CHARACTER GROUPS OF DIHEDRAL AND GENERALIZED QUATERNION GROUPS

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ABSTRACT. Finite abelian group duality appears in the discrete Fourier transform, or as the finite fragment of Pontryagin duality. The dual or character group of an abelian group encodes the products of its (linear) irreducible characters. Now, recent developments in combinatorics enable the construction of character groups for finite dihedral and generalized quaternion groups, encoding the products of all the irreducible characters (linear and non-linear) in purely multiplicative fashion. In particular, just as in the abelian case, each dihedral group may serve as its own character group. Furthermore, certain Adams operations on the characters are shown to correspond to powers in the character group of a dihedral group.

1. Introduction

For a topological abelian group A, the *character group* $\widehat{A}$ or $A\widehat{\ }$ is the set of continuous homomorphisms $\chi\colon A \to T$ from A to the torus T, the multiplicative group of complex numbers of modulus 1. The character group forms an abelian group under pointwise multiplication, with $(\chi_1 \cdot \chi_2)(a) = \chi_1(a)\chi_2(a)$ for $\chi_1, \chi_2 \in \widehat{A}$ and $a \in A$. According to *Pontryagin duality* [7, 12], if A is locally compact, then so is $\widehat{A}$, and the group A is naturally isomorphic to its double dual $A\widehat{\ }\widehat{\ }$.

A finite abelian group A, as a discrete topological group, is (locally) compact, and in this case A and its character group $\widehat{A}$ are isomorphic, albeit not canonically within the framework of Pontryagin duality. The set $\widehat{A}$ forms a basis for the space of complex-valued functions on A. In an expression of such a function as a linear combination of elements of $\widehat{A}$, the (possibly normalized) coefficients are interpreted as constituting the *discrete Fourier transform* of the function.

For finite nonabelian groups, no direct analog of the notion of the character group of a finite abelian group has hitherto been available, the characters instead being taken within the linear algebraic settings of the character ring, or of the span of the

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coefficient functions of matrix representations. The purpose of the current paper is to consider character groups of finite dihedral groups and generalized quaternion groups which do naturally extend the notion of the character group of an abelian group. These character groups produce a finite and purely multiplicative encoding of the multiplicative structure of all the irreducible characters, both linear and nonlinear.

Construction of the character groups rests on recent developments in combinatorics which originated in the work of Hilton *et al.* [1, 2, 8, 9, 10] on the lifting of finite *weighted quasigroups* to finite quasigroups. In [14, Cor. 5.10] it was shown that the set $\widetilde{\Gamma}$ or $\widetilde{\Gamma}(G)$ of irreducible characters of a finite group G of order n forms a weighted quasigroup $(\widetilde{\Gamma}, w, \alpha)$ in the sense of Hilton *et al.*, with structure defined as in (2.4), (2.5) below. This weighted quasigroup lifts to a (not necessarily uniquely determined) quasigroup $\widetilde{G}$ of order n [14, Th. 5.11], known as a *character quasigroup* of the group G. Each finite abelian group is its own lift, so character quasigroups for arbitary finite groups may be viewed as generalizations of the character groups of finite abelian groups. Note that any two groups that share the same character table, such as the dihedral group D_d and the generalized quaternion group Q_{2d} for $d = 2^n$ with $1 < n \in \mathbb{Z}$ (§3.4), will have all their character quasigroups in common. In other words, character quasigroups should not be expected to feature in a full duality like Pontryagin duality. Nevertheless, vestiges of duality do still remain. For example, Theorem 2.2 in effect shows that abelian quotients of a finite group G correspond to abelian subgroups of a character quasigroup $\widetilde{G}$.

The current paper is concerned primarily with the existence question for associative character quasigroups, *character groups*, of dihedral groups D_d with finite degree d (order $2d$), and of the corresponding generalized quaternion groups Q_{2d} of order $2d$ if $d = 2^n$ for $1 < n \in \mathbb{Z}$. Theorems 4.1 and 4.3 characterize the character quasigroups of D_d in the respective cases of even and odd degree d. Theorem 4.6 and Corollary 4.3 show that both $\mathbb{Z}/d \oplus \mathbb{Z}/2$ and D_d serve as character groups for the dihedral group D_d. By contrast, Corollary 4.5 observes that no generalized quaternion group can serve as its own character group.

Theorem 4.7 shows that, if $d \neq 4$, the dihedral group D_d has $\mathbb{Z}/d \oplus \mathbb{Z}/2$ as its unique abelian character group. On the other hand, the Boolean cube $(\mathbb{Z}/2)^3$ is also a character group of D_4 and Q_8 (§4.5). Given that Theorem 4.7 classifies all the abelian character groups of dihedral groups, Problem 4.8 asks for a classification of their nonabelian character groups.

In the final section of the paper, Theorem 5.1 relates power maps in character groups of dihedral groups to Adams operations on their characters.

2. Quasigroups, weighted quasigroups and character quasigroups

2.1. **Quasigroups.** A *quasigroup* is a set Q equipped with a binary operation $\cdot$ of *multiplication* such that for all $a, b \in Q$, there exist unique solutions $x, y \in Q$ to the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b\,. \tag{2.1}$$

In particular, groups are quasigroups, and indeed any nonempty associative quasigroup is a group. Small finite quasigroups are conveniently displayed by their Cayley or multiplication tables. The body of the multiplication table of a finite quasigroup is a Latin square, and conversely each Latin square may serve in such a role.

Algebraically, a quasigroup is a set together with 3 binary operations $\cdot, /, \backslash$, the latter described respectively as *right* and *left division*, satisfying:

$$a \cdot (a \backslash b) = b\,; \qquad (a/b) \cdot b = a\,;$$
$$a \backslash (a \cdot b) = b\,; \qquad (a \cdot b)/b = a\,.$$

Thus the solutions x, y to the equations (2.1) may be written as $x = a\backslash b$ and $y = a/b$, in a notation that may be familiar to users of mathematical software. For further reading and background on the theory of quasigroups, see [15].

2.2. **Weighted quasigroups.** A generalization of the notion of a finite quasigroup appears in [9, 10]. Let (X, w, α) consist of a finite set X with a *weight function*

$$w \colon X \to \mathbb{N}$$

and a *multiplication function*

$$\alpha \colon X \times X \times X \to \mathbb{N}; (x, y, z) \mapsto \alpha_z(x, y)\,.$$

The multiplication function may be displayed in the Cayley table format

$$\begin{array}{c||ccc} * & \dots & y & \dots \\ \hline\hline \vdots & & \vdots & \\ x & \dots & \sum_{z\in X} \alpha_z(x,y) z & \dots \\ \vdots & & \vdots & \end{array} \tag{2.2}$$

as in (4.7), for example.

The structure (X, w, α) forms a *weighted quasigroup* if

$$\sum_{z\in X} \alpha_z(x, y) = \sum_{z\in X} \alpha_x(z, y) = \sum_{z\in X} \alpha_y(x, z) = w(x)w(y) \tag{2.3}$$

for all $x, y \in X$. We define the *gross weight* of a weighted quasigroup (X, w, α) to be $\sum_{z\in X} w(z)$.

A quasigroup $(Q,\cdot)$ of finite order n may be regarded as a weighted quasigroup of gross weight n, with $w(x)=1$ and $\alpha(x,y,z)=\delta_{x\cdot y,z}$ for $x,y,z\in Q$. In this case, the conditions of (2.3) amount to the Latin square conditions that each symbol from Q appears exactly once in each row and column of the body of the multiplication table of Q.

2.3. **Covering weighted quasigroups with quasigroups.** A quasigroup Q of finite order n is said to *cover* a weighted quasigroup (X,w,α) of gross weight n if there is a surjective function $f\colon Q\to X$ such that

$$w(x)=\left|f^{-1}\{\,x\,\}\right|$$

and

$$\alpha(x,y,z)=\left|\,\{\,(a,b)\in f^{-1}\{\,x\,\}\times f^{-1}\{\,y\,\}\mid f(a\cdot b)=z\,\}\,\right|$$

for all $x,y,z\in X$. Equivalently, (X,w,α) is said to *lift* to Q in these circumstances. The function f is called a *covering* or *lifting*.

The following theorem was presented in the given form as [14, Th. 5.3].

Theorem 2.1. *Each weighted quasigroup of gross weight n lifts to a quasigroup of order n.*

The original formulation appeared in [8], and then again in different language as [9, Th. 1], [10, Th. 1]. The proofs were purely combinatorial, relying on regularity results from graph theory to build up a Latin square stepwise by row, column, and symbol.

