

# RICCI CURVATURE, CIRCULANTS, AND A MATCHING CONDITION

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ABSTRACT. The Ricci curvature of graphs, as recently introduced by Lin, Lu, and Yau following a general concept due to Ollivier, provides a new and promising isomorphism invariant. This paper presents a simplified exposition of the concept, including the so-called *logistic diagram* as a computational or visualization aid. Two new infinite classes of graphs with positive Ricci curvature are identified. A local graph-theoretical condition, known as the *matching condition*, provides a general formula for Ricci curvatures. The paper initiates a longer-term program of classifying the Ricci curvatures of circulant graphs. Aspects of this program may prove useful in tackling the problem of showing when twisted tori are not isomorphic to circulants.

## 1. INTRODUCTION

Various analogues of Ricci curvature have been extended from the context of Riemannian manifolds in differential geometry to more general metric spaces [1, 11, 12], and in particular to connected locally finite graphs [2, 3, 5, 6, 8, 9]. In graph theory, the concept appears to be a significant isomorphism invariant (in addition to other uses such as those discussed in [8, §4]). A typical problem where it may be of interest concerns the class of valency 4 graphs known as (*rectangular*) *twisted tori*, which have gained attention through their applications to computer architecture [4]. The apparently difficult open problem is to prove that, with finitely many exceptions, these graphs are not circulant (since this would show that they provide a genuinely new range of architectures). Now it is known that the twisted tori are quotients of cartesian products of cycles. As such, they are flat — their Ricci curvature is zero (again with a small finite number of exceptions) [8]. It follows that all but the smallest twisted tori cannot be isomorphic to any circulant possessing an edge with positive Ricci curvature.

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If the concept of Ricci curvature is to become a useful tool in graph theory, it is necessary to build up a catalogue of graphs whose curvature is known, and in particular to identify graphs of positive curvature. Currently, complete graphs (compare Corollary 6.5), Hamming hypercubes (compare Corollary 6.7 and [13]), and classical Erdős-Rényi random graphs have been established as infinite classes of graphs of positive curvature [8]. In this paper, further new classes are identified. These include complete bipartite graphs  $K_{l,m}$  (Theorem 5.1), and so-called *subcomplete bipartite graphs*  $S_l$ , the complements of two disjoint cliques  $K_l$  with a matching between them (Theorem 7.4).

In addition to exhibiting concrete classes of graphs with known Ricci curvature, it is also desirable to pursue a theoretical analysis of the concept, in order to facilitate future computations. An example of such analysis is the theorem of Lin–Lu–Yau invoked earlier, showing how the Ricci curvature of a cartesian product is related to the Ricci curvature of its (regular) factors [8, Th. 3.1]. In this paper, so-called *matching conditions* are introduced — locally for edges, and globally for graphs (Definition 6.1)<sup>1</sup>. In a (simple) graph  $G = (V, E)$ , with a vertex  $x$ , set  $x^E = \{y \in V \mid \{x, y\} \in E\}$ . For  $\{x, y\} \in E$ , the Matching Condition requires a local matching between  $x^E \setminus (\{y\} \cup y^E)$  and  $y^E \setminus (\{x\} \cup x^E)$ . Connected graphs satisfying the Global Matching Condition are regular (Lemma 6.2). Theorem 6.3 then gives an explicit formula for the Ricci curvature of an edge satisfying the Local Matching Condition within a graph. Complete graphs, Hamming hypercubes, and subcomplete bipartite graphs all satisfy the Global Matching Condition, so Theorem 6.3 may be used to determine their curvature. While these curvatures are all positive, another theoretical result (Theorem 8.1) gives a general condition for flatness, in particular covering the cases of long cycles (as in [8, Ex. 2.2]), interior edges of paths, and the Durbar Plate graph of Figure 4.

Classification of circulants from the standpoint of Ricci curvature appears to be a long-term research program, which may involve certain number-theoretical issues. In this paper, a few tentative initial steps are taken, with conventions as follows. For a positive integer  $n$ , consider the additive group  $(\mathbb{Z}/n, +, 0)$  of integers modulo  $n$ . Let  $J$  be a subset of  $\{1, 2, \dots, \lfloor n/2 \rfloor\}$ . The *circulant graph*  $C_n(J)$  is then defined as the (simple, regular) graph on the vertex set  $\mathbb{Z}/n$  whose edge set is  $E = \{\{x, x + d\} \mid x \in \mathbb{Z}/n, d \in J\}$ . In this context,  $J$  is known as the *jump set*. Throughout, it is implicitly assumed that  $J$  is chosen so that

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<sup>1</sup>In this context, it is worth noting the role of matchings in very recent work of Bhattacharya and Mukherjee on Ollivier’s coarse Ricci curvature for graphs [3].

$C_n(J)$  is connected, e.g. by containing a residue coprime to  $n$ . The following table summarizes the circulants  $C_n(J)$  that are shown to have a constant Ricci curvature  $\kappa$ , at the given locations in the paper:

$J$	$\kappa$	Location
$\{1\}, n = 4$	1	<sup>1</sup> Cor. 6.7
$\{1\}, n = 5$	1/2	<sup>2</sup> Prop. 4.2
$\{1\}, n > 5$	0	<sup>3</sup> Cor. 8.2
$\{2k + 1 \mid 0 \leq k < \lceil n/4 \rceil\}, n \text{ even}^2$	$4/n$	Cor. 5.2
$\{2k + 1 \mid 0 \leq k < \lfloor n/4 \rfloor\}, 6 < n \equiv 2 \pmod{4}$	$4/(n - 2)$	Cor. 7.5

Notes: 1. [8, Ex. 2.3]. 2. [8, Ex. 2.2]. 3. [8, Ex. 2.2].

Furthermore, Proposition 6.8 uses the Local Matching Condition to exhibit edges of positive Ricci curvature in circulants whose jump set is  $\{r, s\}$  with residues  $0 < r \neq s < n/2$  and  $s \in \{\pm 3r\}$ .

For completeness, an additional table summarizes other general classes of graphs whose (constant) Ricci curvatures  $\kappa$  are computed using the techniques of the paper, at the given locations in the paper:

Graph	$\kappa$	Location
Complete $K_n$	$n/(n - 1)$	<sup>1</sup> Cor. 6.5
Complete bipartite $K_{l,m}$ ( $l \geq m > 0$ )	$2/l$	Th. 5.1
Subcomplete bipartite $S_l$ ( $l > 3$ )	$2/(l - 1)$	Th. 7.4
Hypercubes $H_n$	$2/n$	<sup>2</sup> Cor. 6.7

Notes: 1. [8, Ex. 2.1]. 2. [8, Ex. 2.3].

