EQUATIONAL QUANTUM QUASIGROUPS

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Abstract. As a self-dual framework to unify the study of quasigroups and Hopf algebras, quantum quasigroups are defined using a quantum analogue of the combinatorial approach to classical quasigroups, merely requiring invertibility of the left and right composites. In this paper, quantum quasigroups are redefined with a quantum analogue of the equational approach to classical quasigroups. Here, the left and right composites of auxiliary quantum quasigroups participate in diagrams whose commutativity witnesses the required invertibilities. Whenever the original and two auxiliary quantum quasigroups appear on an equal footing, the triality symmetry of the language of equational quasigroups is replicated. In particular, the problem arises as to when this triality emerges in the Hopf algebra context.

1. Introduction

1.1. **Quantum quasigroups.** Quasigroups and Hopf algebras represent two distinct extensions of the concept of a group. Like groups, quasigroups are set-theoretical objects with a cancellative multiplication. Unlike group multiplications, however, quasigroup multiplications are not required to be associative. On the other hand, Hopf algebras extend the group concept to a linear setting, say to a vector space A, with a linear *multiplication* $\nabla\colon A \otimes A \to A; x \otimes y \mapsto x \cdot y$ required to be associative. The concept of a Hopf algebra is self-dual, so along with the multiplication, there is a *comultiplication* $\Delta\colon A \to A \otimes A$ that is coassociative and compatible (mutually homomorphic) with the multiplication.

Initial extensions of the concept of a Hopf algebra to comprise a non-associative multiplication, such as [5, 8, 20, 21, 24] for example, took what are now regarded as *semi-classical* approaches. Restricting themselves to the linearization of certain equationally defined classes of quasigroups, such as inverse-property loops [32, §I.4.1], these approaches impose a linearized version of the defining equations on the non-associative multiplication. For example, the inverse property is linearized by the *Hopf quasigroups* of Klim and Majid [20, Prop. 4.2(1)], while the *Moufang-Hopf algebras* of Benkhart

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et al. linearize Moufang properties [5, Def'n 1.2]. All of these various semi-classical approaches lack the self-duality that is characteristic of Hopf algebras.

Quantum quasigroups were introduced [31] as a self-dual framework for the unification of quasigroups and Hopf algebras, within the general setting of any symmetric monoidal category. Viewed from the quasigroup side, they linearize an elegant characterization of quasigroups that was presented by the topologist I.M. James in the nineteen-sixties [15]. Viewed from the Hopf algebra side, they abstract the property (well-known to experts, but usually obscured under cohomological conditions) that each Hopf algebra A is an A-A-bi-Galois object [6, Ex. 1.2]. Much as groups are characterized as (non-empty) associative quasigroups, finite-dimensional Hopf algebras may be characterized as (co-)associative, (co-)unital quantum quasigroups [31, Th. 4.5]. Beyond the original work of [31], the theory of quantum quasigroups was analyzed further from the semi-classical viewpoint in [13]. Operadic aspects were discussed in [10], and symmetry aspects were treated in [14].

As an object A of a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, a *quantum quasigroup* (A, ∇, Δ) carries a *bimagma* structure (2.12), equipped with mutually homomorphic multiplication $\nabla \colon A \otimes A \to A$ and comultiplication $\Delta \colon A \to A \otimes A$. The *left composite* morphism $\mathsf{G} \colon A \otimes A \to A \otimes A$ (2.14) and *right composite* $\reflectbox{$\mathsf{G}$} \colon A \otimes A \to A \otimes A$ (2.15) are required to be invertible. From a model-theoretic standpoint, this requirement is analogous to the combinatorial definition of a quasigroup, which demands the existence of unique solutions to certain equations (Definition 2.1). For many algebraic purposes, the combinatorial definition is unsatisfactory, since existentially quantified conditions do not transfer nicely to homomorphic images. The problems with the combinatorial definition are solved by the process of *Skolemization* [9, 28], which encodes the unique equation solutions in the additional structure of right and left divisions (§2.1) to yield what are known as *equational quasigroups* (Definition 2.2).

1.2. **Equational quantum quasigroups.** The aim of the current work is to conduct a comparable Skolemization of quantum quasigroups, in which explicit inverses for the original left and right composites are provided by the composites of additional multiplications and comultiplications. Generically, the structures that result are known as *equational quantum quasigroups*. They come in three flavors, respectively described as *quantum* S-, T-, and U-*quasigroups* (Figure 2).

The **quantum** S-**quasigroups** embody the Skolemization of the original quantum quasigroups (which in the current context of equational quantum quasigroups may be described as *combinatorial*). A Hopf algebra furnishes quantum S-quasigroups (Theorem 3.25). Indeed, a single Hopf algebra may be Skolemized by multiple quantum S-quasigroups (Corollary 3.26).

The language of equational quasigroups, involving the three operations of multiplication, left and right division with their respective opposites, has an S_3-symmetry known as *triality* displayed in Figure 1. This symmetry is richer than the S_2-symmetry of the language of groups. It is related[1] to, but distinct from, the S_3-symmetry of the Dynkin diagram D_4 (manifest, for example, in the structures discussed in [5]), which is also described as *triality*. The latter is a symmetry of unary left, right, and direct or inverted bi-multiplication operations that are obtained from the binary operation of multiplication in a Moufang loop or an Okubo quasigroup [32, Ex. I.4.1M], [33]. If necessary, the two respective forms of triality may be disambiguated as *binary triality* and *unary triality*.

The **quantum** T-**quasigroups** have a full and exact triality symmetry, as shown in Figure 4. This symmetry nicely matches the triality symmetry of classical equational quasigroups that appears in Figure 1. The search for such an exact binary triality of quantum quasigroups was the prime initial motivation for the current study of equational quantum quasigroups, aiming to overcome the uncertainties inherent to the earlier triality for quantum quasigroups presented in [14, (3.12)]. The uncertainties were identified there by the pairs of opposed single-headed double-shafted arrows that replaced the double-headed double-shafted arrows of Figure 1, essentially saying that the operation of taking a certain kind of conjugate is not invertible.

One of the basic themes in the theory of equational quantum quasigroups is to determine when a quantum S-quasigroup has triality, or in other words is part of a quantum T-quasigroup. For example, bearing in mind that (combinatorial quantum quasigroup reducts of) Hopf algebras may sustain multiple quantum S-quasigroup structures, Problem 3.27 asks if any of these in fact constitutes a quantum T-quasigroup structure. Example 3.28 notes that this is the case for group algebras, and also (incidentally, or then as a consequence of Proposition 3.16) for the dual group algebras of finite groups.

The **quantum U-quasigroups** are unilateral (one-sided), like the *left* and *right quantum quasigroups*: bimagmas (Q, ∇, Δ) for which just one of the composites is invertible [30]. However, while these latter structures break the chiral symmetry between left and right, the unilateral quantum quasigroups do not. The situation is clearly exhibited by trivial classical models. Consider the projections $\pi_i \colon Q \times Q \to Q; (x_l, x_r) \mapsto x_i$ for $i \in \{ l, r \}$ on a set Q. Here, (Q, π_l) is a classical right quasigroup — a right quantum quasigroup in the symmetric monoidal category $(\mathbf{Set}, \times, \{ 1 \})$, and (Q, π_r) is a classical left quasigroup. Then (Q, π_l, π_r) is a quantum U-quasigroup in

[1]For a sample relationship, see Remark 2.6.

the symmetric monoidal category $(\mathbf{Set}, \times, \{1\})$ — compare Example 3.31 for the case of an abelian group G. The general case of Example 3.31 shows how classical unilateral quasigroups may offer a symmetric approach to the *quandles* used in knot theory [17], while Examples 3.32 and 3.33 suggest quantum U-quasigroup analogues for quandles and related algebras such as *racks* (compare [2]).

1.3. **Plan of the paper.** Chapter 2 presents some requisite background material on classical quasigroups and combinatorial quantum quasigroups, including the triality of classical equational quasigroups. The majority of this material is standard, except for the way the left and right divisions are derived directly from the Skolemization of the solutions s and t to the two equations $sy = x = yt$ for fixed x, y, and the introduction of classical unilateral quasigroups in §2.3.

The core of the paper is formed by Chapter 3, which introduces and examines equational quantum quasigroups in general symmetric monoidal categories. The basic definitions, which were summarized above, appear in detail in §3.1. In the subsequent section, classical equational quasigroups are interpreted as quantum equational quasigroups in the Cartesian category $(\mathbf{Set}, \times, \top)$ of sets, using a diagonal comuliplication. Next, the important example of a *quantum couple* from [31, §3.6], an associative quantum quasigroup forming a not necessarily coassociative generalization of the quantum double of a finite group [19], [22, Ex. 6.1.8], [34], is interpreted as a quantum S-quasigroup which may specialize to a quantum T-quasigroup (compare Theorem 3.23). Hopf algebras are construed as quantum S-quasigroups in §3.4. Theorem 3.25 gives the generic interpretation, while Corollary 3.24 shows that other interpretations may be possible. Quantum U-quasigroups, including *quantum quandles*, are discussed in §3.5. Then, symmetries and conjugates of equational quantum quasigroups are treated in §3.6, leading to the full and exact triality symmetry for quantum T-quasigroups displayed in Figure 4.

When S is a unital, commutative ring, two-sided equational quantum quasigroups in the monoidal category $\big(\underline{\underline{S}}, \oplus, \{0\}\big)$ of S-modules under the biproduct $\oplus$ are especially well-behaved. These so-called *linear quantum quasigroups* always form quantum T-quasigroups. Chapter 4 is dedicated to a detailed study of their structure, essentially that of a bimodule [13, Cor. 3.8] [31, Prop. 3.39]. Based on a subset $\mathcal{S} = \{R_0, R_1, R_2, \rho_0, \rho_1, \rho_2\}$ of the automorphism group $\underline{\underline{S}}(Q, Q)^*$ of the S-module Q, an initial approach (Theorem 4.7) uses a redundant specification (4.19) of the bimodule. This specification is called the *Six-Parameter* or *Triality Representation*, since as displayed in Corollary 4.9, permutation of the indices in the set $\mathcal{S}$ tracks nicely with the triality symmetry of the quantum T-quasigroup.

The redundancy that is inherent to the Six-Parameter Representation is described by the equations of Lemma 4.6(a). As rewritten in Lemma 4.6(b), these equations specify a central element Ω of the subgroup of $\underline{\underline{S}}(Q,Q)^*$ generated by $\mathcal{S}$. The central element Ω, together with the automorphisms ρ, λ, L, R of Q that appear in the multiplication

$$\nabla_0 \colon Q \oplus Q \to Q; \begin{bmatrix} x & y \end{bmatrix} \mapsto \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \rho \\ \lambda \end{bmatrix} \tag{1.1}$$

and comultiplication

$$\Delta_0 \colon Q \to Q \oplus Q; \begin{bmatrix} x \end{bmatrix} \mapsto \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} L & R \end{bmatrix} \tag{1.2}$$

of the leading quantum quasigroup (Q, ∇_0, Δ_0), yield the irredundant *Five-Parameter Representation* of a linear quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ in Theorem 4.10. Theorem 4.14 supplies a converse, building a quantum T-quasigroup structure on the group algebra SG of a group which is the product $G = MW$ of mutually commuting subgroups $M = \langle \rho, \lambda, \Omega \rangle$ and $W = \langle L, R, \Omega \rangle$.

Theorem 4.10 provides a fundamental insight into the nature of the Skolemizations of quantum quasigroups that are realized by equational quantum quasigroups. Consider a linear quantum quasigroup equipped with multiplication (1.1) and comultiplication (1.2). Theorem 4.10 classifies its linear Skolemizations. The most trivial is obtained by setting $\Omega = 1_Q$. More general linear Skolemizations are parametrized by other compatible choices of Ω. In other words, the following problem, which is open in general, is solved by Theorem 4.10 for the Cartesian module categories $(\underline{\underline{S}}, \oplus, \{0\})$.