2.4. **Weighted character quasigroups.** Suppose that G is a group of finite order n, with set $\widetilde{\Gamma}(G)$ or $\widetilde{\Gamma}=\{\,\theta_1,\dots,\theta_t\,\}$ of irreducible characters.[1] Then a weighted quasigroup structure is defined on $\widetilde{\Gamma}$ by

$$w\colon\widetilde{\Gamma}\to\mathbb{N};\theta_i\mapsto\theta_i(1)^2 \tag{2.4}$$

and

$$\alpha\colon\widetilde{\Gamma}^3\to\mathbb{N};(\theta_i,\theta_j,\theta_k)\mapsto\theta_i(1)\theta_j(1)\theta_k(1)\,\langle\theta_i\cdot\theta_j|\theta_k\rangle \tag{2.5}$$

with scalar product

$$\langle\theta|\varphi\rangle=\frac{1}{n}\sum_{g\in G}\theta(g)\overline{\varphi(g)} \tag{2.6}$$

[1]The notation $\widetilde{\Gamma}$ refers to the dual of the conjugacy class association scheme Γ of G, as discussed in [3, Exs. 2.2, 3.1] [4, Ex. 2.1(2),§2.5], [14, §2.2.3]

for functions $\theta, \varphi \colon G \to \mathbb{C}$ [14, Cor. 5.10]. The structure $(\widetilde{\Gamma}, w, \alpha)$ is called the *weighted character quasigroup* of G. Its gross weight is $\sum_{\theta_i \in \widetilde{\Gamma}} \theta_i(1)^2 = n$.

The Cayley table (2.2) of the weighted character quasigroup $\big(\widetilde{\Gamma}(G), w, \alpha\big)$ has the form

$$
\begin{array}{c||ccccc|ccccc}
* & \lambda_1 & \dots & \lambda_i & \dots & \lambda_l & \nu_1 & \dots & \nu_j & \dots & \nu_n \\
\hline\hline
\lambda_1 & \lambda_1 & \dots & \lambda_i & \dots & \lambda_l & & \dots & \nu_j(1)^2\nu_j & \dots & \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\
\lambda_{i'} & \lambda_{i'} & \dots & \lambda_{i'}\lambda_i & \dots & \lambda_{i'}\lambda_l & & \dots & \nu_j(1)^2\lambda_{i'}\nu_j & \dots & \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\
\lambda_l & \lambda_l & \dots & \lambda_l\lambda_i & \dots & \lambda_l\lambda_l & & \dots & \nu_j(1)^2\lambda_l\nu_j & \dots & \\
\hline
\nu_1 & \nu_1(1)^2\nu_1 & \dots & \nu_1(1)^2\nu_1\lambda_i & \dots & & & \dots & & \dots & \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\
\nu_{j'} & \nu_{j'}(1)^2\nu_{j'} & \dots & \nu_{j'}(1)^2\nu_{j'}\lambda_i & \dots & & & \dots & N_{j',j} & \dots & \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\
\nu_m & \nu_m(1)^2\nu_m & \dots & \nu_m(1)^2\nu_m\lambda_i & \dots & & & \dots & & \dots &
\end{array}
\tag{2.7}
$$

where $\{ \lambda_1, \dots, \lambda_l \}$ is the set of linear characters (with l being the order of the commutator quotient group $G/[G,G]$), and $\{ \nu_1, \dots, \nu_m \}$ is the set of non-linear irreducible characters (with $m = t - l$) of G. Here, the entries of the bottom right $m \times m$ square, the *non-linear square*, are given according to (2.5) by

$$
N_{j',j} = \sum_{k=1}^{l} \nu_{j'}(1)\nu_j(1) \left\langle \nu_{j'}\nu_j | \lambda_k \right\rangle \lambda_k + \sum_{k=1}^{m} \nu_{j'}(1)\nu_j(1)\nu_k(1) \left\langle \nu_{j'}\nu_j | \nu_k \right\rangle \nu_k
$$

for $1 \leq j', j \leq m$.

Since character multiplication is commutative, the table (2.7) is symmetric about its main diagonal. The top left $l \times l$-square or *linear square* is just the Cayley table of the Pontryagin dual $\widehat{G/[G,G]}$ of the commutator quotient group of G.

The top right $l \times m$-rectangle is called the *mixed rectangle*. Each of its entries (and thus also each entry of the unnamed bottom left $m \times l$-rectangle) is just a positive integer multiple of a single non-linear character, because multiplication of irreducible characters by a linear character permutes the linear, and thus also the non-linear characters. (Compare [11, Th. 4.3], [15, Th. 7.9], for example, using a purely character-theoretic argument that also works in the non-associative quasigroup case.)

Consider linear character $\lambda_{i'}$ and non-linear character ν_j. Suppose $\lambda_{i'}\nu_j = \nu_k$. Then according to (2.5),

$$\alpha(\lambda_{i'}, \nu_j, \nu_k) = \lambda_{i'}(1)\nu_j(1)\nu_k(1)\,\langle \lambda_{i'}\nu_j | \nu_k \rangle$$
$$= \nu_j(1)(\lambda_{i'}\nu_j)(1)\,\langle \nu_k | \nu_k \rangle = \nu_j(1)^2\,,$$

verifying the form displayed in (2.7) for the elements of the mixed rectangle and its reflection in the main diagonal.

2.5. **Character quasigroups.** Specializing the conditions of §2.3, the weighted character quasigroup lifts to a quasigroup Q if there exists a surjection $f\colon Q \to \widetilde{\Gamma}$ such that $\theta_i(1)^2 = |f^{-1}\{\theta_i\}|$ for $\theta_i \in \widetilde{\Gamma}$ and

$$\alpha(\theta_i, \theta_j, \theta_l) = \big|\, \{\, (a, b) \in f^{-1}\{\theta_i\} \times f^{-1}\{\theta_j\} \;\big|\; f(a \cdot b) = \theta_l \,\}\, \big| \tag{2.8}$$

for all $\theta_i, \theta_j, \theta_l \in \widetilde{\Gamma}$. A quasigroup cover Q of $\big(\widetilde{\Gamma}(G), w, \alpha\big)$ is called a *character quasigroup* of G, and a *character group* of G if the cover is associative. The existence of character quasigroups is indeed guaranteed for any finite group G [14, Th. 5.11], but the existence of character groups is an open question in general, one of the prime motivations for the current paper. The following result is worthy of note.

Theorem 2.2. *Let Q be a character quasigroup of a finite group G. Then the Pontryagin dual $\widehat{G/[G,G]}$ of the commutator quotient group of G is a subgroup of the quasigroup Q.*

Proof. Suppose $f\colon Q \to \widetilde{\Gamma}$ is a covering. Consider a linear character λ of G. Then $|f^{-1}\{\lambda\}| = \lambda(1)^2 = 1$ implies that the corestriction of f to the set Λ of linear characters of G is bijective. Furthermore, (2.5) and (2.8) imply that

$$f^{-1}(\lambda)f^{-1}(\lambda') = f^{-1}(\lambda \cdot \lambda')$$

for $\lambda, \lambda' \in \Lambda$, so the corestriction of f to Λ is an isomorphism from the subquasigroup $f^{-1}(\Lambda)$ of Q to the group $\widehat{G/[G,G]}$. $\square$

Remark 2.3. As noted in the introduction, the relationship between finite groups and their character quasigroups cannot be expected to produce a full duality like Pontryagin duality. Nevertheless, Theorem 2.2 does give at least one indication that aspects of a duality may appear. In this case, abelian quotients of the primal group G correspond to certain abelian subgroups of the dual character quasigroup Q.

3. Characters of dihedral and generalized quaternion groups

3.1. **Character tables of dihedral groups.** We establish notation for conjugacy classes, irreducible representations and characters of the dihedral groups

$$D_d = \langle r, s | r^d = s^2 = (rs)^2 = 1 \rangle \tag{3.1}$$

of degree d (order $2d$) (compare [13, §5.3], say), depending on the parity of d.

