{\infty}\alpha_k(v/2)^{2k}/k!^2,\end{aligned}$$

whence on setting $v=2z\sqrt{(t(1-t))}$, integrating from $t=0$ to $t=1$, and multiplying with z one obtains

$$\sum_{k=1}^{\infty}\alpha_k z^{2k+1}/(2k+1)!=\int_{t=0}^{1} z(J_0(2z(t(1-t))^{1/2})^{-1}-z^2t(1-t)-1)\,dt.$$

The fermion part of (8.2), the first term on the right hand side, is then just

$$\left(\sum_{l=0}^{\infty}2^l z^l/l!\right)\left(\sum_{k=1}^{\infty}\alpha_k z^{2k+1}/(2k+1)!\right)=\sum_{n=3}^{\infty}\frac{z^n}{n!}\sum_{k=1}^{[(n-1)/2]}\binom{n}{2k+1}2^{n-2k-1}\alpha_k,$$

as required.

Derivation of the boson part starts from the formulae

$$(1+x\pm y)^n=\sum_{0\le p,\,0\le l}^{p+l\le n}\sum x^p(\pm y)^l n!/p!\,l!(n-p-l)!$$

and

$$t^3(1-t^2)^{-(p+1)}=\sum_{r=1}^{\infty}\binom{p+r-1}{p}t^{2r+1}$$

(cf. [9, p. 10]), which yield

$$\begin{aligned}&t^3(2(1-t^2))^{-1}((1+(1-t^2)^{-1}+y)^n-(1+(1-t^2)^{-1}-y)^n\\&=\sum_{0\le p,\,0\le 2k+1}^{p+2k+1\le n}\sum\frac{n!}{p!(2k+1)!(n-p-2k-1)!}\sum_{r=1}^{\infty}\binom{p+r-1}{p}t^{2r+1}y^{2k+1}.\end{aligned}$$

Thus by the argument of the Hadamard multiplication theorem [15, §4.6]

$$\begin{aligned}(8.5)\qquad&\frac{1}{2\pi i}\oint_{0<|t|=\rho<1}\frac{t^3}{2(1-t^2)}\left\{\left(1+\frac{1}{1-t^2}+\frac{v}{t}\right)^n-\left(1+\frac{1}{1-t^2}-\frac{v}{t}\right)^n\right\}dt\\&=\sum_{0\le p,\,0\le 2k+1}^{p+2k+1\le n}\sum\frac{n!}{p!(2k+1)!(n-p-2k-1)!}\binom{p+k-1}{p}v^{2k+1}.\end{aligned}$$

Note that each side of (8.5) is 0 for $n=0$, 1, 2. Setting $v=1$ in (8.5) gives the boson part of $\delta(n)$. (This causes no convergence problems since the expressions under consideration are polynomials in y and v.) The boson part of (8.2), the second term on the right hand side, is then obtained on multiplying by $z^n/n!$ and summing from $n=0$ to ∞.

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