Problem 1.1. Suppose that $(\mathbf{V}, \otimes, \mathbf{1})$ is a symmetric monoidal category. Suppose that (Q, ∇_0, Δ_0) is a quantum quasigroup in $(\mathbf{V}, \otimes, \mathbf{1})$. Classify the quantum S-quasigroups $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ in $(\mathbf{V}, \otimes, \mathbf{1})$ which are Skolemizations of (Q, ∇_0, Δ_0).

Theorem 3.18 solves Problem 1.1 in the Cartesian symmetric monoidal category of sets for the class of those quantum quasigroups in $(\mathbf{Set}, \times, \top)$ where the comultiplications Δ_i are all diagonal.

1.4. **Notational and other conventions.** As the default options, this paper follows the notational conventions of [32]. For example, if (X, S) denotes structure S on an object X, then a *reduct* of (X, S) is (X, R) for a subset R of S, or more generally for a subset of the full set of structure on X derived from S. In order to minimize the occurrence of parentheses in our non-associative contexts, we adopt the "algebraic" or "diagrammatic" convention which composes functions in the natural reading order from left to right. Thus functions are placed to the right of their arguments, either on the line or as a superfix (as in $n!$ or x^2, for example).

2. Background

2.1. **Combinatorial and equational quasigroups.** A *magma* (Q, ∇) or $(Q, \cdot)$ is a set Q with a binary *multiplication*

$$\nabla \colon Q \times Q \to Q; (x, y) \mapsto x \cdot y \tag{2.1}$$

[27, Pt. I, §IV.4.1]. For use of the nabla notation ∇ outside of the context of duality with a comultiplication Δ, compare [1, §II.3.1].

Definition 2.1. A magma $(Q, \cdot)$ is a (*combinatorial*) *quasigroup* if, for all elements x, y in Q, there exist unique solutions s, t to the equations

$$s \cdot y = x = y \cdot t \tag{2.2}$$

in the set Q.

Definition 2.1 is quite satisfactory from a combinatorial point of view, since it shows that the bodies of multiplication tables of (finite, nonempty) quasigroups are Latin squares [18, Th. 1.1.1], [29, §1.1]. On the other hand, the definition is most unsatisfactory from an algebraic point of view, since the magma image of a combinatorial quasigroup under a homomorphism of magmas need not be a combinatorial quasigroup — compare [4, pp.1182–3], [29, Ex. 1.2], [32, Ex. I.2.2.1].

From a model-theoretic perspective, the issue arises from the existential quantifiers in the definition. The process of *Skolemization* (cf. [9, 28]) may be invoked to eliminate the troublesome existential quantifiers. Extra structure is added to the magma, producing the desired solutions s and t explicitly (i.e., functionally) in terms of the arguments x and y.

Indeed, using the *left multiplication* $L(y) \colon Q \to Q; q \mapsto yq$ and *right multiplication* $R(y) \colon Q \to Q; q \mapsto qy$ by each element y of Q,[2] the equations (2.2) may be rewritten as

$$sR(y) = x = tL(y)\,. \tag{2.3}$$

Defining the extra structure

$$Q \times Q \to Q; (x, y) \mapsto xR(y)^{-1} =: x/y$$

of *right division* and

$$Q \times Q \to Q; (y, x) \mapsto xL(y)^{-1} =: y\backslash x$$

of *left division*, the solutions are given as $s = x/y$ and $t = y\backslash x$. The left division $x\backslash y$ may be vocalized as "x dividing y", while the right division

[2] Here, we follow the well-established notational conventions of [7, §II.9]. The left and right multiplications may be regarded as "Currying" the binary multiplication (2.1) — compare [32, §O.3.4].

x/y is vocalized as "x divided by y". In a group, we have $x\backslash y = x^{-1}y$ and $x/y = xy^{-1}$. Universally quantified identities

$$\text{(2.4)} \qquad \begin{array}{llll} \text{(SL)} & x\cdot(x\backslash y)=y\,, & \text{(SR)} & y=(y/x)\cdot x\,,\\ \text{(IL)} & x\backslash(x\cdot y)=y\,, & \text{(IR)} & y=(y\cdot x)/x \end{array}$$

are imposed on the full structure $(Q,\cdot,/,\backslash)$. The labels of the identities signify that (SL), (SR) give the surjectivity of the left multiplications and right multiplications, while (IL), (IR) give their injectivity.

Definition 2.2. A structure $(Q,\cdot,/,\backslash)$, equipped with multiplication, right division, and left division, is a (*classical*) (*equational*) *quasigroup* if, for all elements x, y in Q, the identities (2.4) are satisfied.

Remark 2.3. The optional adjective "classical" in Definition 2.2 is used in the context of this paper to distinguish from the "quantum" counterpart that appears later — Definition 3.6.

Lemma 2.4. *The identities*

$$\text{(2.5)} \qquad \text{(DL)}\quad x/(y\backslash x)=y\,; \qquad \text{(DR)}\quad y=(x/y)\backslash x$$

hold in an equational quasigroup $(Q,\cdot,/,\backslash)$.

Proof. The identities (2.5) follow from

$$[x/(y\backslash x)]R(y\backslash x) \overset{\text{(SR)}}{=} x \overset{\text{(SL)}}{=} yR(y\backslash x)$$

and

$$yL(x/y) \overset{\text{(SR)}}{=} x \overset{\text{(SL)}}{=} [(x/y)\backslash x]L(x/y)$$

using the injectivity of the right and left multiplications. □

2.2. **Triality symmetry of equational quasigroups.** Let $(Q,\cdot,/,\backslash)$ be an equational quasigroup. According to Definition 2.2, the set Q supports magmas under each of the multiplication, right division, and left division operations. Additional magma structures are furnished by the respective *opposites*, creating a full set of six combinatorial quasigroup structures with names and notations (based on permutations of $\{1, 2, 3\}$) as follows:

$$(Q,\cdot)^{(1)} = (Q,\cdot), \tag{2.6}$$
where $x_1 \cdot x_2 = x_3$ with the *original multiplication*;

$$(Q,\cdot)^{(1\ 2)} = (Q,\circ), \tag{2.7}$$
where $x_2 \circ x_1 = x_3$ with the *opposite multiplication*;

$$(Q,\cdot)^{(2\ 3)} = (Q,\backslash), \tag{2.8}$$
where $x_1 \backslash x_3 = x_2$ with the *left division*;

$$(Q,\cdot)^{(3\ 1)} = (Q,/), \tag{2.9}$$
where $x_3 / x_2 = x_1$ with the *right division*;

$$(Q,\cdot)^{(1\ 2\ 3)} = (Q,//), \tag{2.10}$$
where $x_2 // x_3 = x_1$ with the *opposite right division*;

$$(Q,\cdot)^{(1\ 3\ 2)} = (Q,\backslash\backslash), \tag{2.11}$$
where $x_3 \backslash\backslash x_1 = x_2$ with the *opposite left division*

[26, §1.2.2.1]. For each permutation $\pi \in S_3$, the operation of the magma $(Q, *) = (Q,\cdot)^\pi$ satisfies $x_{1\pi} * x_{2\pi} = x_{3\pi}$. Here, as usual, we are using cycle representations for permutations, reading their action from left to right (as should be apparent from (2.10) and (2.11) above).

Definition 2.5. Let $(Q,\cdot)$ be a quasigroup.

(a) The elements $(Q,\cdot)^\pi$ of the orbit of $(Q,\cdot)$ under $\pi \in S_3$ are called the *conjugates* or "parastrophes" of $(Q,\cdot)$.
(b) The quasigroup $(Q,\cdot)^\pi$ is called the π-*conjugate* of $(Q,\cdot)$.
(c) The (1 2)-conjugate is the *opposite* quasigroup $(Q,\circ)$, as in (2.7).

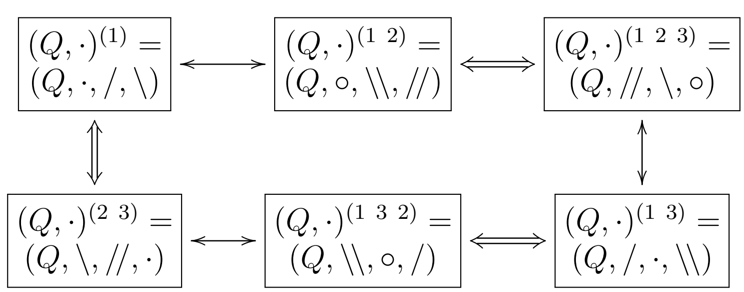


FIGURE 1. The *triality diagram* for quasigroups [29, Fig. 1.4], here based on the Cayley diagram of S_3 for left multiplications by the generating transpositions (1 2) and (2 3).

Figure 1 displays the Cayley diagram for the presentation

$$\langle s_1, s_2 | s_1^2 = s_2^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$

of S_3. Single-shafted arrows correspond to left multiplication by $s_1 = (1\ 2)$, and double-shafted arrows to left multiplication by $s_2 = (2\ 3)$. The figure integrates the Cayley diagram, with the permutations $\pi \in S_3$ that act on a combinatorial quasigroup $(Q, \cdot)$, into a corresponding record of the six conjugates $(Q, \cdot)^\pi$ of $(Q, \cdot)$, here exhibited as full equational quasigroups. The figure directly matches the subsequent treatment of quantum triality in §3.6.2, based on Definition 3.8. In that context, it also becomes more natural to follow a C_3-orbit in Figure 1 for selecting the three operations chosen as basic. For typographic convenience, we select the orbit $(\circ, \backslash, /)$ — compare (3.7)–(3.9).

Remark 2.6. Suppose that $(Q, \cdot)$ is a quasigroup. An *autotopy* of $(Q, \cdot)$ is a triple $(\alpha_1, \alpha_2, \alpha_3)$ of bijections $\alpha_i \colon Q \to Q$ such that $x_1^{\alpha_1} \cdot x_2^{\alpha_2} = (x_1 \cdot x_2)^{\alpha_3}$ for all $x_1, x_2 \in Q$. In particular, an autotopy with equal components is an automorphism. The full set $\mathrm{Atp}(Q, \cdot)$ of all autotopies of $(Q, \cdot)$ forms a group under composition, the *autotopy group* of $(Q, \cdot)$ [32, p.91]. It is the automorphism group of the object $(Q, \cdot)$ in the category **Qtp** of *quasigroup homotopies* [32, p.198].

If $(\alpha_1, \alpha_2, \alpha_3)$ is an autotopy of $(Q, \cdot)$, then for each $\pi \in S_3$, the triple $(\alpha_{1\pi}, \alpha_{2\pi}, \alpha_{3\pi})$ is an autotopy of $(Q, \cdot)^\pi$. Thus, (binary) quasigroup triality permutes the components of autotopies of conjugate quasigroups. On the other hand, unary quasigroup triality permutes components of autotopies of a single quasigroup, which has to be an inverse-property loop [11, §8.3], [32, Prop. I.4.1.1, Ex. I.4.1M].

2.3. **Unilateral quasigroups and chiral symmetry.** Perusal of (2.4) leads to the reduction of equational quasigroup structure into two chirally dual halves.

Definition 2.7. Let Q be a set.

(a) A(n *equational*) *left quasigroup* or L-*quasigroup* structure $(Q, \cdot, \backslash)$ on Q carries a multiplication and left division with the identities (SL) and (IL).
(b) A(n *equational*) *right quasigroup* or R-*quasigroup* structure $(Q, \cdot, /)$ on Q carries a multiplication and right division with the identities (SR) and (IR).

The symmetries of (2.4) provide the following *duality symmetry* of the languages of left and right quasigroups.

Lemma 2.8. *Let Q be a set.*

(a) *If* $(Q, \cdot, \backslash)$ *is a left quasigroup, then so is* $(Q, \backslash, \cdot)$.
(b) *If* $(Q, \cdot, /)$ *is a right quasigroup, then so is* $(Q, /, \cdot)$.