3.1.1. *Even degree.* When d is even, say $d = 2k$, the conjugacy classes are

$$\begin{aligned} [sr] &= \left\{ sr^{2i+1} \mid 0 \le i < k \right\} , \\ [s] &= \left\{ sr^{2i} \mid 0 \le i < k \right\} , \text{ and} \\ [r^i] &= \left\{ r^{\pm i} \right\} \text{ for } 0 \le i \le k \,. \end{aligned}$$

There are four linear irreducible representations/characters:

$$\begin{aligned} &\chi_1 \colon D_d \to \mathbb{C}; r \mapsto 1, s \mapsto 1 \,, \\ &\chi_2 \colon D_d \to \mathbb{C}; r \mapsto 1, s \mapsto -1 \,, \\ &\chi_3 \colon D_d \to \mathbb{C}; r \mapsto -1, s \mapsto 1 \,, \\ &\chi_4 \colon D_d \to \mathbb{C}; r \mapsto -1, s \mapsto -1 \,. \end{aligned}$$

The remaining $k-1$ irreducible representations are two-dimensional, given by

$$\rho_h \colon D_d \to \mathbb{C}_2^2; r \mapsto \begin{bmatrix} \omega^h & 0 \\ 0 & \overline{\omega^h} \end{bmatrix}, \ s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.2}$$

for $0 < h < k$, where ω is the d-th root of unity $\exp(2\pi i/d)$. We will follow the usual abuse of notation by using the same symbol ρ_h for the corresponding character $\operatorname{tr} \rho_h$ (compare [17, §3.5], for example), disambiguating verbally on the rare occasions when that becomes necessary.

The character table is

	$[e]$	$[r]$	$[r^2]$	$\cdots$	$[r^k]$	$[sr]$	$[sr^2]$
χ_1	1	1	1	$\cdots$	1	1	1
χ_2	1	1	1	$\cdots$	1	-1	-1
χ_3	1	-1	$(-1)^2$	$\cdots$	$(-1)^k$	-1	1
χ_4	1	-1	$(-1)^2$	$\cdots$	$(-1)^k$	1	-1
ρ_1	2	$\omega^1 + \overline{\omega^1}$	$\omega^2 + \overline{\omega^2}$	$\cdots$	$\omega^k + \overline{\omega^k}$	0	0
ρ_2	2	$\omega^2 + \overline{\omega^2}$	$\omega^4 + \overline{\omega^4}$	$\cdots$	$\omega^{2k} + \overline{\omega^{2k}}$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
ρ_{k-1}	2	$\omega^{k-1} + \overline{\omega^{k-1}}$	$\omega^{2k-2} + \overline{\omega^{2k-2}}$	$\cdots$	$\omega^{k(k-1)} + \overline{\omega^{k(k-1)}}$	0	0

3.1.2. *Odd degree.* When d is odd, say $d = 2k+1$, the conjugacy classes are

$$[sr] = \left\{ sr^i \mid 1 \le i \le d \right\} ,$$
$$[r^i] = \left\{ r^{\pm i} \right\} \text{ for } 0 \le i \le k .$$

In this case there are only two linear characters

$$\chi_1 \colon D_d \to \mathbb{C}; r \mapsto 1, s \mapsto 1 ,$$
$$\chi_2 \colon D_d \to \mathbb{C}; r \mapsto 1, s \mapsto -1 .$$

There are k two-dimensional irreducible representations, again given by (3.2), but this time for $0 < h \le k$. The character table is

	$[e]$	$[r]$	$[r^2]$	$\cdots$	$[r^k]$	$[sr]$
χ_1	1	1	1	$\cdots$	1	1
χ_2	1	1	1	$\cdots$	1	-1
ρ_1	2	$\omega^1 + \overline{\omega^1}$	$\omega^2 + \overline{\omega^2}$	$\cdots$	$\omega^k + \overline{\omega^k}$	0
ρ_2	2	$\omega^2 + \overline{\omega^2}$	$\omega^4 + \overline{\omega^4}$	$\cdots$	$\omega^{2k} + \overline{\omega^{2k}}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	0
ρ_k	2	$\omega^k + \overline{\omega^k}$	$\omega^{2k+2} + \overline{\omega^{2k+2}}$	$\cdots$	$\omega^{k^2} + \overline{\omega^{k^2}}$	0

3.2. **Products of characters.** It will be convenient to extend the scope of the index h in (3.2) to arbitrary integers interpreted modulo d, as is consistent with its actual appearance there in terms of ω^h. Furthermore, when discussing representations up to equivalence, or characters, the convention $\rho_h = \rho_{-h}$ will be invoked. Again, this is seen to be consistent.

When d is even, say $d = 2k$, the multiplication of the non-linear characters $\rho_1, \dots, \rho_{k-1}$ is commutative and satisfies $\rho_n \cdot \rho_m = \rho_{k-n} \cdot \rho_{k-m}$. Assuming w.l.o.g. that $n \geq m$, it is given by

$$\rho_n \cdot \rho_m = \begin{cases} \rho_{n-m} + \rho_{n+m} & \text{if } n > m \text{ and } n + m \neq k\,, \\ \rho_{n-m} + \chi_3 + \chi_4 & \text{if } n > m \text{ and } n + m = k\,, \\ \chi_1 + \chi_2 + \rho_{n+m} & \text{if } n = m \neq k/2\,, \\ \chi_1 + \chi_2 + \chi_3 + \chi_4 & \text{if } n = m = k/2\,. \end{cases}$$

If d is odd, say $d = 2k+1$, the multiplication of the non-linear characters $\rho_1, \dots, \rho_k$ is commutative and satisfies $\rho_n \cdot \rho_m = \rho_{k+1-n} \cdot \rho_{k+1-m}$. Assuming w.l.o.g. that $n \geq m$, it is given by

$$\rho_n \cdot \rho_m = \begin{cases} \rho_{n-m} + \rho_{n+m} & \text{if } n > m\,, \\ \chi_1 + \chi_2 + \rho_{n+m} & \text{if } n = m\,, \end{cases}$$

We would like to be able to treat all of these cases together, which can be done as follows. Always write the product as

$$\rho_n \cdot \rho_m = \rho_{n-m} + \rho_{n+m}\,, \tag{3.3}$$

with the convention $\rho_0 = \chi_1 + \chi_2$. In addition, if $d = 2k$ is even, then we also make the convention that $\rho_k = \chi_3 + \chi_4$. The products work out as they should in all cases. Indeed, our conventions are consistent with (3.2), noting only that the corresponding two-dimensional representations for $h = 0$, or for $h = k$ when $d = 2k$, are no longer irreducible (compare [13]). Then (3.3) may be verified directly on the basis of (3.2).

3.3. **Weighted character quasigroups of D_d.** Using computational observations from the preceding section, this section interprets the general description from §2.4 in the specific context of the dihedral groups.

3.3.1. *The linear square.* The commutator quotient of the dihedral group D_d is $\mathbb{Z}/2$ in the odd degree case and $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ in the even degree case.

Lemma 3.1. *The linear square of the weighted character quasigroup of D_d takes the respective forms*

χ_1	χ_2	χ_3	χ_4
χ_2	χ_1	χ_4	χ_3
χ_3	χ_4	χ_1	χ_2
χ_4	χ_3	χ_2	χ_1

and

χ_1	χ_2
χ_2	χ_1

in the even and odd degree cases.

3.3.2. *The mixed rectangle.* We have the relations

$$\chi_1 \cdot \rho_h = \chi_2 \cdot \rho_h = \rho_h$$

in all cases, and

$$\chi_3 \cdot \rho_h = \chi_4 \cdot \rho_h = \rho_{k-h}$$

in the case of even degree $d = 2k$.

Lemma 3.2. *The labeled mixed rectangle of the weighted character quasigroup of D_d takes the respective forms*

$*$	ρ_1	ρ_2	$\cdots$	ρ_{k-1}
χ_1	$4\rho_1$	$4\rho_2$	$\cdots$	$4\rho_{k-1}$
χ_2	$4\rho_1$	$4\rho_2$	$\cdots$	$4\rho_{k-1}$
χ_3	$4\rho_{k-1}$	$4\rho_{k-2}$	$\cdots$	$4\rho_1$
χ_4	$4\rho_{k-1}$	$4\rho_{k-2}$	$\cdots$	$4\rho_1$

and

$*$	ρ_1	ρ_2	$\cdots$	ρ_k
χ_1	$4\rho_1$	$4\rho_2$	$\cdots$	$4\rho_k$
χ_2	$4\rho_1$	$4\rho_2$	$\cdots$	$4\rho_k$

in the case of even degree $d = 2k$ and odd degree $d = 2k + 1$.