In addition to the new results detailed above, Sections 2–4 of the paper offer a simplified exposition of the Lin–Lu–Yau graph-theoretical Ricci curvature following [8], with a notation more directly tailored to the current applications. (The notation of the paper [8] stays closer to the notation used earlier for more general metric spaces.) Novel aspects of the exposition include a direct proof of the existence of the limit defining the Ricci curvature (Corollary 3.3), and the so-called *logistic diagram* introduced in Section 4 as a computational or visualization aid for determining Ricci curvature.

## 2. RICCI CURVATURE

Imagine two points  $x, y$  on the equator (or a great circle) of a sphere, as illustrated in Figure 1.

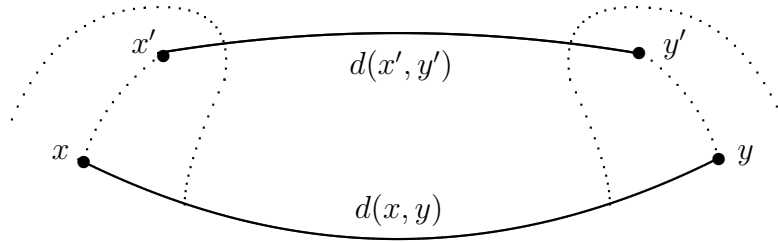


FIGURE 1. Ricci curvature in differential geometry.

The distance  $d(x, y)$  between them is traced along the equator. Now consider small neighborhoods of  $x$  and  $y$ , with typical points  $x'$  and  $y'$  respectively. The positive curvature of the sphere exhibits itself in an effect whereby the average of the distance  $d(x', y')$  between points of these neighborhoods works out to be less than the distance  $d(x, y)$  from  $x$  to  $y$ . It is this aspect of classical Ricci curvature in differential geometry which was adapted for use in graph theory by Lin, Lu, and Yau [8].

Let  $G = (V, E)$  be a connected (simple, undirected) graph, with vertex set  $V$  and edge set  $E$ . For a vertex  $x$  of  $G$ , and a positive real number  $\varepsilon < 1$ , the  $\varepsilon$ -ball  $b_x^\varepsilon$  (centered at  $x$ ) is the probability distribution on  $V$  defined by  $b_x^\varepsilon(x) = 1 - \varepsilon$ ,  $b_x^\varepsilon(v) = \varepsilon \cdot |x^E|^{-1}$  for  $v$  in  $x^E$ , and  $b_x^\varepsilon(v) = 0$  for the remaining vertices  $v$  of  $G$ . (Here, the parameter  $\varepsilon$  corresponds to  $1 - \alpha$  in the notation of [8], while  $b_x^\varepsilon$  corresponds to the measure  $m_x^\alpha$ . The current notation facilitates recognition of curvatures as derivatives, for example in Corollary 3.3 below.)

Now consider an edge  $\{x, y\}$  of  $G$ . This is regarded as the graph-theoretical analogue of the great circle segment from  $x$  to  $y$  displayed in Figure 1. The balls  $b_x^\varepsilon$  and  $b_y^\varepsilon$  are the analogues of the neighborhoods of  $x$  and  $y$  there. A proxy for the average distance between the neighborhoods is needed.

That proxy is the so-called *Wasserstein distance*, *Mallows distance*, or *earthmover's distance*  $W(b_x^\varepsilon, b_y^\varepsilon)$  between  $b_x^\varepsilon$  and  $b_y^\varepsilon$  [7, 10, 14]. Intuitively, the measure  $b_x^\varepsilon$  is represented by grains of sand located at the

vertices in  $\{x\} \cup x^E$ , where the amount of sand present at a vertex  $v$  is directly proportional to the measure  $b_x^\varepsilon(v)$  of that vertex. One then seeks the cheapest solution to the logistic problem of moving the sand to the distribution represented by  $b_y^\varepsilon$ . If a quantity  $q$  of sand has to be moved a distance  $d$ , the cost of that movement is  $qd$ . A *coupling*  $A$  between  $b_x^\varepsilon$  and  $b_y^\varepsilon$  represents the broad requirements for a transportation plan, specified by the amount  $A_{uv}$  moved from vertex  $u$  to vertex  $v$  for each pair  $(u, v)$  in  $V^2$ . The initial distribution is encoded by the *input constraint*

$$(2.1) \quad \sum_{v \in V} A_{uv} = b_x^\varepsilon(u)$$

for each vertex  $u$  of  $V$ . The final distribution is encoded by the *output constraint*

$$(2.2) \quad \sum_{u \in V} A_{uv} = b_y^\varepsilon(v).$$

for each vertex  $v$  of  $V$ . The coupling  $A$  is a probability distribution on  $V^2$  whose respective marginals on  $V$  are  $b_x^\varepsilon$  and  $b_y^\varepsilon$ .

Let  $\mathcal{C}$  denote the set of all couplings from  $b_x^\varepsilon$  to  $b_y^\varepsilon$ . The Wasserstein distance is then defined as

$$(2.3) \quad W(b_x^\varepsilon, b_y^\varepsilon) = \inf_{A \in \mathcal{C}} \sum_{(u,v) \in V^2} d(u, v) A_{uv},$$

the lowest cost for transporting the “sand” from  $b_x^\varepsilon$  to  $b_y^\varepsilon$ . For vertices  $u$  and  $v$  of  $G$ , the distance  $d(u, v)$  is taken within the graph  $G$ , as the length of a shortest path from  $u$  to  $v$ . Given the Wasserstein distance (2.3), the (*Ricci*)  $\varepsilon$ -*curvature* is defined as

$$\kappa_\varepsilon(x, y) = 1 - W(b_x^\varepsilon, b_y^\varepsilon).$$

The *Ricci curvature* is the limit

$$(2.4) \quad \kappa(x, y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \kappa_\varepsilon(x, y).$$

The general existence of this limit is established in [8, §2]. However, the  $\varepsilon$ -curvature is just a linear function  $k(\varepsilon)$  of  $\varepsilon$  near zero, and as such is differentiable at zero, so  $\kappa(x, y) = k'(0)$ . See Corollary 3.3 below.

The graph  $G$  is said to have *constant Ricci curvature*  $r$  if  $\kappa(x, y) = r$  for each edge  $\{x, y\}$  of  $G$ . The graph  $G$  is said to be *flat* if it has constant curvature 0.