Remark 2.9. (a) A combinatorial left quasigroup may be defined as a magma $(Q, \cdot)$ within which, for each ordered pair (x, y) of elements of Q, the equation $y \cdot t = x$ has a unique solution t. In an equational left quasigroup, the existence of these solutions $t = xL(y)^{-1} = y\backslash x$ is equivalent to (SL), while the uniqueness is equivalent to (IL). Combinatorial right quasigroups stand in a similar relationship to their equational counterparts.

(b) Right projection $\pi_r \colon Q \times Q \to Q; (x, y) \mapsto y$ on any set Q yields the associative multiplication of a combinatorial left quasigroup, where each left multiplication is the identity 1_Q on Q. The corresponding equational left quasigroup is (Q, π_r, π_r). Dually, left projection gives a right quasigroup multiplication.

A symmetric complement to the chirally dual concepts of Definition 2.7 is provided by the following definition.

Definition 2.10. Consider a set Q. The structure of a U-*quasigroup* or *unilateral quasigroup* $(Q, \backslash, /)$ consists of a left and right division satisfying the identities (DL) and (DR).

Lemma 2.11. *Consider a set* $(Q, \backslash, /)$ *with a left and right division. Then:*
(a) $(Q, \backslash, /)$ *is a unilateral quasigroup iff* $(Q, //, \backslash)$ *is a right quasigroup;*
(b) $(Q, \backslash, /)$ *is a unilateral quasigroup iff* $(Q, \backslash\backslash, /)$ *is a left quasigroup;*
(c) $(Q, \backslash, /)$ *is a unilateral quasigroup iff* $(Q, \backslash, //)$ *is a right quasigroup;*
(d) $(Q, \backslash, /)$ *is a unilateral quasigroup iff* $(Q, /, \backslash\backslash)$ *is a left quasigroup.*

Proof. For (a), we have $x/(y\backslash x) = (y\backslash x)//x$ and $(x/y)\backslash x = (y//x)\backslash x$. For (b), we have $x/(y\backslash x) = x/(x\backslash\backslash y)$ and $(x/y)\backslash x = x\backslash\backslash(x/y)$. By Lemma 2.8, (c) is equivalent to (a), while (d) is equivalent to (b). □

Remark 2.12. (a) In the context of Lemma 2.11, it becomes apparent that $(Q, \backslash, /)$ will be a unilateral quasigroup iff $(Q, \backslash)$ is a combinatorial right quasigroup, or equivalently iff $(Q, /)$ is a combinatorial left quasigroup. Since the full (chirally symmetric) set of combinatorial conditions requires as much structure as is required for the equational conditions (DL) and (DR), there is no structural distinction to be made between combinatorial and equational unilateral quasigroups.

(b) It follows from (a) and Remark 2.9(b), or alternatively may simply be verified directly using (DL) and (DR), that a set Q furnishes a unilateral quasigroup (Q, π_l, π_r). This trivial but important example illustrates how unilateral quasigroups restore the chiral symmetry that is broken by left and right quasigroups.

2.4. **Quantum quasigroups.** Quantum quasigroups provide a self-dual unification of quasigroups and Hopf algebras [31].

Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with swap morphism $\tau\colon A\otimes A \to A \otimes A$. A *weak bimagma* (A, ∇, Δ) is a $\mathbf{V}$-object A, equipped with a *multiplication* $\nabla\colon A\otimes A \to A$ and *comultiplication* $\Delta\colon A \to A\otimes A$. A *bimagma* is a weak bimagma (A, ∇, Δ) in which the multiplication and comultiplication are mutually homomorphic. The mutual homomorphism of the multiplication and comultiplication is expressed either by the *bimagma diagram*

$$
\begin{array}{ccccc}
a\otimes b & \xmapsto{\nabla} & a\cdot b & \xmapsto{\Delta} & (a\cdot b)^L \otimes (a\cdot b)^R \\
{\scriptstyle \Delta\otimes\Delta}\downarrow & & & & \uparrow{\scriptstyle \nabla\otimes\nabla} \\
a^L\otimes a^R\otimes b^L\otimes b^R & & \xmapsto[1_A\otimes\tau\otimes 1_A]{} & & a^L\otimes b^L\otimes a^R\otimes b^R
\end{array}
\tag{2.12}
$$

[13, (2.1)] [31, (2.4)], or equationally as

$$x^L\cdot y^L = (x\cdot y)^L \quad \text{and} \quad x^R\cdot y^R = (x\cdot y)^R \tag{2.13}$$

in elementary form with $\nabla\colon a\otimes b \mapsto a\cdot b$ and the "non-coassociative Sweedler notation" $\Delta\colon a \mapsto a^L\otimes a^R$ (cf. [13, Rem. 2.2(b)], [16], [31]).

Definition 2.13. Let (A, ∇, Δ) be a bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) The *left composite* is

$$
\mathsf{G}\colon A\otimes A \xrightarrow{\Delta\otimes 1_A} A\otimes A\otimes A \xrightarrow{1_A\otimes\nabla} A\otimes A\,;
\tag{2.14}
$$

$$x\otimes y \longmapsto x^L\otimes x^R\otimes y \longmapsto x^L\otimes x^R y$$

("G" for "Gauche").

(b) The *right composite* is

$$
\mathsf{Ↄ}\colon A\otimes A \xrightarrow{1_A\otimes\Delta} A\otimes A\otimes A \xrightarrow{\nabla\otimes 1_A} A\otimes A\,;
\tag{2.15}
$$

$$x\otimes y \longmapsto x\otimes y^L\otimes y^R \longmapsto xy^L\otimes y^R$$

("Ↄ" for "Droite"), the dual of the left composite.

(c) The bimagma (A, ∇, Δ) called is a *quantum quasigroup* if the left composite and right composite are invertible.

(d) The bimagma (A, ∇, Δ) is called a *left quantum quasigroup* if the left composite is invertible.

(e) The bimagma (A, ∇, Δ) is called a *right quantum quasigroup* if the right composite is invertible.

On the one hand, quasigroups taken with a diagonal comultiplication $\Delta\colon x \mapsto x \otimes x$ are quantum quasigroups within the Cartesian category of sets with direct products (in which context an ordered pair (x, y) is written as $x \otimes y$) [15]. On the other hand, any Hopf algebra $(A, \nabla, \eta, \Delta, \varepsilon, S)$ reduces to a quantum quasigroup (A, ∇, Δ). Most previously studied nonassociative generalizations of Hopf algebras, including the *Hopf quasigroups* of Majid *et al.*, also reduce to quantum quasigroups [5, 8, 20, 21, 24]. However, these earlier concepts are not self-dual. With the term "Hopf quasigroup" already taken, the term "quantum quasigroup" has been adopted for the general concept, in line with one of the many senses of the term "quantum group" (compare [22], for example). In this context, as noted previously in Remark 2.3, it is convenient to refer to "ordinary" quasigroups as *classical quasigroups*.

Futhermore, given the subsequent refinement of the original quantum quasigroup concepts of Definition 2.13 to include explicit structure that will exhibit inverses for the composites, those original concepts may now be described as *combinatorial*, by analogy with the relationship between combinatorial and equational classical quasigroups in §2.1.

3. Equational quantum quasigroups

The basic setting for this chapter is provided by a symmetric monoidal category $\mathbf{V}$, with involutive swap $\tau\colon A \otimes B \to B \otimes A$ for objects A, B of $\mathbf{V}$. Since the swap is involutive, it may appear as

$$A \otimes B \overset{\tau}{\text{———}} B \otimes A \qquad \text{or} \qquad A \otimes B \overset{\tau}{\longleftrightarrow} B \otimes A$$

on morphism diagrams within the category $\mathbf{V}$.

3.1. **Fundamental definitions.**

3.1.1. *Quantum tri- and bi-quasigroups.*

Definition 3.1. A *quantum triquasigroup* $(Q, \nabla_i, \Delta_i)_{i \in \mathbb{Z}/3}$ or

$$(Q, \nabla_i, \Delta_i)_{i=0}^{i=2} = (Q, \nabla_0, \nabla_1, \nabla_2, \Delta_0, \Delta_1, \Delta_2) \tag{3.1}$$

consists of

(0) a quantum quasigroup (Q, ∇_0, Δ_0);
(1) a left quantum quasigroup (Q, ∇_1, Δ_1); and
(2) a right quantum quasigroup (Q, ∇_2, Δ_2)

on an object Q of $\mathbf{V}$.

A quantum triquasigroup is *associative* if all three of its multiplications are associative. Dually, it is *coassociative* if its three comultiplications are all coassociative. In similar fashion, the *commutativity* and *cocommutativity* of quantum triquasigroups are defined.

Use of residues modulo 3 for labelling constituent quantum quasigroups of a quantum triquasigroup implies equations such as $2+1=0$ and $2=-1$. There is an implication that the constituent quantum quasigroup labelled by 0 is more fundamental than the other two constituents, which in general are only one-sided quantum quasigroups. This implicit understanding manifests itself, for example, in Definition 3.6 below. On the other hand, the following definition (which is designed to handle the duality symmetry of one-sided quasigroups) reduces Definition 3.1 by excising the quantum quasigroup (Q, ∇_0, Δ_0). The two remaining indices 1 and 2, now more naturally written as ± 1, are nevertheless still interpreted as residues modulo 3.

Definition 3.2. A *quantum biquasigroup* $(Q, \nabla_i, \Delta_i)_{i=\pm 1}$ or

$$(Q, \nabla_1, \nabla_{-1}, \Delta_1, \Delta_{-1}) \tag{3.2}$$

consists of

(1) a left quantum quasigroup (Q, ∇_1, Δ_1) and
(2) a right quantum quasigroup $(Q, \nabla_{-1}, \Delta_{-1})$

on an object Q of $\mathbf{V}$.

3.1.2. *Composite diagrams.* In a quantum triquasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ or quantum biquasigroup $(Q, \nabla_i, \Delta_i)_{i=\pm 1}$, each bimagma (Q, ∇_i, Δ_i) has its respective left and right composites G_i and $\reflectbox{$\mathsf{G}$}_i$. The individual *composite diagrams*

$$\begin{array}{ccc} Q \otimes Q & \xrightarrow{\reflectbox{G}_i} & Q \otimes Q \\ \tau \updownarrow & & \updownarrow \tau \\ Q \otimes Q & \xleftarrow[\mathsf{G}_{i+1}]{} & Q \otimes Q \end{array} \tag{3.3}$$

in $\mathbf{V}$ play an important role in the theory that initially arises from the following fundamental lemma.

Lemma 3.3. *Commutativity of the diagram* (3.3) *means that:*

(a) *the inverse of* $\reflectbox{$\mathsf{G}$}_i$ *is* $\tau \mathsf{G}_{i+1} \tau$*, and*
(b) *the inverse of* G_{i+1} *is* $\tau \reflectbox{$\mathsf{G}$}_i \tau$

for each $i \in \mathbb{Z}/_3$.

Proof. Starting from the top left corner of (3.3), we have

$$\reflectbox{G}_i \tau \mathsf{G}_{i+1} \tau = 1_{Q \otimes Q} \,,$$

so $\tau \mathsf{G}_{i+1}\tau$ is a retract of ∂_i. Starting from the top right corner of (3.3), we have

$$\tau \mathsf{G}_{i+1}\tau\partial_i = 1_{Q\otimes Q} ,$$

so $\tau \mathsf{G}_{i+1}\tau$ is a section of ∂_i. Thus (a) holds. The proof of (b) is similar. □

It is helpful to have distinctive individual names for the three composite diagrams (3.3), using the pair $(i+1)i$ of respective residues labeling the left and right composites that appear within them.

Definition 3.4. Consider a quantum triquasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$.

(a) The diagram (3.3) is called the $(i + 1)i$-*diagram* of the quantum triquasigroup.
(b) The 02-diagram is called the *deuce* diagram.
(c) The 10-diagram is called the *tenner* diagram.
(d) The 21-diagram is called the *blackjack* diagram.