3.3.3. *The non-linear square.*

Lemma 3.3. *If the degree $d = 2k$ is even, the labeled non-linear square has the form:*

$*$	ρ_1	ρ_2		ρ_{k-2}	ρ_{k-1}
ρ_1	$4\chi_1 + 4\chi_2 + 8\rho_2$	$8\rho_1 + 8\rho_3$	$\cdots$	$8\rho_{k-3} + 8\rho_{k-1}$	$4\chi_3 + 4\chi_4 + 8\rho_{k-2}$
ρ_2	$8\rho_1 + 8\rho_3$	$\ddots$		$\iddots$	$8\rho_{k-3} + 8\rho_{k-1}$
	$\vdots$				$\vdots$
ρ_{k-2}	$8\rho_{k-3} + 8\rho_{k-1}$	$\iddots$		$\ddots$	$8\rho_1 + 8\rho_3$
ρ_{k-1}	$4\chi_3 + 4\chi_4 + 8\rho_{k-2}$	$8\rho_{k-3} + 8\rho_{k-1}$	$\cdots$	$8\rho_1 + 8\rho_3$	$4\chi_1 + 4\chi_2 + 8\rho_2$

If the degree $d = 2k + 1$ is odd, the labeled non-linear square has the form:

$*$	ρ_1	ρ_2		ρ_{k-1}	ρ_k
ρ_1	$4\chi_1 + 4\chi_2 + 8\rho_2$	$8\rho_1 + 8\rho_3$	$\cdots$	$8\rho_{k-2} + 8\rho_k$	$8\rho_{k-1} + 8\rho_k$
ρ_2	$8\rho_1 + 8\rho_3$	$\ddots$		$\iddots$	$8\rho_{k-2} + 8\rho_{k-1}$
	$\vdots$				$\vdots$
ρ_{k-1}	$8\rho_{k-2} + 8\rho_k$	$\iddots$		$\ddots$	$8\rho_1 + 8\rho_2$
ρ_k	$8\rho_{k-1} + 8\rho_k$	$8\rho_{k-2} + 8\rho_{k-1}$	$\cdots$	$8\rho_1 + 8\rho_2$	$4\chi_1 + 4\chi_2 + 8\rho_1$

Remark 3.1. In the even degree case, the non-linear square is symmetric about the off diagonal:

$$\rho_{k-n} * \rho_{k-m} = 8\rho_{(k-n)-(k-m)} + 8\rho_{(k-n)+(k-m)} = 8\rho_{m-n} + 8\rho_{-(m+n)} = \rho_n * \rho_m \,.$$

for $n \neq m$. In the odd degree case,

$$\rho_{k+1-n} * \rho_{k+1-m} = 8\rho_{(k+1-n)-(k+1-m)} + 8\rho_{(k+1-n)+(k+1-m)} = 8\rho_{m-n} + 8\rho_{m+n-1} \,,$$

while $\rho_n * \rho_m = 8\rho_{m-n} + 8\rho_{m+n}$ for $n \neq m$. Thus in this case, reflection in the off-diagonal still preserves the first summand $8\rho_{m-n}$ in the weighted quasigroup entry, and just implements the transposition $\big((m+n-1)\ (m+n)\big)$ on the index of the second summand.

3.4. **Generalized quaternion groups.** Take $1 < n \in \mathbb{Z}$. The presentation (3.1) for $d = 2^n$ is $D_d = \langle r, s | r^d = 1,\ s^2 = 1,\ (rs)^2 = 1\rangle$. The (*generalized*) *quaternion group* of order $2d$ is then defined by

$$Q_{2d} = \langle r, s | r^d = 1,\ s^2 = r^{d/2},\ rs^{-1}rs = 1\rangle$$

[6, p.63]. The groups D_d and Q_{2d} are not isomorphic: s^2 is the unique involution of Q_{2d} [6, p.63]. Nevertheless, the character tables of D_d and Q_{2d} coincide [6, p.64]. Thus the weighted character quasigroups of Q_{2d} are given by the respective even-degree parts of Lemmas 3.1, 3.2, and 3.3.

4. The character groups

4.1. **Quasigroup semialgebras.** Let Q be a quasigroup, say a character quasigroup for a finite group G. Let $\mathbb{N}Q$ be the free $\mathbb{N}$-semimodule over Q, with the product $\cdot$ defined by semilinear extension of the quasigroup multiplication on Q. In particular, if Q is a group, then $\mathbb{N}Q$ forms the fragment of the integral group algebra where all the coefficients are nonnegative.

A multisubset M of Q, where each element q of Q appears in M with muliplicity m_q, is represented in $\mathbb{N}Q$ as

$$M = \sum_{q \in Q} m_q q \,.$$

For example, given two subsets $S_i, S_j \subseteq Q$, the coefficient of an element q of Q in the product $S_i \cdot S_j$ counts the number of times q is expressed as a product of an element of S_i with an element of S_j.

4.2. **Characterizing character quasigroups.**

4.2.1. *The even degree case.*

Theorem 4.1. *Let Q be a finite quasigroup of order $2d$, where $d = 2k$ is even. Then Q functions as a character quasigroup of D_d, or of Q_{2d} for $d = 2^n$ with $1 < n \in \mathbb{Z}$, if and only if there exist elements $q_1, q_2, q_3, q_4 \in Q$ and subsets $S_i \subseteq Q$ for $0 < i < k$, each of size 4, along with $S_0 = \{ q_1, q_2 \}$ and $S_k = \{ q_3, q_4 \}$, such that the conditions*

(0) *Q is the disjoint union of the sets $S_0, \ldots, S_k$;*

(1) *$S_0 \cup S_k \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ as a subgroup of Q;*

$$(2) \begin{cases} \text{(a)} \quad q_j S_i = S_i = S_i q_j & \text{for } 0 < i < k \text{ and } j = 1, 2\,, \\ \text{(b)} \quad q_j S_i = S_{k-i} = S_i q_j & \text{for } 0 < i < k \text{ and } j = 3, 4\,; \end{cases}$$

(3) *$S_j \cdot S_i = S_i \cdot S_j = S_{k-i} \cdot S_{k-j}$ for $0 < i, j, i + j < k$;*

(4) *for $0 < i, j < k$,*

$$S_i \cdot S_j = \begin{cases} 4S_0 + 4S_k & \text{if } \ i = j = k/2, \\ 2S_{i+j} + 4S_0 & \text{if } \ i = j \neq k/2, \\ 4S_k + 2S_{i-j} & \text{if } \ i + j = k \text{ and } i \neq j, \\ 2S_{i+j} + 2S_{i-j} & \text{otherwise} \end{cases}$$

are satisfied. In condition (4), the indices h on the sets S_h are interpreted as residues modulo d, with the convention that $S_h = S_{-h}$.

Proof. By §3.4, it suffices to focus on the dihedral group case. Suppose that Q is a character quasigroup of D_d, with covering $f \colon Q \to \widetilde{\Gamma}(D_d)$. Define the elements $q_i = f^{-1}(\chi_i)$ for $1 \leq i \leq 4$ and sets $S_i = f^{-1}\{\rho_i\}$ for $0 < i < k$. The conditions (1)–(4) follow immediately from the results of the previous sections.

Conversely, suppose that we are given a finite quasigroup Q, with elements q_i for $1 \leq i \leq 4$ and subsets S_j, of cardinality 4 for $0 < j < k$ and 2 for $j = 0, k$, satisfying the conditions (1)–(4). We claim that Q is a character quasigroup of D_d. Define the surjective map $f \colon Q \to \widetilde{\Gamma}(D_d)$ by

$$q_i \mapsto \chi_i \quad \text{ and } \quad S_j \to \{\rho_j\}$$

for $1 \leq i \leq 4$ and $0 < j < k$. This mapping satisfies

$$|f^{-1}\{\chi_i\}| = 1 = \chi_i(1)^2 \ \text{ and } \ |f^{-1}\{\rho_j\}| = 4 = \rho_j(1)^2$$

by assumption. Thus, it remains to check that that (2.8) holds.

Condition (1) implies that for $1 \leq i, j \leq 4$, the equation $q_i \cdot q_j = q_l$ holds for exactly one $1 \leq l \leq 4$, whence (2.8) holds for $((\theta_1, \theta_2, \theta_3) \in \{ \chi_i \mid 1 \leq i \leq 4 \}^2 \times \widetilde{\Gamma}$.