## 3. DUALITY

Consider a graph  $G = (V, E)$ , with edge  $\{x, y\}$ . Let  $\varepsilon$  be a small positive number (for example less than  $1/2$ ). Determination of the infimum in (2.3) represents a (primal) problem  $\mathcal{P}(x, y)$  in linear programming. The linear objective function (of variables  $A_{uv}$  for  $(u, v) \in V^2$ ) to be minimized is

$$(3.1) \quad \sum_{(u,v) \in V^2} d(u, v) A_{uv},$$

the total amount of “sand” moved. Formally, the problem  $\mathcal{P}(x, y)$  is specified as minimization of the objective function (3.1), with non-negative variables  $A_{uv}$ , subject to the constraints (2.1) and (2.2) for each vertex  $u$  or  $v$  of  $G$ .

Now a real-valued function  $f : V \rightarrow \mathbb{R}$  on the vertex set  $V$  of a graph  $G = (V, E)$  is said to be 1-*Lipschitz* if

$$(3.2) \quad |f(u) - f(v)| \leq d(u, v)$$

for each pair of vertices  $u, v$  of  $G$ . Note that (3.2) may be rewritten as a pair of linear inequalities

$$(3.3) \quad f(u) - f(v) \leq d(u, v)$$

and

$$(3.4) \quad f(v) - f(u) \leq d(u, v).$$

The linear programming problem  $\mathcal{D}(x, y)$  dual to the primal problem  $\mathcal{P}(x, y)$  is the maximization of the objective function

$$(3.5) \quad \sum_{v \in V} (b_x^\varepsilon(v) - b_y^\varepsilon(v)) f(v)$$

of variables  $f(v)$  subject to the constraints (3.3) and (3.4) for each pair of vertices  $u, v$  [8, Eqn. (2)]. This linear-programming duality is a special case of a more general duality within a functional-analytic setting [14, §2.2].

Let  $\mathcal{L}$  denote the set of all 1-Lipschitz functions  $f : V \rightarrow \mathbb{R}$ . Note that  $\mathcal{L}$  is invariant under negation:  $-\mathcal{L} = \mathcal{L}$ . Strong duality in linear programming implies that the Wasserstein distance may be expressed in the form

$$(3.6) \quad W(b_x^\varepsilon, b_y^\varepsilon) = \sup_{f \in \mathcal{L}} \sum_{v \in V} (b_x^\varepsilon(v) - b_y^\varepsilon(v)) f(v)$$

as a dual to the previous expression

$$(3.7) \quad W(b_x^\varepsilon, b_y^\varepsilon) = \inf_{A \in \mathcal{C}} \sum_{(u,v) \in V^2} d(u, v) A_{uv}.$$

The following lemma gives a first application of duality. (A proof based on Section 2, running an optimal coupling in the reverse direction, would be somewhat less immediate.)

**Lemma 3.1.** *For an edge  $\{x, y\}$  of a graph  $G = (V, E)$ , one has  $\kappa(x, y) = \kappa(y, x)$ .*

*Proof.* For a positive real number  $\varepsilon$ , consider the measures  $b_x^\varepsilon$  and  $b_y^\varepsilon$ . Then

$$\begin{aligned} W(b_x^\varepsilon, b_y^\varepsilon) &= \sup_{f \in \mathcal{L}} \sum_{v \in V} f(v) (b_x^\varepsilon(v) - b_y^\varepsilon(v)) \\ &= \sup_{-f \in \mathcal{L}} \sum_{v \in V} -f(v) (b_x^\varepsilon(v) - b_y^\varepsilon(v)) \\ &= \sup_{f \in \mathcal{L}} \sum_{v \in V} f(v) (b_y^\varepsilon(v) - b_x^\varepsilon(v)) = W(b_y^\varepsilon, b_x^\varepsilon), \end{aligned}$$

so

$$\kappa_\varepsilon(x, y) = 1 - W(b_x^\varepsilon, b_y^\varepsilon) = 1 - W(b_y^\varepsilon, b_x^\varepsilon) = \kappa_\varepsilon(y, x)$$

and  $\kappa(x, y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \kappa_\varepsilon(x, y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \kappa_\varepsilon(y, x) = \kappa(y, x)$ .  $\square$

In view of Lemma 3.1, one may regard the Ricci curvature  $\kappa(x, y)$  as a property of the edge or unordered pair  $\{x, y\}$ . The second instance of duality gives an immediate proof of the existence of the limit (2.4) that will apply in all the circumstances studied in this paper.

**Proposition 3.2.** *Let  $\{x, y\}$  be an edge in a graph  $G = (V, E)$ . Fix an element  $\varepsilon$  of the interval  $[0, 1/2)$ . Then the Ricci  $\varepsilon$ -curvature  $\kappa_\varepsilon(x, y)$  has the formal expression of a linear function  $k(\varepsilon)$  of  $\varepsilon$ .*

*Proof.* Let  $f : V \rightarrow \mathbb{R}$  be a 1-Lipschitz function that achieves the supremum (3.6). Suppose that the respective degrees of  $x$  and  $y$  are  $l$

and  $m$ . Then

$$\begin{aligned}
\kappa_\varepsilon(x, y) &= 1 - W(b_x^\varepsilon, b_y^\varepsilon) \\
&= 1 - \sum_{v \in V} (b_x^\varepsilon(v) - b_y^\varepsilon(v)) f(v) \\
&= 1 - \left(1 - \left(1 + \frac{1}{m}\right) \varepsilon\right) f(x) + \left(1 - \left(1 + \frac{1}{l}\right) \varepsilon\right) f(y) \\
&\quad - \left(\frac{\varepsilon}{l} - \frac{\varepsilon}{m}\right) \sum_{v \in (x^E \cap y^E)} f(v) \\
&\quad - \frac{\varepsilon}{l} \sum_{v \in (x^E \setminus (\{y\} \cup y^E))} f(v) + \frac{\varepsilon}{m} \sum_{v \in (y^E \setminus (\{x\} \cup x^E))} f(v) \\
&= 1 - (f(x) - f(y))(1 - \varepsilon) - \frac{\varepsilon}{l} \sum_{v \in x^E} f(v) + \frac{\varepsilon}{m} \sum_{v \in y^E} f(v),
\end{aligned}$$

as required.  $\square$

**Corollary 3.3.** *Suppose that a single 1-Lipschitz function  $f : V \rightarrow \mathbb{R}$  achieves the suprema (3.6) for all small values of  $\varepsilon$ . Then the limit  $\kappa(x, y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \kappa_\varepsilon(x, y)$  exists, as  $k'(0)$ , or as the constant negated derivative*

$$-\frac{d}{d\varepsilon} W(b_x^\varepsilon, b_y^\varepsilon)$$

of the linear function  $W(\varepsilon)$ . The limit may be written as

$$\left(f(x) - \frac{1}{|x^E|} \sum_{v \in x^E} f(v)\right) - \left(f(y) - \frac{1}{|y^E|} \sum_{v \in y^E} f(v)\right)$$

in explicit form.