Remark 3.5. The terminology introduced in Definition 3.4(d) arises from the Anglo-Saxon name for versions of the European card game "twenty-one," where a certain kind of exceptional hand is called a "blackjack" [3, p.430]. The terminology matches the exceptional role of the 21-diagram in the subsequent theory: Commuting of the blackjack diagram is required in Definition 3.8 below, but not in Definition 3.6.

3.1.3. *Equational quantum quasigroups.* According to Definition 3.1, the left composites G_i for $i \neq 2$ and right composites ∂_i for $i \neq 1$ are actually invertible. However, the definition of a combinatorial (two- or one-sided) quantum quasigroup does not require the inverses of the composites to arise from specific (two- or one-sided) quantum quasigroup structures. All that is required there is that the composites are invertible. Indeed, for a given quantum quasigroup Q, there may be distinct quantum quasigroups whose composites, following conjugation by τ, are inverses of the composites of Q — compare [14, §3.3.3] and Corollary 3.26 below. On the other hand, the equational quantum quasigroup definition below comprises explicit one-sided quantum quasigroups whose conjugated composites are inverse to the composites of the component quantum quasigroup (Q, ∇_0, Δ_0).

Definition 3.6. A quantum triquasigroup in which the deuce and tenner diagrams commute is defined to be an *equational quantum quasigroup* or *quantum* S-*quasigroup*.

Remark 3.7. The "S" in Definition 3.6 refers to the Skolemization (cf. §2.1) where the inverses of G_0 and ∂_0, posited to exist in the quantum quasigroup definition, are now presented explicitly. Thus, according to Lemma 3.3, the deuce or 02-diagram provides $\tau\partial_2\tau$ as the inverse of G_0, while the tenner or 10-diagram provides $\tau\mathsf{G}_1\tau$ as the inverse of ∂_0. In addition, the 10-diagram provides $\tau \partial_0 \tau$ as the inverse of G_1, while the 02-diagram provides $\tau \mathsf{G}_0 \tau$ as the inverse of ∂_2.

Definition 3.8. A *triality quantum quasigroup* or *quantum* T-*quasigroup* is a quantum triquasigroup where the diagrams (3.3) commute for all $i \in \mathbb{Z}/_3$.

Lemma 3.9. *In a triality quantum quasigroup* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$*, all three bimagmas* (Q, ∇_i, Δ_i) *form two-sided quantum quasigroups.*

Definition 3.10. In the context of Lemma 3.9, the quantum quasigroup (Q, ∇_0, Δ_0) is described as the *leading* quantum quasigroup of the triality quantum quasigroup. The same adjective is applied to its multiplication and comultiplication.

The blackjack diagram, absent from Definition 3.6, fully comes into its own in the following definition.

Definition 3.11. A *unilateral quantum quasigroup* or *quantum* U-*quasigroup* is a quantum biquasigroup where the blackjack diagram commutes.

3.1.4. *Minimal conditions.* The respective Definitions 3.6, 3.8, and 3.11 for quantum S-, T-, and U-quasigroups have been presented on the basis of Definitions 3.1 and 3.2 for quantum tri- and bi-quasigroups. That approach is intended to make the equational quantum quasigroup definitions more immediately understandable, but it does introduce redundancies which will now be removed.

Proposition 3.12. *A triple* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *of bimagmas forms a quantum S-quasigroup if and only if the deuce and tenner diagrams commute.*

Proof. The "only if" direction is immediate from Definition 3.6. Conversely, consider a triple $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ of bimagmas for which the 02- and 10-diagrams commute. Then the left composites G_0 and G_1 are invertible, as are the right composites ∂_0 and ∂_2. Thus, $(Q, \nabla_0, \Delta_0)_0$ is a quantum quasigroup, $(Q, \nabla_1, \Delta_1)_0$ is a left quantum quasigroup, and $(Q, \nabla_2, \Delta_2)_0$ is a quantum quasigroup. It follows that $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ is a quantum triquasigroup, and therefore a quantum S-quasigroup by Definition 3.6. □

The proofs of the remaining propositions of this type are similar.

Proposition 3.13. *A triple* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *of bimagmas forms a quantum T-quasigroup if and only if all three composite diagrams commute.*

Proposition 3.14. *A pair* $(Q, \nabla_i, \Delta_i)_{i=\pm 1}$ *of bimagmas forms a quantum U-quasigroup if and only if the blackjack diagram commutes.*

Figure 2 presents a Venn diagram of the three irredundant definitions.

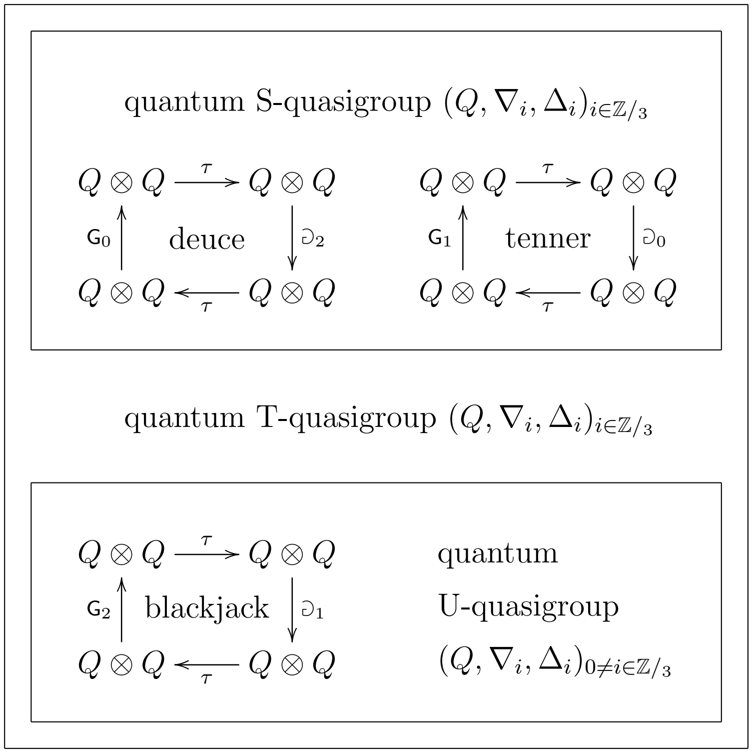
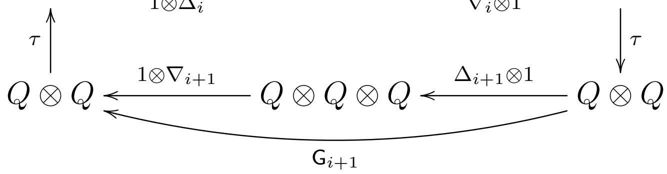


FIGURE 2. The equational quantum quasigroup definitions require commuting of the indicated named diagrams on a pair $(Q, \nabla_i, \Delta_i)_{0\neq i\in\mathbb{Z}/3}$ or triple $(Q, \nabla_i, \Delta_i)_{i\in\mathbb{Z}/3}$ of bimagmas in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with swap τ.

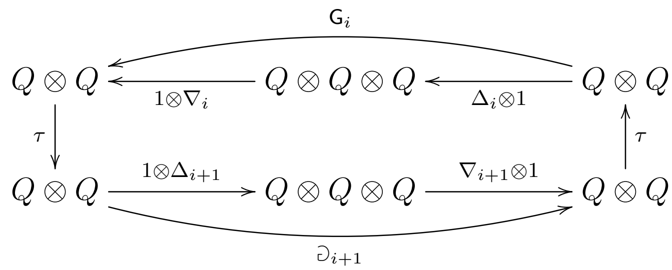
3.1.5. *Self-duality.* It will now be shown that the respective concepts of a quantum S-, T-, and U-quasigroup are self-dual. To this end, note that the composite diagram

(3.4)



presents an expanded version of (3.3). In turn,

(3.5)



exhibits the dual of (3.4). Based directly on the original ordering of the bimagmas in Propositions 3.12–3.14, the three dual diagrams (3.5) are not indexed in agreement with the specifications of Definition 3.4, as shown in Figure 3.

$$\begin{array}{lccccc} i=0: & 10 & \xrightarrow{\text{direct}} & 01 & \xrightarrow{\text{negate}} & 02 \\ i=1: & 21 & \xrightarrow{\text{direct}} & 12 & \xrightarrow{\text{negate}} & 21 \\ i=2: & 02 & \xrightarrow{\text{direct}} & 20 & \xrightarrow{\text{negate}} & 10 \end{array}$$

Figure 3. Indexing of the dual composite diagrams before and after negation of the dual indices.

Elimination of the discrepancy requires the following definitions.

Definition 3.15. (a) Consider a triple $(Q, \nabla_i, \Delta_i)_{i\in\mathbb{Z}/3}$ or $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ of bimagmas, as in Propositions 3.12 and 3.13. Then $(Q, \Delta_{-i}, \nabla_{-i})_{i=0}^{i=2}$ is defined as the corresponding *dual* triple of bimagmas.

(b) Consider a pair $(Q, \nabla_i, \Delta_i)_{0\neq i\in\mathbb{Z}/3}$ or $(Q, \nabla_i, \Delta_i)_{i=\pm 1}$ of bimagmas, as in Proposition 3.14. Then $(Q, \Delta_{-i}, \nabla_{-i})_{i=\pm 1}$ is defined as the corresponding *dual* pair of bimagmas.

Inspection of Figure 3 now shows that with these negated indexings of the duals, duality fixes the blackjack diagram, while transposing the deuce and tenner diagrams. The self-duality of the basic definitions follows.

Proposition 3.16. *The concepts of a quantum S-, T-, and U-quasigroup are self-dual.*

3.2. **Classical equational quasigroups.** Suppose that $(Q, \circ, /, \backslash)$ is a classical equational quasigroup. Consider the Cartesian symmetric monoidal category $(\mathbf{Set}, \times, \top)$ of sets under the direct product, with a singleton set $\top$ (the terminal object) as monoidal unit. Take diagonal comultiplications

$$\Delta_i \colon Q \to Q \times Q; x \mapsto (x, x) \tag{3.6}$$

for $i \in \mathbb{Z}/_3$, and multiplications

$$\nabla_0 \colon Q \times Q \to Q; (x, y) \mapsto x \circ y\,, \tag{3.7}$$

$$\nabla_1 \colon Q \times Q \to Q; (x, y) \mapsto x \backslash y\,, \tag{3.8}$$

$$\nabla_2 \colon Q \times Q \to Q; (x, y) \mapsto x / y\,. \tag{3.9}$$

It follows from [31, Prop. 3.11(a)] that each individual (Q, ∇_i, Δ_i) forms a quantum quasigroup. By (2.14) and (2.15), the composites are

$$\begin{array}{ll} (x,y) \xmapsto{\mathsf{G}_0} (x, x \circ y)\,, & (x,y) \xmapsto{\partial_0} (x \circ y, y)\,, \\ (x,y) \xmapsto{\mathsf{G}_1} (x, x \backslash y)\,, & (x,y) \xmapsto{\partial_1} (x \backslash y, y)\,, \\ (x,y) \xmapsto{\mathsf{G}_2} (x, x / y)\,, & (x,y) \xmapsto{\partial_2} (x / y, y)\,. \end{array} \tag{3.10}$$

Cocommutative and coassociative quantum T-quasigroups in $(\mathbf{Set}, \times, \top)$, where the multiplications and comultiplications are chosen as in (3.6)–(3.9), are equivalent to classical equational quasigroups.