By condition (2), we have

$$\{\rho_j\} = f\{S_j\} = f\{q_1 \cdot S_j\} = f\{q_2 \cdot S_j\} \text{ and}$$
$$\{\rho_{k-j}\} = f\{S_{k-j}\} = f\{q_3 \cdot S_j\} = f\{q_4 \cdot S_j\} .$$

These equations say that there are exactly four pairs $(a, b) \in f^{-1}\{\chi_i\} \times f^{-1}\{\rho_j\}$ such that $f\{ab\} = \{\rho_j\}$. Thus for

$$(\theta_1, \theta_2, \theta_3), (\theta_2, \theta_1, \theta_3) \in \{ \chi_i \mid 1 \leq i \leq 4 \} \times \{ \rho_j \mid 0 < j < k \} \times \widetilde{\Gamma} ,$$

(2.8) holds

In the remaining cases, the condition of (2.8) is equivalent to saying that the sum of the multiplicities of elements of $S_i \cdot S_j$ that are contained in S_l is equal to $\alpha(i, j, l)$. The condition (3) allows us to restrict to $i \geq j \geq 1$ and $1 \leq i + j \leq k$. There are now four cases to consider:

(i) $i = j = k/2$;

(ii) $i = j \neq k/2$;

(iii) $i + j = k$ and $i \neq j$;

(iv) none of the above.

Case (i): If $i = j = k/2$, then condition (4) says that $S_{k/2}^2 = 4S_0 + 4S_k$, which agrees with the associated product $\rho_{k/2}^2 = \chi_1 + \chi_2 + \chi_3 + \chi_4$ in the character ring after scaling appropriately with the numerical factors.

Case (ii): If $i = j \neq k/2$, then condition (4) says that $S_i^2 = 2S_{2i} + 4S_0$, which agrees with the associated product $\rho_i^2 = \rho_{2i} + \chi_1 + \chi_2$ in the character ring after scaling.

Case (iii): If $i + j = k$ and $i \neq j$, then condition (4) says that $S_i \cdot S_j = 2S_{i-j} + 4S_k$ which again agrees with the product $\rho_i \cdot \rho_j = \rho_{i-j} + \chi_3 + \chi_4$ in the character ring after scaling.

Case (iv): If none of the previous cases apply, then condition (4) says that $S_i \cdot S_j = 2S_{i+j} + 2S_{i-j}$, which once again agrees with the product $\rho_i \cdot \rho_j = \rho_{i+j} + \rho_{i-j}$ in the character ring after scaling.

Therefore, (2.8) is satisfied in all cases, and so f is indeed a covering, making Q a character quasigroup as desired. $\square$

Definition 4.2. In the context of Theorem 4.1, with even degree $d = 2k$, the sets S_i for $0 < i < k$ will be called *quartets.*

4.2.2. *The odd degree case.* Analogous arguments give the corresponding theorem for the odd case.

Theorem 4.3. *Let Q be a finite quasigroup of order $2d$, where $d = 2k+1$ is odd. Then Q functions as a character quasigroup of D_d if and only if there exist elements $q_1, q_2 \in Q$ and subsets $S_i \subseteq Q$ for $1 \le i \le k$, each of size 4, along with $S_0 = \{ q_1, q_2 \}$, such that the conditions*

(0) *Q is the disjoint union of the sets $S_0, \dots, S_k$;*

(1) *$S_0 \cong \mathbb{Z}/2$ as a subquasigroup of Q;*

(2) *$q_j S_i = S_i = S_i q_j$ for $1 \le i \le k$ and $j = 1, 2$;*

(3) *$S_j \cdot S_i = S_i \cdot S_j = S_{k+1-i} \cdot S_{k+1-j}$ for $1 \le i + j \le k+1$;*

(4) *for $1 \le i, j \le k$,*

$$S_i \cdot S_j = \begin{cases} 2S_k + 4S_0 & \text{if } i = j = \frac{k+1}{2}\,, \\ 2S_{i+j} + 4S_0 & \text{if } \ i = j \neq \frac{k+1}{2}\,, \\ 2S_k + 2S_{i-j} & \text{if } \ i + j = k+1 \text{ and } \ i \neq j\,, \\ 2S_{i+j} + 2S_{i-j} & \text{otherwise} \end{cases}$$

are satisfied. In condition (4), the indices h on the sets S_h are interpreted as residues modulo d, with the convention that $S_h = S_{-h}$.

Definition 4.4. In the context of Theorem 4.3, with odd degree $d = 2k+1$, the sets S_i for $1 \le i \le k$ will be called *quartets.*

4.2.3. *Symmetry breaking.* In the statements of Theorems 4.1 and 4.3, the role of the elements q_1 and q_2 is symmetric. Henceforth, the symmetry will be broken in standard fashion as follows.

Corollary 4.1. *Suppose that, in the context of Theorems 4.1 and 4.3, the quasigroup Q is a character quasigroup of D_d. Then the elements q_1 and q_2 may be chosen without loss of generality so that $q_1 q_2 = q_2$. In particular, if Q is a group, then $q_1 = 1$.*

Proof. The corollary follows from the choices $f\colon q_i \mapsto \chi_i$ made in the proofs of the theorems, on the basis of Lemma 3.1. $\square$

4.2.4. *Form of the quartets.*

Corollary 4.2. *In the context of Theorems 4.1 and 4.3, suppose that the quasigroup Q is a character quasigroup of D_d.*

(a) *The quartets S_i (for $i \neq d/4$) are of the form*

$$S_i = \{ x_i, y_i, q_2 x_i, q_2 y_i \} \tag{4.1}$$

for distinct x_i and y_i not in S_0.

(b) *If $d = 2k$ and k is even, then $S_{k/2}$ has the form*

$$\{ x_{k/2}, q_2 x_{k/2}, q_3 x_{k/2}, q_4 x_{k/2} \}$$

for some $x_{k/2} \in Q \smallsetminus (S_0 \cup S_k)$ satisfying $x_{k/2}^2 = q_j$ for some $j \in \{ 1, 2, 3, 4 \}$.

Proof. (a) The form (4.1) of S_i is immediate from condition (2) of Theorems 4.1 and 4.3.

(b) Choose an element $x_{k/2}$ of $S_{k/2}$. Note that $x_{k/2}^2 \in S_0 \cup S_k$ by Theorem 4.1(2) and the first case of Theorem 4.1(4). Then $x_{k/2}^2 = q_j$ for some $j \in \{ 1, 2, 3, 4 \}$. □

4.3. **Character groups.**

4.3.1. *Quartets of character groups.* If a character quasigroup Q of D_d is associative, i.e., a group, then more precise information on the sets S_i of the preceding theorems is available. The specification of q_2 in the following (as distinct from the identity element) is consistent with the symmetry-breaking convention of Corollary 4.1.

Proposition 4.1. *Let Q be a character group for D_d.*

(a) *If $d = 2k$ is even, consider $0 < i < k$, and $i \neq k/2$ if k is even.*

(b) *If $d = 2k + 1$ is odd, consider $1 \leq i \leq k$.*

If the element x_i of the quartet S_i from (4.1) commutes with q_2, then exactly one of the conditions

(1) $x_i^2 \in S_0$ *and* $S_{2i} = \{ x_i y_i, y_i x_i, q_2 x_i y_i, q_2 y_i x_i \}$,

(2) $x_i y_i \in S_0$ *and* $S_{2i} = \{ x_i^2, y_i^2, q_2 x_i^2, q_2 y_i^2 \}$

holds.

Proof. To begin, note that the respective first statements of conditions (1) and (2) are mutually exclusive. Indeed, suppose that

$$\{ x_i^2, x_i y_i \} = \{ 1, q_2 \} . \tag{4.2}$$

- If $x_i^2 = 1$, then $y_i = x_i^{-1} q_2 = x_i q_2 = q_2 x_i$.
- If $x_i y_i = 1$, then $q_2 = x_i^2$ and $q_2 y_i = x_i$.

Either case contradicts the fact that S_i has cardinality 4, as would the coalescence $x_i = y_i$ of the conditions (1) and (2).