#### 4. THE LOGISTIC DIAGRAM

Let  $\{x, y\}$  be an edge of a graph  $G = (V, E)$ . In order to determine the Wasserstein distance  $W(b_x^\varepsilon, b_y^\varepsilon)$  for a small positive number  $\varepsilon$ , one needs to solve at least one of the optimization problems  $\mathcal{P}(x, y)$  or  $\mathcal{D}(x, y)$  discussed in the previous section. For a given 1-Lipschitz function  $f : V \rightarrow \mathbb{R}$ , (3.6) shows that

$$\sum_{v \in V} f(v) (b_x^\varepsilon(v) - b_y^\varepsilon(v)) \leq W(b_x^\varepsilon, b_y^\varepsilon).$$



On the other hand, for a given coupling  $A$  from  $b_x^\varepsilon$  to  $b_y^\varepsilon$ , (3.7) shows that

$$W(b_x^\varepsilon, b_y^\varepsilon) \leq \sum_{(u,v) \in V^2} d(u,v) A_{uv}.$$

An optimal 1-Lipschitz function  $f : V \rightarrow \mathbb{R}$  and coupling  $A$  from  $b_x^\varepsilon$  to  $b_y^\varepsilon$  are achieved if

$$(4.1) \quad \sum_{v \in V} f(v) (b_x^\varepsilon(v) - b_y^\varepsilon(v)) = \sum_{(u,v) \in V^2} d(u,v) A_{uv},$$

in which case the Wasserstein distance is the common value of these quantities. To compute the Ricci curvature of an edge  $\{x, y\}$  in a graph  $G = (V, E)$ , the main technique used in this paper thus consists of exhibiting a specific 1-Lipschitz function  $f : V \rightarrow \mathbb{R}$  and coupling  $A$  such that the equality (4.1) is realized for small values of  $\varepsilon$ .

A useful aid is provided by a so-called *logistic diagram* (compare Figures 2 and 3), which displays the quantities relevant to (4.1) for a given edge  $\{x, y\}$ . The central part of the logistic diagram has two rows: an upper one comprising the vertices in  $y^E$ , and a lower one comprising the vertices in  $x^E$ . The central part is bordered, both above and below, by a pair of rows. The upper row of the pair records the value of the measure  $b_x^\varepsilon$  at each vertex, while the lower records the value of  $b_y^\varepsilon$ . These values are separated by a dash, which may also be read as a subtraction sign to give the quantities  $b_x^\varepsilon(v) - b_y^\varepsilon(v)$  appearing on the left-hand side of (4.1). For a vertex  $v$  in  $y^E$ , the value  $f(v)$  of the optimal 1-Lipschitz function  $f : V \rightarrow \mathbb{R}$  appears above it in the uppermost row of the diagram. For a vertex  $v$  in  $x^E$ , the value  $f(v)$  appears below it in the lowermost row of the diagram.

In Section 2, couplings  $A$  from  $b_x^\varepsilon$  to  $b_y^\varepsilon$  were described as presenting the broad requirements for a transportation plan from  $b_x^\varepsilon$  to  $b_y^\varepsilon$ , with  $A_{uv}$  as the amount (of “sand”) to be moved from vertex  $u$  to vertex  $v$ . However, the coupling does not identify the path that each such movement has to follow through the graph  $G = (V, E)$ . The logistic diagram is more specific. Let  $\vec{G} = (V, \vec{E})$  denote the directed graph on the vertex set  $V$ , with directed edges  $(s, t)$  and  $(t, s)$  for each edge  $\{s, t\}$  in  $E$ . Given the edge  $\{x, y\}$  of  $G$ , a *transportation plan* is a non-negative real-valued function  $T : \vec{E} \rightarrow [0, \infty[; (s, t) \mapsto T(s, t)$  such that

$$(4.2) \quad \sum_{\{u,t\} \in E} (T(u, t) - T(t, u)) = b_x^\varepsilon(u) - b_y^\varepsilon(u)$$

for each  $u \in V$ . The logistic diagram displays a transportation plan which is optimal in the sense that

$$\sum_{v \in (x^E \cup y^E)} f(v) (b_x^\varepsilon(v) - b_y^\varepsilon(v)) = \sum_{(s,t) \in \vec{E}} T(s,t),$$

corresponding to (4.1). Normally, the diagram presents each directed edge  $(s,t)$  such that  $T(s,t) \neq 0$  in the weighted format  $s \xrightarrow{T(s,t)} t$ .

Each non-zero off-diagonal value  $A_{uv}$  of the optimal coupling, for vertices  $u \neq v \in (x^E \cup y^E)$ , is formally recovered from the optimal transportation plan  $T$  as the maximal flow from  $u$  to  $v$  through the network  $\vec{G}$  with capacities  $T(s,t)$  for  $(s,t) \in \vec{E}$ . Full specification of the optimal coupling is then completed by defining diagonal values  $A_{uu}$  so that the input constraints (2.1) and output constraints (2.2) are both satisfied. The compatibility of these constraints is verified in the following lemma. Note that the diagonal values  $A_{uu}$  make no contribution to the objective function (3.1), since they are multiplied there by zero in the form  $d(u,u)$ .

**Lemma 4.1.** *Let  $A$  denote the optimal coupling. For a vertex  $u$  of  $x^E \cup y^E$ , the diagonal value  $A_{uu}$  is determined equally as*

$$(4.3) \quad A_{uu} = b_x^\varepsilon(u) - \sum_{u \neq v \in V} A_{uv}$$

for satisfaction of the input constraint (2.1) and as

$$(4.4) \quad A_{uu} = b_y^\varepsilon(u) - \sum_{u \neq v \in V} A_{vu}$$

for satisfaction of the output constraint (2.2).