Proposition 3.17. *Making assignments* (3.6)–(3.9) *for the multiplications and comultiplications, the commuting of the diagrams* (3.3) *are equivalent to the classical equational quasigroup identities.*

Proof. By (3.10), we have the series

$$\begin{array}{ccccccc} (x,y) & & & \xmapsto{\quad\partial_0\quad} & & & (x \circ y, y) \\ {\scriptstyle \tau}\downarrow & & & & & & \downarrow{\scriptstyle \tau} \\ (y,x) & \overset{\text{(IL)}}{=\!=} & (y, y \backslash (y \cdot x)) & =\!= & (y, y \backslash (x \circ y)) & \xleftarrow[\mathsf{G}_1]{} & (y, x \circ y)\,, \end{array} \tag{3.11}$$

$$\begin{array}{ccccc} (x,y) & & \xmapsto{\quad\partial_1\quad} & & (x \backslash y, y) \\ {\scriptstyle \tau}\downarrow & & & & \downarrow{\scriptstyle \tau} \\ (y,x) & \overset{\text{(DL)}}{=\!=} & (y, y / (x \backslash y)) & \xleftarrow[\mathsf{G}_2]{} & (y, x \backslash y)\,, \end{array} \tag{3.12}$$

$$
\begin{array}{ccccccc}
(x,y) & & \overset{\partial_2}{\longmapsto} & & & & (x/y,y) \\
\downarrow\tau & & & & & & \downarrow\tau \\
(y,x) & \overset{(\mathrm{SR})}{=\!=} & (y,(x/y)\cdot y) & =\!= & (y,y\circ(x/y)) & \underset{\mathsf{G}_0}{\longleftarrow\!\!\!\dashv} & (y,x/y)\,,
\end{array}
\tag{3.13}
$$

$$
\begin{array}{ccccccc}
(x\backslash y,x) & \overset{\partial_0}{\longmapsto} & ((x\backslash y)\circ x,x) & =\!= & (x\cdot(x\backslash y),x) & \overset{(\mathrm{SL})}{=\!=} & (y,x) \\
\uparrow\tau & & & & & & \uparrow\tau \\
(x,x\backslash y) & & & \underset{\mathsf{G}_1}{\longleftarrow\!\!\!\dashv} & & & (x,y)\,,
\end{array}
\tag{3.14}
$$

$$
\begin{array}{ccccc}
(x/y,x) & \overset{\partial_1}{\longmapsto} & ((x/y)\backslash x,x) & \overset{(\mathrm{DR})}{=\!=} & (y,x) \\
\uparrow\tau & & & & \uparrow\tau \\
(x,x/y) & & \underset{\mathsf{G}_2}{\longleftarrow\!\!\!\dashv} & & (x,y)\,,
\end{array}
\tag{3.15}
$$

$$
\begin{array}{ccccccc}
(x\circ y,x) & \overset{\partial_2}{\longmapsto} & ((x\circ y)/x,x) & =\!= & ((y\cdot x)/x),x) & \overset{(\mathrm{IR})}{=\!=} & (y,x) \\
\uparrow\tau & & & & & & \uparrow\tau \\
(x,x\circ y) & & & \underset{\mathsf{G}_0}{\longleftarrow\!\!\!\dashv} & & & (x,y)
\end{array}
\tag{3.16}
$$

of six commutativities respectively and collectively embracing the classical equational quasigroup identities. ☐

As a corollary, we obtain the following.

Theorem 3.18. *Under the assignments* (3.6)–(3.9) *for the multiplications and comultiplications on an object* Q *of* $(\mathbf{Set},\times,\top)$*, the following structures on* Q*:*

(a) *A classical equational quasigroup* $(Q,\circ,/,\backslash)$;
(b) *A quantum* S*-quasigroup* $(Q,\nabla_i,\Delta_i)_{i=0}^{i=2}$;
(c) *A quantum* T*-quasigroup* $(Q,\nabla_i,\Delta_i)_{i=0}^{i=2}$

are equivalent.

Proof. The equivalence between (a) and (c) is immediately apparent from Proposition 3.17. The equivalence between (b) and (c) then follows, since the two identities (2.5) are a consequence of the initial four from (2.4). ☐

3.3. **The quantum couple.** The concept of a *quantum couple* GQ, over a commutative, unital ring S, was introduced as an example of an associative quantum quasigroup in [31, §3.6]. Here, Q is a finite quasigroup, and G is a (not necessarily finite) group with a (not necessarily faithful) automorphic right action on Q. The concept unifies the following examples.

Example 3.19. (a) If Q is a group, and G is the group Q providing the adjoint action $g\colon Q \to Q; q \mapsto g^{-1}qg$ for each $g \in Q = G$, then the quantum couple is the (quantum quasigroup reduct of the) usual *group double* Hopf algebra of the finite group [22, Ex. 6.1.8], [34].

(b) If Q is the trivial singleton quasigroup $\top$, then the quantum couple $G\top$ is the (quasi)group algebra SG of the group G [31, Ex. 3.34].

(c) If G is the trivial group $\top$, then the quantum couple $\top Q$ reduces to the dual quasigroup algebra S^Q [31, Ex. 3.33].

The goal of this section is to exhibit a quantum S-quasigroup structure on the quantum couple. Recall that the underlying S-module of the quantum couple is the free S-module $SG \otimes SQ$ over the set

$$G \times Q = \{g \otimes q \mid g \in G\,,\ q \in Q\}\,,$$

where the notation $g|q$ is used as an abbreviation for the basic element $g\otimes q$ of $SG \otimes SQ$. The multiplication on the quantum couple is then given by

$$\nabla\colon f|p \otimes g|q \mapsto \delta_{pg,q} fg|q\,,$$

on the basic elements [31, (3.9)], while

$$\Delta\colon g|q \mapsto \sum_{q^L q^R = q} g|q^L \otimes g|q^R$$

gives the action of the comultiplication on the basic elements. The opposite multiplication is

$$\tau\nabla = \nabla_0\colon f|p \otimes g|q \mapsto \delta_{qf,p}(f \circ g)|p\,, \tag{3.17}$$

while the opposite comultiplication is

$$\Delta\tau = \Delta_0\colon g|q \mapsto \sum_{q^{L_0}\circ q^{R_0} = q} g|q^{L_0} \otimes g|q^{R_0}\,. \tag{3.18}$$

These are incorporated as part of the quantum triquasigroup structure being defined on the quantum couple. The equation (2.14) produces

$$\overbrace{f|p \otimes g|q \xmapsto[\Delta_0\otimes 1]{} \sum\nolimits_{p^{L_0}\circ p^{R_0}=p} f|p^{L_0} \otimes f|p^{R_0} \otimes g|q \xmapsto[1\otimes\nabla_0]{}}^{\mathsf{G}_0} f|(q^f\backslash p) \otimes (f\circ g)|q^f$$

as the zero-th left composite in the quantum triquasigroup, since $q^f = p^{R_0}$ iff $p = p^{L_0} \circ p^{R_0} = p^{L_0} \circ q^f = q^f \cdot p^{L_0}$ iff $p^{L_0} = q^f \backslash p$. Similarly, the equation (2.15) produces

$$f|p \otimes g|q \xmapsto[1\otimes\Delta_0]{} \textstyle\sum_{q^{L_0}\circ q^{R_0}=q} f|p \otimes g|q^{L_0} \otimes g|q^{R_0} \xmapsto[\nabla_0\otimes 1]{} (f\circ g)|p \otimes g|(q/p^{f^{-1}})$$

$$f|p \otimes g|q \xmapsto{\partial_0} (f\circ g)|p \otimes g|(q/p^{f^{-1}})$$

as the zero-th right composite in the quantum tqriuasigroup, since $q^{L_0} f = p$ iff $q^{L_0} = p^{f^{-1}}$ iff $q = q^{L_0} \circ q^{R_0} = p^{f^{-1}} \circ q^{R_0} = q^{R_0} \cdot p^{f^{-1}}$ iff $q^{R_0} = q/p^{f^{-1}}$.

Now, define multiplications

$$\nabla_1\colon f|p \otimes g|q \mapsto \delta_{p(f\backslash g),q}(f\backslash g)|q \tag{3.19}$$

$$\nabla_2\colon f|p \otimes g|q \mapsto \delta_{qg,p}(f/g)|q \tag{3.20}$$

and comultiplications

$$\Delta_1\colon g|q \mapsto \sum_{q^{L_1}/q^{R_1}=q} g|q^{L_1} \otimes g|q^{R_1} \tag{3.21}$$

$$\Delta_2\colon g|q \mapsto \sum_{q^{L_2}\backslash q^{R_2}=q} g|q^{L_2} \otimes g|q^{R_2}\,. \tag{3.22}$$

Here, starting with the initial definitions of ∇_0 and Δ_0 as the respective opposite multiplication and comultiplication from the quantum quasigroup structure on the quantum couple, the remaining two quantum quasigroup multiplications ∇_i and comultiplications Δ_i emerge from the requirement that the deuce and tenner diagrams commute, as illustrated by the following lemma for the case of the tenner diagram. Note that, for typographical reasons arising from the proof of the lemma, the tenner diagram has been flipped about its diagonal from the top left to the bottom right to produce the diagram (3.23).

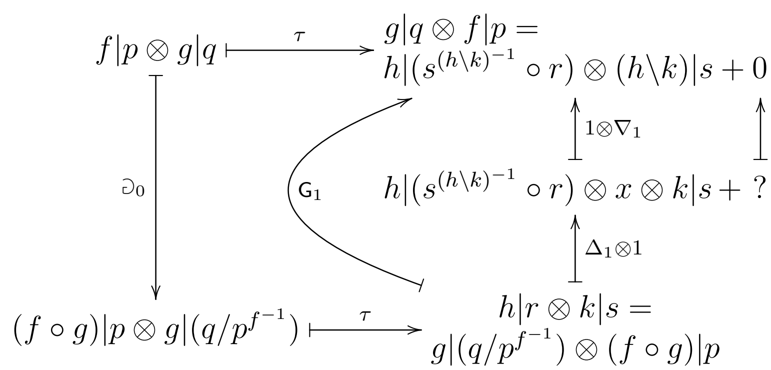
Lemma 3.20. *The commutation of the tenner diagram*

$$\begin{array}{ccc} GQ\otimes GQ & \xrightarrow{\tau} & GQ\otimes GQ \\ {\scriptstyle \partial_0}\big\downarrow & & \big\uparrow{\scriptstyle \mathsf{G}_1} \\ GQ\otimes GQ & \xrightarrow[\tau]{} & GQ\otimes GQ \end{array} \tag{3.23}$$

yields the definitions (3.19) *and* (3.21).

Proof. The diagram (3.23) appears as the left hand cell of

(3.24)



at the elementary level.

The equations for g, q, f, p implicit in the bottom right hand corner of the elementary diagram are solved consecutively in terms of h, r, k, s as

$$\begin{aligned} g = h\,, \qquad & f = h\backslash k \;\Leftarrow\; k = f \circ g = gf = hf\,, \\ p = s\,, \qquad & q = s^{(h\backslash k)^{-1}} \circ r = r \cdot s^{(h\backslash k)^{-1}} = r \cdot p^{f^{-1}} \;\Leftarrow\; r = p^{f^{-1}} \backslash q\,, \end{aligned}$$

yielding the image of $1 \otimes \nabla_1$ in the diagram. The first component of the central term $h|(r \circ s^{(h\backslash k)^{-1}}) \otimes x \otimes k|s$ on the right hand side of the diagram is then obtained by tracing back along the identity component of $1 \otimes \nabla_1$. The third component of the central term $h|(s^{(h\backslash k)^{-1}} \circ r) \otimes x \otimes k|s$ on the right hand side of the diagram is obtained by tracing forward along the identity component of $\Delta_1 \otimes 1$.

As an Ansatz based on the requirement to conform with Corollary 3.24 below, (3.21) is proposed as a choice for the comultiplication Δ_1. The image of $h|r \otimes k|s$ under $\Delta_1 \otimes 1$ in (3.24) will then become

$$\sum_{r^{L_1}/r^{R_1}=r} h|r^{L_1} \otimes h|r^{R_1} \otimes k|s = h|(s^{(h\backslash k)^{-1}} \circ r) \otimes h|s^{(h\backslash k)^{-1}} \otimes k|s + \sum_{r^{L_1} \neq s^{(h\backslash k)^{-1}} \circ r} ?\,,$$

since $r = (s^{(h\backslash k)^{-1}} \circ r)/r^{R_1} = (r \cdot s^{(h\backslash k)^{-1}})/r^{R_1}$ iff $r^{R_1} = s^{(h\backslash k)^{-1}}$.