The second part of the condition of Theorem 4.1(4) in case (a), or the first two parts of the condition of Theorem 4.3(4) in case (b), imply that $S_i \cdot S_i$ as displayed in the body of the Cayley table

$\cdot$	x_i	y_i	$q_2 x_i$	$q_2 y_i$
x_i	x_i^2	$x_i y_i$	$q_2 x_i^2$	$q_2 x_i y_i$
y_i	$y_i x_i$	y_i^2	$y_i q_2 x_i$	$y_i q_2 y_i$
$q_2 x_i$	$q_2 x_i^2$	$q_2 x_i y_i$	x_i^2	$x_i y_i$
$q_2 y_i$	$q_2 y_i x_i$	$q_2 y_i^2$	$q_2 y_i q_2 x_i$	$q_2 y_i q_2 y_i$

should contain 4 copies of both 1 and q_2. For the completion of the table, recall $q_2^2 = 1$ by condition (1) of Theorems 4.1 or 4.3.

If neither of the respective first statements of conditions (1) and (2) were to hold, the need to have four copies of S_0 in the body of the Cayley table would force $\{ q_2 x_i^2, q_2 x_i y_i \} = \{ 1, q_2 \}$. Multiplication of both sides of this equation by q_2 would then lead back to the contradiction (4.2). The rest of the proof thus reduces to the following two cases.

Case I: $x_i^2 \in S_0$. Among the remaining elements of the Cayley table body that are not in S_0, there are two copies of each element in the set

$$\{ x_i y_i, y_i x_i, q_2 x_i y_i, q_2 y_i x_i \} ,$$

which must be S_{2i}.

Case II: $x_i y_i \in S_0$. Then the remaining elements of the Cayley table body are

$$\{ x_i^2, q_2 x_i^2, y_i q_2 y_i, q_2 y_i^2, q_2 y_i q_2 y_i, y_i^2 \} ,$$

comprising two copies of x_i^2 and $q_2x_i^2$, and one copy of the rest. This implies that the set consisting of the last four elements above must have size 2. If $y_iq_2y_i = q_2y_iq_2y_i$ or $y_iq_2y_i = y_i^2$, then $q_2 = 1$, a contradiction. Thus $y_iq_2y_i = q_2y_i^2$, and

$$S_{2i} = \{ x_i^2, y_i^2, q_2x_i^2, q_2y_i^2 \}$$

as required. □

4.4. **Semidirect products.** For each integer $m > 1$, take the additive group $\mathbb{Z}/m$ of residues modulo m. Consider a group homomorphism $\sigma\colon \mathbb{Z}/2 \to \mathrm{Aut}\,(\mathbb{Z}/d)$.

Theorem 4.5. *The semidirect product $\mathbb{Z}/d \rtimes_\sigma \mathbb{Z}/2$ is a character group for D_d.*

Proof. We use the notational conventions that $a := (a, 0)$ and $\overline{a} := (a, 1)$ for $a \in \mathbb{Z}/d$. When $d = 2k + 1$ is odd, define the elements $q_1 = 0$, $q_2 = \overline{0}$ and k quartets

$$S_i = \left\{ i, \overline{i}, \sigma_1(i), \overline{\sigma_1(i)} \right\} \qquad \text{for } 1 \le i \le k \tag{4.3}$$

of $\mathbb{Z}/d \rtimes_\sigma \mathbb{Z}/2$. When $d = 2k$ is even, define the additional elements $q_3 = k$, $q_4 = \overline{k}$, and restrict the quartet definition (4.5) to the range $0 < i < k$.

The conditions (1),(2) of Theorems 4.1 and 4.3 follow immediately. Next, we verify the condition (3) by noting the identity $i + j = (k + i) + (k + j)$ in Z/d for the even case and $i + j = (k + i) + (k + 1 + j)$ for the odd case. With condition (3) verified and using commutativity it suffices to check condition (4) for $i \ge j$ and $i + j \le k$. To do this, we consider the typical 4×4 segment

(4.4)

$\cdot$	j	$\overline{j}$	$\sigma_1(j)$	$\overline{\sigma_1(j)}$
i	$i+j$	$\overline{i+j}$	$i-j$	$\overline{i-j}$
$\overline{i}$	$\overline{i+\sigma_1(j)}$	$i+\sigma_1(j)$	$\overline{i+\sigma_1(-j)}$	$i+\sigma_1(-j)$
$\sigma_1(i)$	$-i+j$	$\overline{-i+j}$	$-i-j$	$\overline{-i-j}$
$\overline{\sigma_1(i)}$	$\overline{-i+\sigma_1(j)}$	$-i+\sigma_1(j)$	$\overline{-i+\sigma_1(-j)}$	$-i+\sigma_1(-j)$

of the Cayley table. Note that regardless of whether σ_1 fixes or negates j, both actions amount to a permutation of the symbols of the table. □

Theorem 4.6. *The semidirect product $\mathbb{Z}/d \rtimes_\sigma \mathbb{Z}/2$ is a character group for D_d.*

Proof. We use the notational conventions that $a := (a, 0)$ and $\overline{a} := (a, 1)$ for $a \in \mathbb{Z}/d$. When $d = 2k + 1$ is odd, define the elements $q_1 = 0$, $q_2 = \overline{0}$ and k quartets

$$S_i = \{ i, \overline{i}, -i, \overline{-i} \} \qquad \text{for } 1 \le i \le k \tag{4.5}$$

of $\mathbb{Z}/d \rtimes_\sigma \mathbb{Z}/2$. When $d = 2k$ is even, define the additional elements $q_3 = k$, $q_4 = \overline{k}$, and restrict the quartet definition (4.5) to the range $0 < i < k$.

The conditions (1),(2) of Theorems 4.1 and 4.3 follow immediately. Next, we verify the condition (3) by noting the identity $i + j = (k + i) + (k + j)$ in Z/d for the even case and $i + j = (k + i) + (k + 1 + j)$ for the odd case. With condition (3) verified and using commutativity it suffices to check condition (4) for $i \geq j$ and $i + j \leq k$. To do this, we consider the typical 4×4 segment

(4.6)

$\cdot$	j	$\overline{j}$	$-j$	$\overline{-j}$
i	$i+j$	$\overline{i+j}$	$i-j$	$\overline{i-j}$
$\overline{i}$	$\overline{i+\sigma_1(j)}$	$i+\sigma_1(j)$	$\overline{i+\sigma_1(-j)}$	$i+\sigma_1(-j)$
$-i$	$-i+j$	$\overline{-i+j}$	$-i-j$	$\overline{-i-j}$
$\overline{-i}$	$\overline{-i+\sigma_1(j)}$	$-i+\sigma_1(j)$	$\overline{-i+\sigma_1(-j)}$	$-i+\sigma_1(-j)$

of the Cayley table. Note that regardless of whether σ_1 fixes or negates j, both actions amount to a permutation of the symbols of the table. □

Corollary 4.3. *The groups $\mathbb{Z}/d \oplus \mathbb{Z}/2$ and D_d may both serve as character groups of D_d, and of Q_{2d} if $d = 2^n$ with $1 < n \in \mathbb{Z}$.*

Proof. The two given character groups correspond to the respective cases of the theorem where the action of σ is trivial or nontrivial. □

4.5. **The dihedral group D_4 and quaternion group Q_8.** Illustrating much of the work done so far, consider the dihedral group D_4 of degree 4 (order 8). Of course, in tandem, the quaternion group Q_8 is also treated.

The weighted quasigroup $(\widetilde{\Gamma}, *)$ of irreducible characters is

(4.7)

$*$	χ_1	χ_2	χ_3	χ_4	ρ_5
χ_1	χ_1	χ_2	χ_3	χ_4	$4\rho_5$
χ_2	χ_2	χ_1	χ_4	χ_3	$4\rho_5$
χ_3	χ_3	χ_4	χ_1	χ_2	$4\rho_5$
χ_4	χ_4	χ_3	χ_2	χ_1	$4\rho_5$
ρ_5	$4\rho_5$	$4\rho_5$	$4\rho_5$	$4\rho_5$	$4\chi_1+4\chi_2+4\chi_3+4\chi_4$

With the coverings

$$0 \mapsto \chi_1,\ \bar{0} \mapsto \chi_2,\ 2 \mapsto \chi_3,\ \bar{2} \mapsto \chi_4, \quad \{ 1, 3, \bar{1}, \bar{3} \} \to \{ \rho_5 \} \ ,$$

Corollary 4.3 offers

+	0	$\bar{0}$	2	$\bar{2}$	1	$\bar{1}$	3	$\bar{3}$
0	0	$\bar{0}$	2	$\bar{2}$	1	$\bar{1}$	3	$\bar{3}$
$\bar{0}$	$\bar{0}$	0	$\bar{2}$	2	$\bar{1}$	1	$\bar{3}$	3
2	2	$\bar{2}$	0	$\bar{0}$	3	$\bar{3}$	1	$\bar{1}$
$\bar{2}$	$\bar{2}$	2	$\bar{0}$	0	$\bar{3}$	3	$\bar{1}$	1
1	1	$\bar{1}$	3	$\bar{3}$	2	$\bar{2}$	0	$\bar{0}$
$\bar{1}$	$\bar{1}$	1	$\bar{3}$	3	$\bar{2}$	2	$\bar{0}$	0
3	3	$\bar{3}$	1	$\bar{1}$	0	$\bar{0}$	2	$\bar{2}$
$\bar{3}$	$\bar{3}$	3	$\bar{1}$	1	$\bar{0}$	0	$\bar{2}$	2

to realize $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ as a character group of D_4.