*Proof.* The flow out of  $u$  is

$$\sum_{u \neq v \in V} A_{uv} = \sum_{\{u,t\} \in E} T(u,t),$$

while the flow in to  $u$  is

$$\sum_{u \neq v \in V} A_{vu} = \sum_{\{u,t\} \in E} T(t,u).$$

The difference between the values (4.3) and (4.4) is

$$\begin{aligned} & \left( b_x^\varepsilon(u) - \sum_{u \neq v \in V} A_{uv} \right) - \left( b_y^\varepsilon(u) - \sum_{u \neq v \in V} A_{vu} \right) \\ &= \left( b_x^\varepsilon(u) - b_y^\varepsilon(u) \right) - \left( \sum_{\{u,t\} \in E} T(u,t) - \sum_{\{u,t\} \in E} T(t,u) \right), \end{aligned}$$

which vanishes by (4.2). □

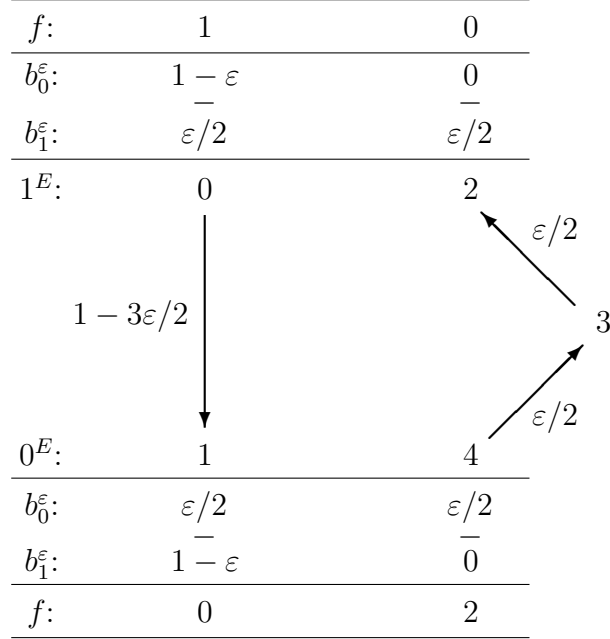


FIGURE 2. Logistic diagram for Proposition 4.2.

The details of a typical logistic diagram are displayed in Figure 2, which applies to the edge  $\{0, 1\}$  in the cycle  $(\mathbb{Z}/5, E)$  of length 5 implemented as the circulant  $C_5(\{1\})$ . The rows of this diagram are labeled on the left, although these labels are usually suppressed as the conventions are understood. The following proposition (stated without proof in [8]) is intended here as an initial illustration of the general technique being used.

**Proposition 4.2.** [8, Example 2.2] *A cycle of length 5 has constant Ricci curvature  $1/2$ .*

*Proof.* Realize the cycle  $(\mathbb{Z}/5, E)$  as the circulant  $C_5(\{1\})$ . Since the automorphism  $x \mapsto x + 1$  is edge-transitive, it suffices to compute  $\kappa(0, 1)$ . As illustrated by Figure 2, one has

$$W(b_0^\varepsilon, b_1^\varepsilon) \geq f(1) \left(1 - \frac{3\varepsilon}{2}\right) + f(4) \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2}$$

and

$$W(b_0^\varepsilon, b_1^\varepsilon) \leq 1 - \frac{3\varepsilon}{2} + 2 \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2},$$

so  $W(b_0^\varepsilon, b_1^\varepsilon) = 1 - \varepsilon/2$ , and then  $\kappa(0, 1) = 1/2$  by Corollary 3.3. (Note that  $f(3) = 1$ , not marked on the diagram.)  $\square$

## 5. COMPLETE BIPARTITE GRAPHS

**Theorem 5.1.** *Suppose that  $l$  and  $m$  are integers, with  $l \geq m > 0$ . Then the complete bipartite graph  $K_{l,m}$  has constant Ricci curvature  $2/l$ .*

*Proof.* Realize  $K_{l,m}$  with disjoint parts  $L$  and  $M$ , of respective cardinalities  $l$  and  $m$ . Each vertex of  $L$ , being joined to each vertex of  $M$  and to no vertex of  $L$ , has degree  $m$ . Similarly, each vertex of  $M$ , being joined to each vertex of  $L$  and to no vertex of  $M$ , has degree  $l$ .

Take an edge  $\{x, y\}$  of  $K_{l,m}$ , with  $x$  in  $L$  and  $y$  in  $M$ . For a small positive real number  $\varepsilon$ , the measure  $b_x^\varepsilon$  assigns weight  $\varepsilon/m$  to each vertex in  $M$ , weight  $1 - \varepsilon$  to  $x$ , and weight 0 to the remaining vertices in  $L$ . The measure  $b_y^\varepsilon$  assigns weight  $\varepsilon/n$  to each vertex in  $L$ , weight  $1 - \varepsilon$  to  $y$ , and weight 0 to the remaining vertices in  $M$ . A logistic diagram for the case with  $L = \{x, x_1, x_2\}$  and  $M = \{y, y_1\}$  appears as Figure 3.

Consider the function  $f: K_{l,m} \rightarrow \mathbb{R}$ , supported on  $L$ , with  $f(x) = 1$  and  $f(x') = -1$  for  $x \neq x' \in L$ . Note that  $f$  is a 1-Lipschitz function. Thus

$$W(b_x^\varepsilon, b_y^\varepsilon) \geq 1 - (1 + 1/l)\varepsilon + (l - 1)\varepsilon/l = 1 - 2\varepsilon/l.$$

Dually, a coupling from  $b_x^\varepsilon$  to  $b_y^\varepsilon$  is obtained by transporting a weight of  $1 - (1 + 1/l)\varepsilon$  from  $x$  to  $y$ , a weight of

$$\frac{\varepsilon}{l - 1} \left( \frac{1}{m} - \frac{1}{l} \right)$$

from  $y$  to each of the  $(l - 1)$  vertices of  $L$  distinct from  $x$ , and a weight of  $\varepsilon/(l - 1)m$  from each vertex of  $M$  other than  $y$  to each vertex of  $L$  other than  $x$ . Thus

$$\begin{aligned} W(b_x^\varepsilon, b_y^\varepsilon) &\leq 1 - \left(1 + \frac{1}{l}\right)\varepsilon + \varepsilon \left(\frac{1}{m} - \frac{1}{l}\right) + (l - 1)(m - 1)\varepsilon/(l - 1)m \\ &= 1 - \varepsilon - \frac{\varepsilon}{l} + \varepsilon \left(\frac{1}{m} - \frac{1}{l}\right) + \left(1 - \frac{1}{m}\right)\varepsilon = 1 - 2\varepsilon/l. \end{aligned}$$

The equality  $W(b_x^\varepsilon, b_y^\varepsilon) = 1 - 2\varepsilon/l$  follows, and then  $\kappa(x, y) = 2/l$  by Corollary 3.3. Finally,  $\kappa(y, x) = 2/l$  by Lemma 3.1.  $\square$