For the commuting of the diagram (3.24), the multiplication ∇_1 may then map $h|s^{(h\backslash k)^{-1}} \otimes k|s \mapsto (h\backslash k)|s$, thereby annihilating the remaining summands from the image of $h|r \otimes k|s$ under $\Delta_1 \otimes 1$. Since $r(h\backslash k) = s$ iff $r = s^{(h\backslash k)^{-1}}$, the multiplication ∇_1 indicated in (3.19) indeed behaves as required. $\square$

Using the definitions (3.19)–(3.20) and (3.21)–(3.22), the equations (2.14) and (2.15) produce the full set of composites as

$$a \overset{\mathsf{G}_0}{\longmapsto} f|(q^f\backslash p)\otimes(f\circ g)|q^f, \qquad a \overset{\Game_0}{\longmapsto} (f\circ g)|p\otimes g|(p^{f^{-1}}\backslash q),$$

$$a \overset{\mathsf{G}_1}{\longmapsto} f|(q^{(f\backslash g)^{-1}}\circ p)\otimes(f\backslash g)|q, \qquad a \overset{\Game_1}{\longmapsto} (f\backslash g)|p^{(f\backslash g)}\otimes g|(q\backslash p^{(f\backslash g)}),$$

$$a \overset{\mathsf{G}_2}{\longmapsto} f|(q^g/p)\otimes(f/g)|q, \qquad a \overset{\Game_2}{\longmapsto} (f/g)|q\otimes g|(q\circ p^{g^{-1}})$$

with common argument $a = f|p\otimes g|q$. At this point, the multiplications, comultiplications, and composites have been constructed from the opposite of the quantum couple structure and the commuting of the deuce and tenner diagrams. Thus, we have obtained the following result.

Theorem 3.21. *Each quantum couple GQ supports a quantum* S*-quasigroup structure using* (3.17)–(3.22) *for the multiplications and comultiplications.*

Proof. It remains to check the bimagma conditions for $i \neq 0$. The diagram

$$\begin{array}{ccc}
f|p\otimes g|q & \overset{\Delta_1\otimes\Delta_1}{\longmapsto} & f|p^{L_1}\otimes f|p^{R_1}\otimes g|q^{L_1}\otimes g|q^{R_1} \\
\nabla_1 \downarrow & & \\
\delta_{p(f\backslash g),q}(f\backslash g)|q & & \downarrow 1_{GQ}\otimes\tau\otimes 1_{GQ} \\
\Delta_1 \downarrow & & \\
\delta_{p(f\backslash g),q}(f\backslash g)|q^{L_1}\otimes fg|q^{R_1} & \underset{\nabla_1\otimes\nabla_1}{\longleftarrow\!\shortmid} & f|p^{L_1}\otimes g|q^{L_1}\otimes f|p^{R_1}\otimes g|q^{R_1}
\end{array}$$

(where the summations in the comultiplications are suppressed — compare the proof of [31, Th. 3.31]) verifies the case $i = 1$. Note that $p(f\backslash g) = q$ and $p^{L_1}p^{R_1} = p$ imply $p^{L_1}(f\backslash g)\cdot p^{R_1}(f\backslash g) = p(f\backslash g)$, so $p^{L_1}(f\backslash g) = q^{L_1}$ and $p^{R_1}(f\backslash g) = q^{R_1}$ with $q^{L_1}q^{R_1} = q$. The case $i = 2$ is similar. □

For a quantum T-quasigroup, the commuting of the blackjack diagram would be required. The following lemma determines when that happens.

Lemma 3.22. *The blackjack diagram*

$$\begin{array}{ccc}
GQ\otimes GQ & \overset{\tau}{\longrightarrow} & GQ\otimes GQ \\
\Game_1 \downarrow & & \uparrow \mathsf{G}_2 \\
GQ\otimes GQ & \underset{\tau}{\longrightarrow} & GQ\otimes GQ
\end{array}$$

commutes if and only if the action of G on Q is trivial.

Proof. The diagram takes the form

$$\begin{array}{ccc}
f|p \otimes g|q & \overset{\tau}{\longmapsto} & x = g|q \otimes f|p \\
 & & \\
 & & y = g|\big(p^{(f\backslash g)^2}/(q\backslash p^{(f\backslash g)})\big) \otimes f|p^{f\backslash g)} \\
\Big\downarrow \partial_1 & & \Big\| \text{(DL)} \\
 & & g|\big(p^{(f\backslash g)^2}/(q\backslash p^{(f\backslash g)})\big) \otimes (g/(f\backslash g))|p^{(f\backslash g)} \\
 & & \Big\uparrow \mathsf{G}_2 \\
(f\backslash g)|p^{(f\backslash g)} \otimes g|(q\backslash p^{(f\backslash g)}) & \underset{\tau}{\longmapsto} & g|(q\backslash p^{(f\backslash g)}) \otimes (f\backslash g)|p^{(f\backslash g)}
\end{array}$$

at the elementary level. Comparing the respective right hand quasigroup components $p^{(f\backslash g)}$ and p of the two terms x and y at the top right hand corner of the diagram, it is immediately clear that the diagram will not commute if the action of G on Q is not trivial. On the other hand, if the action is trivial, then $p^{(f\backslash g)} = p$, and also $p^{(f\backslash g)^2}/(q\backslash p^{(f\backslash g)}) = p/(q\backslash p) \overset{\text{(DL)}}{=} q$, so the diagram does commute in this case. □

We thus obtain the following.

Theorem 3.23. *A quantum couple GQ supports a quantum* T*-quasigroup structure using* (3.17)–(3.22) *for the multiplications and comultiplications if and only if G acts trivially on Q.*

Example 3.19(c) yields the following. (Compare [23, §4.8] for the loop case.)

Corollary 3.24. *Consider a quasigroup Q. Then, under the respective multiplications $\nabla_0 = \nabla_1 = \nabla_2 \colon p \otimes q \mapsto \delta_{pq} q$ and comultiplications*

$$\Delta_0 \colon q \mapsto \sum_{q^{L_0} \circ q^{R_0} = q} q^{L_0} \otimes q^{R_0}\,,$$

$$\Delta_1 \colon q \mapsto \sum_{q^{L_1} / q^{R_1} = q} q^{L_1} \otimes q^{R_1}\,,$$

$$\Delta_2 \colon q \mapsto \sum_{q^{L_2} \backslash q^{R_2} = q} q^{L_2} \otimes q^{R_2}\,,$$

the dual quasigroup algebra S^Q forms an associative quantum T*-quasigroup.*

3.4. **Hopf algebras.** Consider a symmetric monoidal category $\mathbf{V}$. In [31, §4.1], it was shown that a Hopf algebra $(A, \nabla, \Delta, \eta, \varepsilon, S)$ in $\mathbf{V}$ reduces to a quantum quasigroup (A, ∇, Δ) in $\mathbf{V}$. Specifically, the left composite G

has $(\Delta \otimes 1_A)(1_A \otimes S \otimes 1_A)(1_A \otimes \nabla)$ as its inverse, while the the right composite $\reflectbox{$\partial$}$ has $(1_A \otimes \Delta)(1_A \otimes S \otimes 1_A)(\nabla \otimes 1_A)$ as its inverse. It will be now be shown that a quantum S-quasigroup $(A, \nabla_i, \Delta_i)_{i=0}^{i=2}$ is associated with the Hopf algebra. In order to match with the conventions that were adopted earlier, the quantum quasigroup (A, ∇_0, Δ_0) will be taken from the opposite/co-opposite Hopf algebra $(A, \tau\nabla, \Delta\tau, \eta, \varepsilon, S)$ of $(A, \nabla, \Delta, \eta, \varepsilon, S)$. If the antipode S is invertible, it provides an isomorphism between these two Hopf algebras [22, Ex. 1.3.3], [25, Prop. 7.1.9(c)].

Theorem 3.25. *Suppose that $(A, \nabla, \Delta, \eta, \varepsilon, S)$ is a Hopf algebra. Define comultiplications $\Delta_0 = \Delta\tau$ and $\Delta_i = \Delta$ for $i = 1, 2$, and multiplications*

$$\nabla_0 = \tau\nabla\,, \quad \nabla_1 = (1_A \otimes S)\nabla\,, \quad \text{and} \quad \nabla_2 = (S \otimes 1_A)\nabla\,. \tag{3.25}$$

Then $(A, \nabla_i, \Delta_i)_{i=0}^{i=2}$ is a quantum S*-quasigroup.*

Proof. Since $(A, \tau\nabla, \Delta\tau, \eta, \varepsilon, S)$ is a Hopf algebra, it follows that (A, ∇_0, Δ_0) is a quantum quasigroup [31, §4.1]. The bimagma diagram for (A, ∇_1, Δ_1) appears as

$$\begin{array}{ccccc}
x \otimes y & \xrightarrow{(1_A \otimes S)\nabla} & x \cdot y^S & \xrightarrow{\Delta} & (x \cdot y^S) \otimes (x \cdot y^S) \\
 & & & & \| \,? \\
\Big\downarrow {\scriptstyle \Delta \otimes \Delta} & & & & x^L \cdot y^{LS} \otimes x^R \cdot y^{RS} \\
 & & & & \Big\uparrow {\scriptstyle (1_A \otimes S)\nabla \otimes (1_A \otimes S)\nabla} \\
x^L \otimes x^R \otimes y^L \otimes y^R & & \xrightarrow[1_A \otimes \tau \otimes 1_A]{} & & x^L \otimes y^L \otimes x^R \otimes y^R
\end{array}$$

in the elementary notation. The desired equality in the top right corner is obtained by applying the corresponding bimagma diagram for (A, ∇, Δ) to the element $x \otimes y^S$. Commuting of the bimagma diagram for (A, ∇_2, Δ_2) is derived in similar fashion.

The left composite in the bimagma (A, ∇_1, Δ_1) is

$$\begin{aligned}
\mathsf{G}_1 &= (1_A \otimes \Delta_1)(\nabla_1 \otimes 1_A) = (1_A \otimes \Delta)[((1_A \otimes S)\nabla) \otimes 1_A] \\
&= (1_A \otimes \Delta)(1_A \otimes S \otimes 1_A)(\nabla \otimes 1_A) = \reflectbox{∂}^{-1} = (\tau \reflectbox{∂}_0 \tau)^{-1} = \tau \reflectbox{∂}_0^{-1} \tau\,,
\end{aligned} \tag{3.26}$$

where the penultimate equality follows by [13, Lemma 4.4], [14, Lemma 3.8] (compare Propositon 3.35 below). Thus, (3.26) yields the desired tenner diagram. Derivation of the deuce diagram is similar. □

We may now see the relevance of Problem 1.1 to Hopf algebras.

Corollary 3.26. *Consider the quantum quasigroup reduct $(A, \nabla_0, \Delta_0) = (A, \tau\nabla, \Delta\tau)$ of the opposite/co-opposite of a Hopf algebra $(A, \nabla, \Delta, \eta, \varepsilon, S)$.*

Then, there may exist distinct quantum S*-quasigroups* $(A, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *that extend* (A, ∇_0, Δ_0).

Proof. Let G be the cyclic group $\langle x | x^3 = 1 \rangle$. Consider the group double $(GG, \nabla, \Delta, \eta, \varepsilon, S)$ over $\mathbb{Z}$, as described in Example 3.19(a). In the extension of (GG, ∇_0, Δ_0) provided by Theorem 3.25, $\Delta_1 = \Delta_2$. On the other hand, in the extension of (GG, ∇_0, Δ_0) provided by Theorem 3.21, the corresponding comultiplications Δ_1 and Δ_2, given respectively by (3.21) and (3.22), act as

$$\Delta_1 \colon 1|x \mapsto (1|1 \otimes 1|x^{-1}) + (1|x \otimes 1|1) + (1|x^{-1} \otimes 1|x)\,,$$
$$\Delta_2 \colon 1|x \mapsto (1|1 \otimes 1|x) + (1|x \otimes 1|x^{-1}) + (1|x^{-1} \otimes 1|1)\,,$$

and are thus distinct. □

By Theorem 3.25, each Hopf algebra yields a quantum S-quasigroup. On the other hand, Corollary 3.24 goes further, showing that dual group algebras yield quantum T-quasigroups. Thus, the following problem arises.