4.5.1. *The Boolean cube.* Another character group of D_4 is the Boolean group $(\mathbb{Z}/2)^3$, with elements

$$q_1 = (0,0,0), \quad q_2 = (0,1,0) \quad q_3 = (1,0,0), \quad q_4 = (1,1,0)$$

and quartet

$$S_1 = (0,0,1) + \{q_1, q_2, q_3, q_4\}$$

chosen to match the conditions of Theorem 4.1.

4.6. **Abelian character groups.** The next two lemmas show that the Boolean group example does not carry over to larger values of d.

Lemma 4.1. *If an abelian group Q is a character group for D_d, then for each $i \neq \frac{k}{2}$ the quartet S_i must be of the form $S_i = \{ x_i, x_i^{-1}, q_2 x_i, q_2 x_i^{-1} \}$.*

Proof. Recall the general form (4.1) for the quartets. In the present setting, the option of Proposition 4.1(1) is excluded, since $\{ x_i y_i, y_i x_i, q_2 x_i y_i, q_2 y_i x_i \}$ collapses to a two-element set in the abelian case. Thus $x_i y_i \in S_0 = \{ 1, q_2 \}$. If $x_i y_i = 1$, then the claim follows immediately. If $x_i y_i = q_2$, then $y_i = q_2 x_i^{-1}$, which amounts to a permutation of the elements in S_i (swapping x_i^{-1} and $q_2 x_i^{-1}$). □

Lemma 4.2. *Let Q be an abelian character group for D_d. Suppose that $S_1 = \{\, x, x^{-1}, q_2x, q_2x^{-1} \,\}$. Then each quartet is given as $S_i = \{\, x^i, x^{-i}, q_2x^i, q_2x^{-i} \,\}$.*

Proof. The proof proceeds by induction on i. Along with the given form for S_1, the second base case (for $i = 2$) follows from Proposition 4.1(2) with $x_1 = x$ and $y_1 = x^{-1}$. Suppose that the quartet $S_m = \{\, x^m, x^{-m}, q_2x^m, q_2x^{-m} \,\}$ for all $m \leq i$. By condition (4) of Theorems 4.1 and 4.3, we have $2S_{i-1} + 2S_{i+1} = S_1 \cdot S_i =$

$$(x + x^{-1} + q_2x + q_2x^{-1})(x^i + x^{-i} + q_2x^i + q_2x^{-i})$$
$$= 2\left(x^{i-1} + x^{-(i-1)} + q_2x^{i-1} + q_2x^{-(i-1)}\right) + 2\left(x^{i+1} + x^{-(i+1)} + q_2x^{i+1} + q_2x^{-(i+1)}\right)$$

and so $S_{i+1} = \left\{\, x^{i+1}, x^{-(i+1)}, q_2x^{i+1}, q_2x^{-(i+1)} \,\right\}$ as required. □

The following theorem classifies the abelian character groups of finite dihedral groups.

Theorem 4.7. *Consider the dihedral group D_d of degree d.*

(a) *The Boolean cube group $(\mathbb{Z}/2)^3$ is a character group of D_4.*

(b) *With the exception of* (a)*, $\mathbb{Z}/d \oplus \mathbb{Z}/2$ is the unique abelian character group of D_d.*

Proof. When $d > 4$, then $1 \neq k/2$ in the even degree cases. The quartets S_i must be distinct, so by Lemma 4.2, whatever element x is chosen for S_1 must have order d or $2d$. Thus the abelian group is either $\mathbb{Z}/2d$ or $\mathbb{Z}/d \oplus \mathbb{Z}/2$.

- If d is odd, these two cases coincide.
- If d is even, then Theorem 4.1(1) prohibits $\mathbb{Z}/2d$ from being a character group of D_d, because $\mathbb{Z}/2d$ fails to contain $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ as a subgroup.

When $d = 4$, and there is only the one quartet S_1, it is possible for the element x in S_1 to have order 2, leading to (a) as observed in §4.5.1. □

Corollary 4.4. *Consider the generalized quaternion group Q_{2d} for $d = 2^n$ with $1 < n \in \mathbb{Z}$.*

(a) *The Boolean cube group $(\mathbb{Z}/2)^3$ is a character group of Q_8.*

(b) *With the exception of* (a)*, $\mathbb{Z}/d \oplus \mathbb{Z}/2$ is the unique abelian character group of Q_{2d}.*

The following problem remains open.

Problem 4.8. For each degree d, classify the nonabelian character groups of the dihedral group D_d.

It is immediately possible to rule out a generalized quaternion group as a potential candidate for a nonabelian character group of a dihedral group of the same order.

Proposition 4.2. *For $1 < n \in \mathbb{Z}$ and $d = 2^n$, the generalized quaternion group Q_{2d} cannot serve as a character group of D_d.*

Proof. As noted in §3.4, Q_{2d} has a unique involution. But Theorem 4.1(1) requires the presence of at least three involutions in a character group of D_d. □

Corollary 4.5. *No generalized quaternion group can serve as its own character group.*

5. Adams operations and character group powers

5.1. **Adams operations.** Let G be a finite group. Consider the set $\mathbb{C}^G$ of complex-valued functions on G, with pointwise ring structure. For each integer m,

$$\Psi^m \colon \mathbb{C}^G \to \mathbb{C}^G; \varphi \mapsto [g \mapsto \varphi(g^m)] \tag{5.1}$$

defines the *Adams operation* Ψ^m on $\mathbb{C}^G$ [13, Ex. 9.1.3], [16, (4.1.2)], [17, §3.5]. It maps class functions to class functions. Note that if χ is a linear representation and character of the finite group G, then $(\Psi^m\chi)(g) = \chi(g^m) = [\chi(g)]^m = \chi^m(g)$ for $m \in \mathbb{Z}$ and $g \in G$, so $\Psi^m(\chi) = \chi^m$. Thus by Theorem 2.2, power maps $x \mapsto x^m$ in a character group of G will always agree with Adams operations in the linear case.

The main goal of this chapter is to exhibit some connections between the power maps $x \mapsto x^m$ on an abelian character group Q of the dihedral group D_d and the effects of the Adams operations Ψ^m on the nonlinear irreducible characters of D_d. Even Adams operations are typically awkward to handle. For example, the generalized character $\Psi^2\rho_1 = \chi_1 - \chi_2 + \rho_1$ on D_3 is not a character (or in the terminology of [5, p.284], not an *effective* character with non-negative integral coefficients for the irreducible characters). Thus, we restrict to cases where the integer m is odd. Since

$$\left(\Psi^{-m}\chi\right)(g) = \chi\left(g^{-m}\right) = \overline{\chi}(g^m) = \chi(g^m) = \left(\Psi^m\chi\right)(g)$$

for a character χ of D_d and $g \in D_d$, it suffices to consider positive integers m.

5.2. **Adams operations on non-linear characters of D_d.** Abusing notation as discussed after (3.2), and with the conventions from §3.2, we may write

$$\Psi^m \rho_h \colon D_d \to \mathbb{C}_2^2 ; r \mapsto \begin{bmatrix} \omega^{mh} & 0 \\ 0 & \overline{\omega^{mh}} \end{bmatrix}, \ s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.2}$$

on the basis of (3.2) for an odd (positive) integer m, so that

$$\Psi^m \rho_h = \rho_{mh} \tag{5.3}$$

at the character level. The formula (5.3) is interpreted as

$$\Psi^m \rho_h = \chi_1 + \chi_2 \tag{5.4}$$

if $mh \equiv 0 \bmod d$, or

$$\Psi^m \rho_h = \chi_3 + \chi_4 \tag{5.5}$$

if $mh \equiv k \bmod d$ in the case $d = 2k$ of even degree. Otherwise, (5.3) requires no special interpretation.