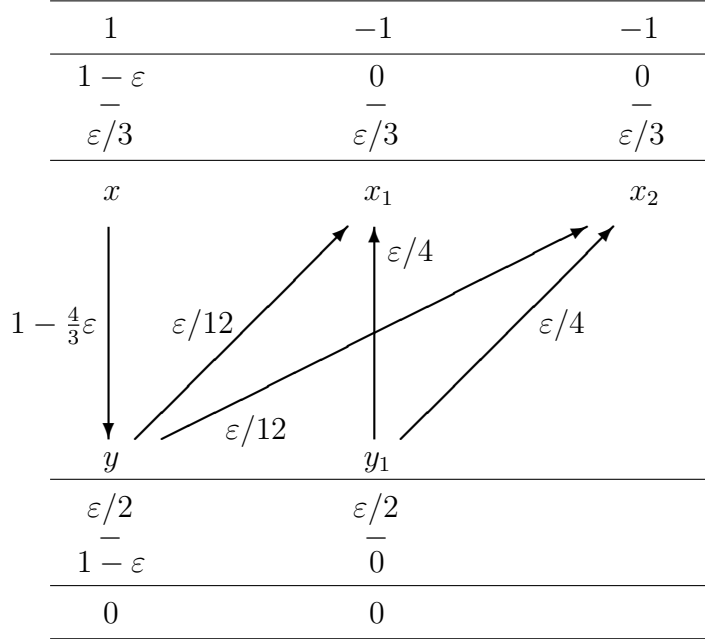


FIGURE 3. Logistic diagram for Theorem 5.1.

**Corollary 5.2.** *Let  $n$  be an even positive integer. Then for*

$$J = \{2k + 1 \mid 0 \leq k < \lceil n/4 \rceil\},$$

*the circulant graph  $C_n(J)$  has constant Ricci curvature  $4/n$ .*

*Proof.* Note that  $C_n(J)$  is the complete bipartite graph  $K_{n/2, n/2}$ . □

### 6. THE MATCHING CONDITION

**Definition 6.1.** Let  $G = (V, E)$  be a graph.

- (a) The *escort* of an edge  $\{x, y\}$  of  $G$  is the subgraph of  $G$  induced on the vertex set  $(x^E \cup y^E) \setminus \{x, y\}$ .
- (b) An edge  $\{x, y\}$  of the graph  $G$  is said to satisfy the (*Local Matching Condition*) if there is a perfect matching (in the graph-theoretical sense) between the two sets  $x^E \setminus (\{y\} \cup y^E)$  and  $y^E \setminus (\{x\} \cup x^E)$  within the escort of  $\{x, y\}$ .
- (c) The graph  $G$  is said to satisfy the (*Global Matching Condition*) if each edge  $\{x, y\}$  satisfies the Local Matching Condition.

**Lemma 6.2.** *Let  $G = (V, E)$  be a graph.*

- (a) *If an edge  $\{x, y\}$  of  $G$  satisfies the Local Matching Condition, then the degrees of  $x$  and  $y$  coincide.*

(b) *A connected graph satisfying the Global Matching Condition is regular.*

*Proof.* For (a), note that

$$\begin{aligned} |x^E| &= |x^E \setminus (\{y\} \cup y^E)| + |x^E \cap y^E| + 1 \\ &= |y^E \setminus (\{x\} \cup x^E)| + |x^E \cap y^E| + 1 = |y^E|. \end{aligned}$$

Thus the degrees of  $x$  and  $y$  coincide. Statement (b) follows.  $\square$

**Theorem 6.3.** *In a graph  $G = (V, E)$ , suppose that  $\{x, y\}$  is an edge satisfying the Local Matching Condition. Then*

$$\kappa(x, y) = (2 + |x^E \cap y^E|) / \delta,$$

where  $\delta$  is the common degree of  $x$  and  $y$ .

*Proof.* For a small positive real number  $\varepsilon$ , the measure  $b_x^\varepsilon$  assigns weight  $\varepsilon/\delta$  to each vertex in  $x^E$ , weight  $1-\varepsilon$  to  $x$ , and weight 0 to the remaining vertices in  $G$ . The measure  $b_y^\varepsilon$  assigns weight  $\varepsilon/\delta$  to each vertex in  $y^E$ , weight  $1-\varepsilon$  to  $y$ , and weight 0 to the remaining vertices in  $G$ . Set  $c = |x^E \cap y^E|$ .

A coupling from  $b_x^\varepsilon$  to  $b_y^\varepsilon$  is obtained as follows. First, transport weight  $1-\varepsilon-\varepsilon/\delta$  from  $x$  to  $y$ . Then, for each vertex in  $x^E \cap y^E$ , leave the weight  $\varepsilon/\delta$  there alone. Finally, consider a matching between  $x^E \setminus (\{y\} \cup y^E)$  and  $y^E \setminus (\{x\} \cup x^E)$  within the escort of  $\{x, y\}$ . From each vertex of  $x^E \setminus (\{y\} \cup y^E)$ , transport the full weight of  $\varepsilon/\delta$  along the corresponding edge of the matching to the vertex of  $y^E \setminus (\{x\} \cup x^E)$  lying at the other end. Note that there are  $\delta-1-c$  edges in the matching. Thus

$$W(b_x^\varepsilon, b_y^\varepsilon) \leq 1 - \left(1 + \frac{1}{\delta}\right) \varepsilon + (\delta - 1 - c) \frac{\varepsilon}{\delta} = 1 - (2 + c)\varepsilon/\delta.$$

Now consider the function  $f : V \rightarrow \mathbb{R}$ , supported on  $\{x\} \cup y^E$ , taking the value 1 at  $x$ , the value  $-1$  at each of the  $\delta-1-c$  vertices of  $y^E \setminus x^E$ , and 0 elsewhere. Note that  $f$  is a 1-Lipschitz function. Then

$$W(b_x^\varepsilon, b_y^\varepsilon) \geq 1 - \left(1 + \frac{1}{\delta}\right) \varepsilon + (\delta - 1 - c) \frac{\varepsilon}{\delta} = 1 - (2 + c)\varepsilon/\delta,$$

so that  $W(b_x^\varepsilon, b_y^\varepsilon) = 1 - (2 + c)\varepsilon/\delta$ . Thus  $\kappa(x, y) = (2 + c)/\delta$  by Corollary 3.3.  $\square$

**Corollary 6.4.** *Let  $G$  be a (regular) triangle-free graph of degree  $\delta$  that satisfies the Global Matching Condition. Then  $G$  has constant curvature  $2/\delta$ .*