Problem 3.27. Which Hopf algebras yield quantum T-quasigroups?

Hopf algebras which do yield quantum T-quasigroups may be described as *triality Hopf algebras* (taking care to distinguish from the "Hopf algebras with triality" of [5]).

Example 3.28. (a) Linearization of the structures from §3.2, for the case where the quasigroup Q is a group, shows that group algebras are triality Hopf algebras.

(b) As confirmed above by Corollary 3.24, dual group algebras form triality Hopf algebras.

Example 3.29. Take the Hopf algebra $(A, \nabla, \Delta, \eta, \varepsilon, S)$ of Theorem 3.25 as the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, where the antipode S negates (the homomorphic copy of) $\mathfrak{g}$, and $\Delta \colon x \mapsto x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$ [22, Ex. 1.5.7], [25, §5.4]. Then $\nabla_1 = \nabla_2 \colon x \otimes y \mapsto -xy$ for $x, y \in \mathfrak{g}$, so $\mathsf{G}_1 = \mathsf{G}_2$ and $\eth_1 = \eth_2$. The blackjack diagram fails to commute, so A is not a triality Hopf algebra by virtue of the corresponding quantum S-quasigroup constructed in the proof of Theorem 3.25.

3.5. **Quantum U-quasigroups.** Here, we will discuss some instances of Definition 3.11.

3.5.1. *Classical U-quasigroups.* Let $(Q, /, \backslash)$ be a classical U-quasigroup. Consider the Cartesian symmetric monoidal category $(\mathbf{Set}, \times, \top)$ of sets under the direct product. Take the diagonal comultiplications of (3.6) for $i = \pm 1$, and the multiplications (3.8) and (3.9) on the object Q. The two diagrams (3.12) and (3.15) of Proposition 3.17 then witness the commuting of the blackjack diagram. Conversely, with the same assignments, (DL) and (DR) are consequences of the commuting of the blackjack diagram. A reduced version of Theorem 3.18 follows.

Theorem 3.30. *Under the assignments* (3.6) *for* $i = \pm 1$*,* (3.8) *and* (3.9) *for the comultiplications and multiplications on an object* Q *of* $(\mathbf{Set}, \times, \top)$*, the following structures on* Q*:*

(a) *A classical U-quasigroup* $(Q, /, \backslash)$;
(b) *A quantum U-quasigroup* $(Q, \nabla_i, \Delta_i)_{i=\pm 1}$

are equivalent.

Example 3.31. Consider a group G. On the underlying set of G, take the diagonal comultiplications $\Delta_1 = \Delta_2 \colon x \mapsto x \otimes x$ and multiplications

$$\Delta_1 \colon x \otimes y \mapsto y^{-1}xy \,,$$
$$\Delta_2 \colon x \otimes y \mapsto xyx^{-1} \,.$$

With the given structure, the quantum biquasigroup $(G, \nabla_i, \Delta_i)_{i=\pm 1}$ is a quantum U-quasigroup in $(\mathbf{Set}, \times, \top)$; compare Lemma 2.11(c) and [17, Q2]. This formulation provides a symmetric approach to Joyce' *quandles.*

3.5.2. *Quantum quandles.* For a field K, consider a Hopf algebra A in the symmetric monoidal category $(\underline{\underline{K}}, \otimes, K)$ of K-vector spaces under the usual tensor product. Use the diagrams

$$\begin{array}{ccccc} x \otimes y & \xmapsto{1\otimes\Delta} & x \otimes y^L \otimes y^R & \xmapsto{1\otimes S\otimes 1} & x \otimes y^{LS} \otimes y^R \\ \downarrow{\scriptstyle \nabla_1} & & & & \downarrow{\scriptstyle \tau\otimes 1} \\ y^{LS}xy^R & \xleftarrow[\nabla]{} & y^{LS}x \otimes y^R & \xleftarrow[\nabla\otimes 1]{} & y^{LS} \otimes x \otimes y^R \end{array}$$

and

$$\begin{array}{ccccc} x \otimes y & \xmapsto{\Delta\otimes 1} & x^L \otimes x^R \otimes y & \xmapsto{1\otimes S\otimes 1} & x^L \otimes x^{RS} \otimes y \\ \downarrow{\scriptstyle \nabla_2} & & & & \downarrow{\scriptstyle 1\otimes\tau} \\ x^Lyx^{RS} & \xleftarrow[\nabla]{} & x^L \otimes yx^{RS} & \xleftarrow[1\otimes\nabla]{} & x^L \otimes y \otimes x^{RS} \end{array} \tag{3.27}$$

to define new multiplications ∇_1 and ∇_2 on A, the respective *right* and *left adjoint actions* of the Hopf algebra A on itself. Compare [22, Ex. 1.6.3] or [25, Def'n. 11.2.4] for the latter.

Example 3.32. The classical U-quasigroup structure that was presented in Example 3.31 linearizes to $(KG, \nabla_1, \nabla_2, \Delta, \Delta)$, a quantum U-quasigroup structure on the group algebra KG of a group G over K. We may refer to this and related structures as *quantum quandles.*

Example 3.33. Consider the dual group algebra K^G of a finite group G (compare Example 3.19(c)). Here,

$$\nabla_1 \colon p \otimes q \mapsto \delta_{1,q} p \quad \text{and} \quad \nabla_2 \colon p \otimes q \mapsto \delta_{p,1} q$$

are the respective right and left actions. The corresponding right and left composites $\partial_1 = \mathsf{G}_2 = 1_{K^G \otimes K^G}$ are clearly invertible; the blackjack diagram commutes trivially. Thus $\big(K^G, \nabla_1, \nabla_2, \Delta, \Delta\big)$ is a quantum U-quasigroup.

Using Eulerian notation $\theta \colon x \mapsto \theta(x)$ for functions from G to K, the equations

$$(\theta\varphi\nabla)(x) = \theta(x)\varphi(x)\,, \quad (\theta\varphi\nabla_1)(x) = \theta(x)\varphi(1)\,, \quad (\theta\varphi\nabla_2)(x) = \theta(1)\varphi(x)$$

summarize the different multiplications on the dual group algebra K^G.

Example 3.34. If the Hopf algebra A is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ (as in Example 3.29), then the left action (3.27) is given by $\nabla_2 \colon x \otimes y \mapsto [x, y]$ for (the images in the universal enveloping algebra of) elements x, y of $\mathfrak{g}$ [22, Ex. 1.6.5]. Since this action is not in general invertible, the structure (A, ∇_1, ∇_2) will not form a quantum U-quasigroup.

Example 3.34 puts Example 3.32 into context, revealing that it is rather special. In particular, we are pushing the limits of the general philosophy that "Hopf algebras are linearized groups."

3.6. **Conjugates of quantum quasigroups.** Each equational quantum quasigroup may yield new equational quantum quasigroups. By analogy with Definition 2.5, these new structures are described as *conjugates* or "parastrophes" in relation to the original structure. We study them in the general setting of a symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

3.6.1. *Opposite equational quantum quasigroups.*

Proposition 3.35. *Consider a quantum* T*-quasigroup*

$$(Q, \nabla_i, \Delta_i)_{i=0}^{i=2} = (Q, \nabla_0, \nabla_1, \nabla_2, \Delta_0, \Delta_1, \Delta_2) \tag{3.28}$$

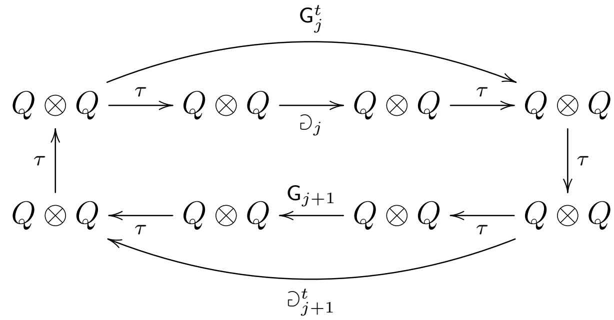
in the symmetric monoidal category $\mathbf{V}$*. Then*

$$(Q, \tau\nabla_{-i}, \Delta_{-i}\tau)_{i=0}^{i=2} = (Q, \tau\nabla_0, \tau\nabla_2, \tau\nabla_1, \Delta_0\tau, \Delta_2\tau, \Delta_1\tau) \tag{3.29}$$

is a quantum T*-quasigroup in* $\mathbf{V}$*.*

Proof. Note that $(Q, \tau\nabla_i, \Delta_i\tau)$ is a quantum quasigroup for each $i \in \mathbb{Z}/_3$ [13, Prop. 4.5], [14, Prop. 3.9]. In terms of the left and right composites G_i and ∂_i of (Q, ∇_i, Δ_i), the respective left and right composites G_i^t and ∂_i^t of $(Q, \tau\nabla_i, \Delta_i\tau)$ are given as $\tau\partial_i\tau$ and $\tau\mathsf{G}_i\tau$ [13, Lemma 4.4], [14, Lemma 3.8]. Since $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ is a quantum T-quasigroup, each of the diagrams (3.3) commutes. Now since the monoidal category $\mathbf{V}$ is symmetric, the rewritten versions

(3.30)



of the diagram (3.3) also commute, for each $j \in \mathbb{Z}/_3$. Setting $j+1=-i$, whence $j=-i-1=-(i+1)$, the outer circuit of (3.30) instantiates the diagram (3.3) for (3.29).[3] It thus follows that the latter is a quantum T-quasigroup, as required. □

Corollary 3.36. *Consider a quantum* S*-quasigroup*

$$(Q,\nabla_i,\Delta_i)_{i=0}^{i=2}=(Q,\nabla_0,\nabla_1,\nabla_2,\Delta_0,\Delta_1,\Delta_2) \tag{3.31}$$

in the symmetric monoidal category $\mathbf{V}$*. Then*

$$(Q,\tau\nabla_{-i},\Delta_{-i}\tau)_{i=0}^{i=2}=(Q,\tau\nabla_0,\tau\nabla_2,\tau\nabla_1,\Delta_0\tau,\Delta_2\tau,\Delta_1\tau) \tag{3.32}$$

is a quantum S*-quasigroup in* $\mathbf{V}$*.*

Proof. In this setting, it is just the cases $j=2,0$ (corresponding to the respective deuce and tenner diagrams) of (3.30) which come into play. □

Corollary 3.37. *Consider a quantum* U*-quasigroup*

$$(Q,\nabla_i,\Delta_i)_{i=\pm1}=(Q,\nabla_1,\nabla_{-1},\Delta_1,\Delta_{-1}) \tag{3.33}$$

in the symmetric monoidal category $\mathbf{V}$*. Then*

$$(Q,\tau\nabla_{-i},\Delta_{-i}\tau)_{i=\pm1}=(Q,\tau\nabla_{-1},\tau\nabla_1,\Delta_{-1}\tau,\Delta_1\tau) \tag{3.34}$$

is a quantum U*-quasigroup in* $\mathbf{V}$*.*

Proof. Here, the sole case $j=1$ (corresponding to the blackjack diagram) of (3.30) is used. □

Definition 3.38. (a) The quantum S-quasigroup (3.32) is defined to be the *opposite* of the quantum S-quasigroup (3.31).

(b) The quantum T-quasigroup (3.29) is defined to be the *opposite* of the quantum T-quasigroup (3.28).

(c) The quantum U-quasigroup (3.34) is defined to be the *opposite* of the quantum U-quasigroup (3.33).

[3] Here, the combinatorics of the indices are similar to the situation in §3.1.5.