5.3. **Powers in character groups of D_d.** The context for the following lemma is provided by Theorem 4.6.

Lemma 5.1. *Consider a semidirect product $\mathbb{Z}/d \rtimes_\sigma \mathbb{Z}/2$ as presented in §4.4. Then $h^m = mh$ and*

$$\left(\overline{h}\right)^m = \begin{cases} \overline{(m/2)h + (m/2)\sigma_1(h)} & \textit{for even } m\,; \\ \lceil m/2 \rceil h + \lfloor m/2 \rfloor \sigma_1(h) & \textit{for odd } m \end{cases} \tag{5.6}$$

for $h \in \mathbb{Z}/d$ and $0 < m \in \mathbb{Z}$.

Proof. The equation $h^m = mh$ is immediate from the top left entry $i \cdot j = i + j$ of the Cayley table fragment (4.6) — i.e., just working in the subgroup $\mathbb{Z}/d$. It is clear that (5.6) holds for $m = 1$. Recall

$$i \cdot \overline{j} = \overline{i + j} \quad \text{ and } \quad \overline{i} \cdot \overline{j} = i + \sigma_1(j)$$

for $i, j \in \mathbb{Z}/d$ from the Cayley table fragment (4.6). Suppose that m is odd and $\left(\overline{h}\right)^m = \overline{\lceil m/2 \rceil h + \lfloor m/2 \rfloor \sigma_1(h)}$. Then

$$\begin{aligned} \left(\overline{h}\right)^{m+1} &= \overline{\lceil m/2 \rceil h + \lfloor m/2 \rfloor \sigma_1(h)} \cdot \overline{h} \\ &= \lceil m/2 \rceil h + \lfloor m/2 \rfloor \sigma_1(h) + \sigma_1(h) \\ &= \left(\frac{m+1}{2}\right) h + \left(\frac{m+1}{2}\right) \sigma_1(h) \\ &= \lceil (m+1)/2 \rceil h + \lfloor (m+1)/2 \rfloor \sigma_1(h) \end{aligned}$$

and

$$\begin{aligned}\left(\overline{h}\right)^{m+2} &= \overline{\left(\frac{m+1}{2}h + \frac{m+1}{2}\sigma_1(h)\right) \cdot \overline{h}} \\ &= \overline{\frac{m+3}{2}h + \frac{m+1}{2}\sigma_1(h)} \\ &= \overline{\lceil (m+2)/2 \rceil h + \lfloor (m+2)/2 \rfloor \sigma_1(h)}\end{aligned}$$

as required. □

Corollary 5.1. *If $\sigma_1(x) = x$ and m is odd, then $\left(\overline{h}\right)^m = \overline{mh}$.*

Proof. Recall that $m = \lceil m/2 \rceil + \lfloor m/2 \rfloor$ for any integer m. □

5.4. **Power maps and Adams operations.** In the following theorem, multisubsets of quasigroups are represented by elements of $\mathbb{N}$-semimodules, as described in §4.1. In particular, covering maps $f \colon Q \to \widetilde{\Gamma}(G)$ extend to maps $f \colon \mathbb{N}Q \to \mathbb{N}\widetilde{\Gamma}$ from the quasigroup algebra to the additive semigroup of characters of a finite group G.

Theorem 5.1. *Consider the abelian character group $Q = \mathbb{Z}/d \oplus \mathbb{Z}/2$ of the dihedral group D_d of degree d. Let m be an odd positive integer. Let θ be an irreducible character of D_d, with*

$$\Psi^m \theta = \sum_{i=1}^{t} n_i \theta_i$$

expressed as an integral linear combination of elements of the full set $\widetilde{\Gamma}(D_d) = \{\, \theta_1, \dots, \theta_t \,\}$ of irreducible characters of D_d. Then the covering map $f \colon Q \to \widetilde{\Gamma}$ acts as

$$f\left(\left[f^{-1}\{\theta\}\right]^m\right) = \theta(1) \left[\sum_{i=1}^{t} n_i \theta_i(1) \theta_i\right] \tag{5.7}$$

at the multiset level.

Proof. The relationship (5.7) has already been established for a linear character θ by the general observations from the beginning of §5.1. Thus it remains to verify (5.7) when θ is a non-linear irreducible character ρ_h, and $\Psi^m \rho_h = \rho_{mh}$ by (5.3). Consider the quartet $S_h = \left\{\, \pm h, \pm \overline{h} \,\right\} = f^{-1}\{\rho_h\}$. By Corollary 5.1, $S_h^m = \left\{\, \pm mh, \pm m\overline{h} \,\right\}$. Three cases arise.

Case I: $mh \equiv 0 \bmod d$. Here $S_h^m = \{\ \pm 0, \pm\overline{0}\ \}$ and

$$f\left(\left[f^{-1}\{\rho_h\}\right]^m\right) = 2(\chi_1 + \chi_2) = \rho_h(1)\left[\chi_1(1)\chi_1 + \chi_2(1)\chi_2\right]$$

as required to match with (5.4).

Case II: $mh \equiv k \bmod d$ for even $d = 2k$. Here $S_h^m = \{\ \pm k, \pm\overline{k}\ \}$ and

$$f\left(\left[f^{-1}\{\rho_h\}\right]^m\right) = 2(\chi_3 + \chi_4) = \rho_h(1)\left[\chi_3(1)\chi_3 + \chi_4(1)\chi_4\right]$$

as required to match with (5.5).

Case III: Otherwise, $S_h^m = \{\ \pm mh, \pm m\overline{h}\ \} = S_{mh}$ and

$$f\left(\left[f^{-1}\{\rho_h\}\right]^m\right) = 4\rho_{mh} = \rho_h(1)\rho_{mh}(1)\rho_{mh}$$

as required to match with (5.3). □

Remark 5.2. In Theorem 5.1, the requirement for Q to be abelian is essential. If Q is nonabelian, so that $\sigma_1(x) = -x$ in terms of §4.4, then Lemma 5.1 shows that the m-th power S_h^m of the quartet S_h is $\{\ \pm mh, \pm\overline{h}\ \}$, which maps to $\rho_h(1)(\rho_{mh} + \rho_h)$ under f at the multiset level.

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References

[1] I.D. Andersen and A.J.W. Hilton, Generalized latin rectangles I: construction and decomposition, *Discrete Math.* **31** (1980), 125–152.

[2] I.D. Andersen and A.J.W. Hilton, Generalized latin rectangles II: embedding, *Discrete Math.* **31** (1980), 235–260.

[3] E. Bannai, Association schemes and fusion algebras (an introduction), *J. Algebraic Combin.* **2** (1993), 327–344.

[4] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, (Benjamin-Cummings, 1984).

[5] D. Bump, *Lie Groups*, second. ed., (Springer, 2004).

[6] W. Feit, *Characters of Finite Groups*, (Benjamin, 1967).

[7] E Hewitt and K. Ross, *Abstract Harmonic Analysis, Vo1ume 1*, (Springer, 1963).

[8] A.J.W. Hilton, The reconstruction of Latin squares with applications to school timetabling and to experimental design, *Math. Programming Stud.* **13** (1980), 68-–77.

[9] A.J.W. Hilton, Outlines of Latin squares, in *Combinatorial Design Theory*, eds. C.J. Colbourn and R.A. Mathon, (North-Holland, 1987) pp. 225—241.

[10] A.J.W. Hilton and J. Wojciechowski, Weighted quasigroups, in *Surveys in Combinatorics, 1993 (Keele)*, ed. K. Walker, (Cambridge Univ. Press, 1993) pp. 137-–171.

[11] K.W. Johnson and J.D.H. Smith, Characters of finite quasigroups V: linear characters, *Europ. J. Combinatorics* **10** (1989), 449–456.
[12] W. Rudin, *Fourier Analysis on Groups*, (Dover Publications, 2017).
[13] J.-P. Serre, *Linear Representations of Finite Groups*, (Springer, 1977).
[14] J.D.H. Smith, Augmented quasigroups and character algebras, *Adv. Math.* 263 (2020), 106983.
[15] J.D.H. Smith, *An Introduction to Quasigroups and Their Representations*, (Chapman & Hall/CRC, 2007).
[16] V.P. Snaith, *Explicit Brauer Induction*, (Cambridge Univ. Press 1994).
[17] T. tom Dieck, *Transformation Groups and Representation Theory*, (Springer, 1979).

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