**Corollary 6.5.** [8, Example 2.1] *The complete graph  $K_n$  has constant curvature  $n/(n-1)$ .*

*Proof.* Trivially,  $K_n$  satisfies the Global Matching Condition. Then Theorem 6.3 gives the constant curvature as  $(2 + (n-2))/(n-1)$ .  $\square$

**Proposition 6.6.** *For each integer  $n > 1$ , the Hamming hypercube  $H_n$  satisfies the Global Matching Condition.*

*Proof.* Consider  $H_n$  as the graph  $(V, E)$ , where  $V$  is the set of (binary representations of) integers  $x$  in the range  $0 \leq x < 2^n$ , with edges  $\{x, y\}$  precisely when  $x$  and  $y$  differ by a power of 2 (i.e., when their binary representations have Hamming distance 1). Consider the edge  $\{0, 1\}$ . Then there is a matching between  $0^E \setminus (\{1\} \cup 1^E)$  and  $1^E \setminus (\{0\} \cup 0^E)$  within the escort of  $\{0, 1\}$ , in which the even integer  $2^i$  (for  $0 \leq i < n$ ) is matched with the odd integer  $2^i + 1$ . Thus  $\{0, 1\}$  satisfies the Matching Condition. By the symmetry of  $H_n$ , it follows that all the edges, and thus the graph  $H_n$  itself, satisfy the Matching Condition.  $\square$

**Corollary 6.7.** [8, Example 2.3] *The Hamming hypercube  $H_n$  has constant curvature  $2/n$ .*

*Proof.* Apply Corollary 6.4, noting that the regular graph  $H_n$  is triangle-free, of degree  $n$ .  $\square$

**Proposition 6.8.** *Suppose that  $n$  is an integer. Consider a circulant  $C_n(\{r, s\})$  with residues  $0 < r \neq s < n/2$  and  $s \in \{\pm 3r\}$ . Then  $\kappa(x, x+r) = 1/2$  for each  $x \in \mathbb{Z}/n$ .*

*Proof.* Since  $r$  and  $s$  are distinct, lying strictly between 0 and  $n/2$ , the circulant  $C_n(\{r, s\})$  has degree 4. The escort of  $\{0, r\}$  has the matching  $\{\{-r, 2r\}, \{s, r+s\}, \{-s, r-s\}\}$ , so the edge  $\{0, r\}$  satisfies the Local Matching Condition. Thus  $\kappa(0, r) = 2/4$  by Theorem 6.3. The result follows, since  $\mathbb{Z}/n \rightarrow \mathbb{Z}/n; x \mapsto x+1$  is an automorphism of  $C_n(\{r, s\})$ .  $\square$

## 7. SUBCOMPLETE BIPARTITE GRAPHS

For a positive integer  $l$ , the complete bipartite graph  $K_{l,l}$  is the complement of two disjoint cliques  $K_l$ . The *subcomplete bipartite graph*  $S_l$  is defined as the complement of the graph consisting of two disjoint cliques  $K_l$  augmented by a matching between them. (For example,  $S_3$  is a cycle of length 6, while  $S_4$  is the 3-dimensional Hamming cube.) The following definition gives a concrete implementation that will be used in the subsequent work.

**Definition 7.1.** For a positive integer  $l$ , the (*standard*) *subcomplete bipartite graph*  $S_l$  is defined as the graph with vertex set

$$V = \{x_0 \mid x \in \mathbb{Z}/l\} \cup \{x_1 \mid x \in \mathbb{Z}/l\}$$

(the disjoint union of two copies of the set of integers modulo  $l$ ), and with

$$E = \{\{x_0, y_1\} \subseteq V \mid 0 < y - x < l\}$$

as its edge set. The summand  $\{x_0 \mid x \in \mathbb{Z}/l\}$  of  $V$  is known as the *0-part*, while the summand  $\{x_1 \mid x \in \mathbb{Z}/l\}$  of  $V$  is known as the *1-part*.

**Proposition 7.2.** *Let  $l$  be a positive integer. Apply the notation of Definition 7.1.*

(a) *The switch map*

$$\sigma: V \rightarrow V; x_0 \mapsto x_1 \mapsto x_0$$

*and translation map*

$$\tau: V \rightarrow V; x_0 \mapsto (x + 1)_0, y_1 \mapsto (y + 1)_1$$

*(for  $x, y \in \mathbb{Z}/l$ ) together generate a commutative group of automorphisms of  $S_l$  that acts transitively on the vertex set  $V$ .*

(b) *The subcomplete bipartite graph  $S_l$  is regular, of degree  $l - 1$ .*

**Proposition 7.3.** *Let  $l > 1$  be a positive integer.*

(a) *If  $l$  is odd, then  $S_l$  is the circulant graph  $C_{2l}(J)$  with jump set*

$$J = \{1, 3, \dots, l - 2\}.$$

(b) *If  $l$  is even, then  $S_l$  is not circulant.*

*Proof.* (a): Use the notation of Definition 7.1. Consider the bijection

$$\theta: V \rightarrow \mathbb{Z}/2l; x_k \mapsto kl + (l + 1)x$$

with two-sided inverse  $\mathbb{Z}/2l \rightarrow V; y \mapsto (y \bmod l)_{(y \bmod 2)}$  (“Chinese Remainder Theorem”). Note that  $0_0\theta = 0$  (using algebraic notation here). The neighborhood

$$\{1_1, \dots, (l - 1)_1\}$$

of  $0_0$  in  $S_l$  maps under  $\theta$  to the neighborhood  $\pm J$  of 0 in  $C_{2l}(J)$ . Then the respective conjugates under  $\theta$  of the switch and translation are  $\theta^{-1}\sigma\theta = \sigma^\theta: \mathbb{Z}/2l \rightarrow \mathbb{Z}/2l; y \mapsto y + l$  and  $\tau^\theta: \mathbb{Z}/2l \rightarrow \mathbb{Z}/2l; y \mapsto y + l + 1$ . Both of these maps are automorphisms of the circulant  $C_{2l}(J)$ .

(b): If  $l$  is even, the number  $l(l - 1)$  of edges in  $S_l$  is not divisible by the number  $2l$  of vertices.  $\square$

**Theorem 7.4.** *For an integer  $l$  with  $l > 3$ , consider the subcomplete bipartite graph  $S_l = (V, E)$ .*



- (a) *The graph  $S_l$  satisfies the Global Matching Condition.*
- (b) *The graph  $S_l$  has constant Ricci curvature  $2/(l-1)$ .*

*Proof.* (a) Take an edge  $\{0_0, r_1\}$  of  $S_l$ , for some  $0 < r < l$ . Consider the set  $Z = \{z \in \mathbb{Z}/l \mid 0 < z < r \text{ or } r < z < l\}$ , with a corresponding subset  $Z_0 = \{z_0 \mid z \in Z\}$  in the 0-part and  $Z_1 = \{z_1 \mid z \in Z\}$  in the 1-part. Note that  $0_0^E = \{r_1\} \cup Z_1$  and  $r_1^E = \{0_0\} \cup Z_0$ . The induced bipartite subgraph  $B$  of  $S_l$  on the vertex set  $Z_0 \cup Z_1$ , as a regular graph with  $2(l-2)$  vertices and degree  $l-3$ , contains a matching between  $Z_1 = 0_0^E \setminus (\{r_1\} \cup r_1^E)$  and  $Z_0 = r_1^E \setminus (\{0_0\} \cup 0_0^E)$ . Thus  $\{0_0, r_1\}$  satisfies the Local Matching Condition. By Proposition 7.2(a), it follows that  $S_l$  satisfies the Global Matching Condition.

(b) Apply Corollary 6.4. □

**Corollary 7.5.** *For an integer  $n$  with  $6 < n \equiv 2 \pmod{4}$ , the circulant graph  $C_n(J)$  with jump set*

$$J = \{2k + 1 \mid 0 \leq k < \lfloor n/4 \rfloor\}$$

*has constant Ricci curvature  $4/(n-2)$ .*

*Proof.* According to Proposition 7.3(a),  $C_n(J)$  is identified as the sub-complete bipartite graph  $S_{n/2}$ . □

## 8. FLATNESS

While positive Ricci curvature has been the main focus of the preceding sections, this final section presents an instance of flatness.

**Theorem 8.1.** *Let  $\{x, y\}$  be an edge of a connected graph  $G = (V, E)$ . Suppose that  $x$  and  $y$  have an even number  $c$  of common neighbors, and a common degree  $l = (4 + 3c)/2$ . If no vertex within  $y^E \setminus (\{x\} \cup x^E)$  has a distance less than 3 from any vertex within  $x^E \setminus (\{y\} \cup y^E)$ , or a distance less than 2 from any vertex within  $x^E \cap y^E$ , then  $\kappa(x, y) = 0$ .*

*Proof.* Set  $y^E \setminus (\{x\} \cup x^E) = \{x_1, \dots, x_{l-c-1}\}$  and  $x^E \setminus (\{y\} \cup y^E) = \{y_1, \dots, y_{l-c-1}\}$ . Set  $x^E \cap y^E = \{z_1, \dots, z_c\}$ . For a small positive real number  $\varepsilon$ , the measure  $b_x^\varepsilon$  assigns weight  $\varepsilon/l$  to the vertices  $y, y_j, z_k$ ; weight  $1 - \varepsilon$  to  $x$ , and weight 0 to all the remaining vertices. The measure  $b_y^\varepsilon$  assigns weight  $\varepsilon/l$  to the vertices  $x, x_i, z_k$ ; weight  $1 - \varepsilon$  to  $y$ , and weight 0 to all the remaining vertices.

Let  $T$  denote the set of vertices of  $G$  at distance 2 from  $x$  that are not neighbors of  $y$ . Within  $T$ , consider the subset  $S$  consisting of vertices that are adjacent to some  $y_j$  for  $0 < j < l - c$ . Note that

$$\forall 0 < i < l - c, \forall s \in S, d(x_i, s) \geq 2$$

by the assumptions of the the theorem. Then a 1-Lipschitz function  $f: V \rightarrow \mathbb{R}$  is defined by

$$f(v) = \begin{cases} 2 & \text{if } v = y_j, \text{ for } 0 < j < l - c; \\ 1 & \text{if } v \in S, \text{ or } v = x \text{ or } z_k, \text{ for } 0 < k \leq c; \\ -1 & \text{if } v = x_i, \text{ for } 0 < i < l - c; \\ 0 & \text{otherwise.} \end{cases}$$

Thus  $W(b_x^\varepsilon, b_y^\varepsilon) \geq$

$$\left(1 - \varepsilon - \frac{\varepsilon}{l}\right) + 2(l - c - 1)\frac{\varepsilon}{l} - (-1)(l - c - 1)\frac{\varepsilon}{l} = 1 + 2\varepsilon - \frac{4 + 3c}{l}\varepsilon = 1.$$

Dually, a coupling from  $b_x^\varepsilon$  to  $b_y^\varepsilon$  is obtained by transporting a weight of  $1 - \varepsilon - \varepsilon/l$  from  $x$  to  $y$ , and a weight of  $\varepsilon/l$  along each path of length 3 from  $y_i$ , through  $x$  and  $y$ , to  $x_i$ , for  $0 < i < l - c$ . Thus  $W(b_x^\varepsilon, b_y^\varepsilon) \leq$

$$\left(1 - \varepsilon - \frac{\varepsilon}{l}\right) + 3(l - c - 1)\frac{\varepsilon}{l} = 1 + 2\varepsilon - \frac{4 + 3c}{l}\varepsilon = 1.$$

It follows that  $W(b_x^\varepsilon, b_y^\varepsilon)$  is the constant 1, and then  $\kappa(x, y) = 0$  by Corollary 3.3.  $\square$

Setting  $c = 0$  in Theorem 8.1 recovers two well-known results.

**Corollary 8.2.** [8, Ex. 2.2] *Each cycle of length 6 or more is flat.*

**Corollary 8.3.** *Let  $\{x, y\}$  be an interior edge of a path (of length at least 3). Then  $\kappa(x, y) = 0$ .*

Note that the path of Corollary 8.3 may appear within a larger graph, such as an irregular tree, as long as the hypotheses of Theorem 8.1 are maintained.

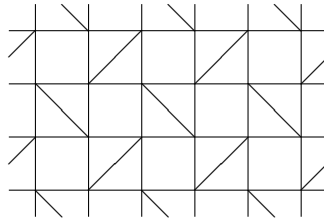


FIGURE 4. A fragment of the Durbar Plate graph.

**Example 8.4.** If  $\{x, y\}$  is any of the diagonal edges within the infinite regular *Durbar Plate graph* of degree 5, a fragment of which is shown in Figure 4, then Theorem 8.1 (with  $c = 2$ ) shows that  $\kappa(x, y) = 0$ .

Note that the remaining edges have curvature  $-1/5$ . The same results apply to sufficiently large toral projections of the Durbar Plate graph (e.g., the illustrated graph on the torus  $\mathbb{Z}/6 \times \mathbb{Z}/4$ ).

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