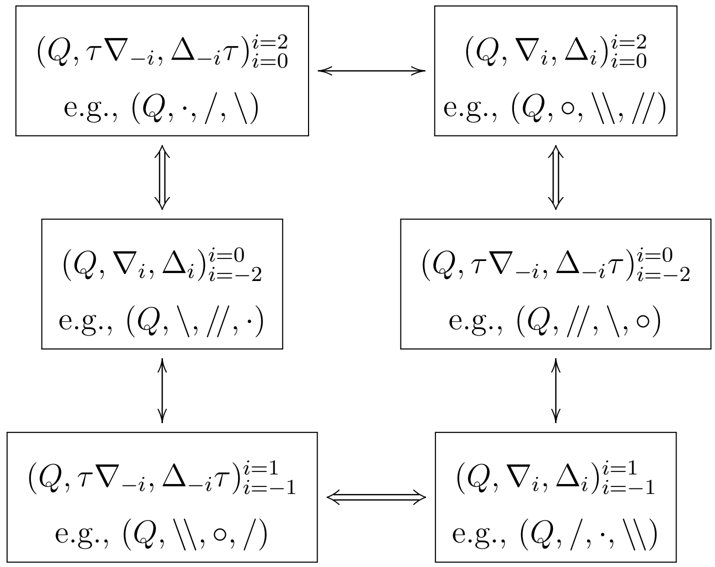


FIGURE 4. Triality symmetry and conjugates of the quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$. The underlying Cayley diagram, and the classical equational quasigroup models, correspond to those in Figure 1.

3.6.2. *Conjugates of quantum* T*-quasigroups.* If $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ is a quantum T-quasigroup, the cyclic symmetry inherent to Definition 3.8 implies that $(Q, \nabla_i, \Delta_i)_{i=-1}^{i=1}$ and $(Q, \nabla_i, \Delta_i)_{i=-2}^{i=0}$ also form quantum T-quasigroups. In turn, Proposition 3.35 implies that the respective opposites are quantum T-quasigroups as well. Thus, the initial quantum T-quasigroup is one of six related quantum T-quasigroups, as displayed in the triality diagram of Figure 4. The diagram expands Figure 1, with the same interpretation of the single- and double-shafted arrows. The boxes forming the vertices of the Cayley diagram also include the full classical equational quasigroup structure of the classical instantiation of the leading quantum quasigroup of the triality quantum quasigoup appearing in the box (cf. Definition 3.10). For example, the leading quantum quasigroup in the bottom left box is $(Q, \tau\nabla_1, \Delta_1\tau)$, corresponding to the first index $i = -1$. In the classical setting of §3.2, its multiplication is the opposite of the left division (3.8).

Definition 3.39. The quantum T-quasigroups appearing in Figure 4 are known as the *conjugates* of the quantum T-quasigroup $(Q, \tau\nabla_{-i}, \Delta_{-i}\tau)_{i=0}^{i=2}$.

4. Linear equational quantum quasigroups

Throughout this section, we will work with a commutative, unital ring S, considering the Cartesian symmetric monoidal category $\big(\underline{\underline{S}}, \oplus, \{\,0\,\}\big)$ of S-modules, with the direct sum as the monoidal product, the trivial module as the monoidal unit, and the swap matrix

$$\texttt{swap}\colon A \oplus B \to B \oplus A; \begin{bmatrix} x & y \end{bmatrix} \mapsto \begin{bmatrix} y & x \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

as the symmetry τ on the direct sum of modules A and B.

4.1. **Linear quantum triquasigroups.** The combinatorial definition of a linear quantum quasigroup [13, §3.5] leads to the following equational version.

Definition 4.1. A quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ in $\big(\underline{\underline{S}}, \oplus, \{\,0\,\}\big)$ is described as being *linear*. The multiplications

$$\nabla_i\colon Q \oplus Q \to Q; \begin{bmatrix} x & y \end{bmatrix} \mapsto \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \rho_i \\ \lambda_i \end{bmatrix} \tag{4.1}$$

and comultiplications

$$\Delta_i\colon Q \to Q \oplus Q; \begin{bmatrix} x \end{bmatrix} \mapsto \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} L_i & R_i \end{bmatrix} \tag{4.2}$$

are given by endomorphisms $\lambda_i, \rho_i, L_i, R_i$ of Q for $i \in \mathbb{Z}/_3$.

For each $i \in \mathbb{Z}/_3$, the bimagma condition (2.13) amounts to the mutual commutativity of the two subalgebras $S(\lambda_i, \rho_i)$ and $S(L_i, R_i)$ within the endomorphism ring $\underline{\underline{S}}(Q,Q)$ of Q [13, Prop. 3.7][31, Prop. 3.39]. Using the standard bimagma notation for the Cartesian symmetric monoidal category $\big(\underline{\underline{S}}, \oplus, \{\,0\,\}\big)$ of S-modules [13, Def'n. 3.5], such a bimagma is written as

$$Q(\rho_i, \lambda_i, L_i, R_i)\,. \tag{4.3}$$

For an alternative viewpoint, take the opposite $S(L_i, R_i)^{\mathsf{op}}$ of the subalgebra $S(L_i, R_i)$ of the endomorphism ring $\underline{\underline{S}}(Q,Q)$ of the S-module Q. Then ${}_{S(L_i,R_i)^{\mathsf{op}}}Q_{S(\rho_i,\lambda_i)}$, or

$$S(L_i, R_i)^{\mathsf{op}} \xrightarrow{\;Q\;} S(\rho_i, \lambda_i) \tag{4.4}$$

in the graphical notation which avoids multiple levels of subscripts, is a bimodule for each $i \in \mathbb{Z}/_3$ [13, Cor. 3.8].

In the quantum T-quasigroup of Definition 4.1, we have

$$\mathsf{G_i} = \big(\begin{bmatrix} L_i & R_i \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}\big) \left(\begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} \rho_i \\ \lambda_i \end{bmatrix}\right) = \begin{bmatrix} L_i & R_i\rho_i \\ 0 & \lambda_i \end{bmatrix} \tag{4.5}$$

and

$$\daleth_i = \big([1] \oplus \begin{bmatrix} L_i & R_i \end{bmatrix}\big)\left(\begin{bmatrix} \rho_i \\ \lambda_i \end{bmatrix} \oplus [1]\right) = \begin{bmatrix} \rho_i & 0 \\ L_i\lambda_i & R_i \end{bmatrix} \tag{4.6}$$

for each $i \in \mathbb{Z}/_3$ [13, Lemma 3.9]. Consideration of the commutativity of the diagrams (3.3) is fundamental to the quantum T-quasigroup setting. We obtain the following.

Lemma 4.2. *For linear quantum quasigroups* (Q, ∇_i, Δ_i) *as presented in Definition 4.1, the commutativity of the diagrams* (3.3) *is equivalent to the equations*

$$R_i^{-1} = L_{i+1} \tag{4.7}$$

$$\rho_i^{-1} = \lambda_{i+1} \tag{4.8}$$

$$R_i R_{i+1} \rho_{i+1} = -L_i \lambda_i \lambda_{i+1} \tag{4.9}$$

$$R_{i+1} \rho_{i+1} \rho_i = -L_{i+1} L_i \lambda_i \tag{4.10}$$

for each $i \in \mathbb{Z}/_3$*. In particular, the endomorphisms* L_i, R_i, λ_i *and* ρ_i *are automorphisms of the module* Q*.*

Proof. The commutativity of the diagrams (3.3) is equivalent to the matrix equations $\tau =$

$$\begin{bmatrix} 0 & \rho_i\lambda_{i+1} \\ R_iL_{i+1} & R_iR_{i+1}\rho_{i+1} + \lambda_i\lambda_{i+1} \end{bmatrix} = \begin{bmatrix} \rho_i & 0 \\ L_i\lambda_i & R_i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L_{i+1} & R_{i+1}\rho_{i+1} \\ 0 & \lambda_{i+1} \end{bmatrix}$$

and $\tau =$

$$\begin{bmatrix} R_{i+1}\rho_{i+1}\rho_i + L_{i+1}L_i\lambda_i & L_{i+1}R_i \\ \lambda_{i+1}\rho_i & 0 \end{bmatrix} = \begin{bmatrix} L_{i+1} & R_{i+1}\rho_{i+1} \\ 0 & \lambda_{i+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \rho_i & 0 \\ L_i\lambda_i & R_i \end{bmatrix}$$

for each $i \in \mathbb{Z}/_3$. The set of four displayed equations is equivalent to this pair of matrix equations, again for each $i \in \mathbb{Z}/_3$. □

The dualities in the respective equation pairs (4.7)–(4.8) and (4.9)–(4.10) of Lemma 4.2 should be noted. The first dual pair constitute the *chiral shift* equations, where the mnemonic “low on the rights, high on the lefts” is useful for keeping track of the indexing. Using the chiral shift equations, we may rewrite the automorphisms L_i and λ_i in terms of R_i and ρ_i. The equations in the second dual pair may then be summarized as

$$R_{i-1}R_iR_{i+1}\rho_{i+1}\rho_i\rho_{i-1} = -1 \tag{4.11}$$

for each $i \in \mathbb{Z}/_3$. The composites (4.5)and (4.6) become

$$\mathsf{G}_\mathsf{i} = \big(\begin{bmatrix} R_{i-1}^{-1} & R_i \end{bmatrix} \oplus [1]\big)\left([1] \oplus \begin{bmatrix} \rho_i \\ \rho_{i-1}^{-1} \end{bmatrix}\right) = \begin{bmatrix} R_{i-1}^{-1} & R_i\rho_i \\ 0 & \rho_{i-1}^{-1} \end{bmatrix} \tag{4.12}$$

and

$$\Game_i = \left([1] \oplus \begin{bmatrix} R_{i-1}^{-1} & R_i \end{bmatrix}\right)\left(\begin{bmatrix} \rho_i \\ \rho_{i-1}^{-1} \end{bmatrix} \oplus [1]\right) = \begin{bmatrix} \rho_i & 0 \\ R_{i-1}\rho_{i-1}^{-1} & R_i \end{bmatrix} \tag{4.13}$$

respectively. The general linear bimagma notations (4.3) for $i \in \mathbb{Z}/_3$ are adequately compressed by the following.

Definition 4.3. Write $Q(R_i, \rho_i)_{i=0}^{i=2}$ or $Q(R_i, \rho_i)$ or $Q(R_0, R_1, R_2, \rho_0, \rho_1, \rho_2)$ as *triality notations* for the linear quantum T-quasigroup of Definition 4.1.

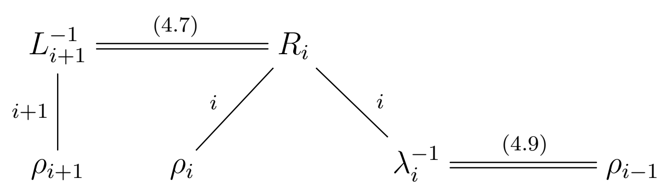
Lemma 4.4. *Given the linear quantum T-quasigroup of Definition 4.3, the subgroups*

$$\langle R_0, R_1, R_2 \rangle \quad \textit{and} \quad \langle \rho_0, \rho_1, \rho_2 \rangle \tag{4.14}$$

of the automorphism group $\underline{\underline{S}}(Q, Q)^*$ *of the* S*-module* Q *commute.*

Proof. Recall that the bimagma condition for each of the three individual quantum quasigroups that constitute the quantum T-quasigroup implies the mutual commutativity of the sets $\{ \lambda_i, \rho_i \}$ and $\{ L_i, R_i \}$. Now consider the system

(4.15)



of relationships in the automorphism group $\underline{\underline{S}}(Q, Q)^*$, where the single edges denote mutual commutativity. The labels on edges of this type indicate the particular constituent quantum quasigroup (Q, ∇_i, Δ_i) whose bimagma property is yielding the claimed commutativity. From (4.15), it is then apparent that each generator R_i of the subgroup on the left hand side of (4.14) commutes with each generator of the subgroup on the right hand side. The mutual commutativity of the two subgroups follows. □

Definition 4.5. (a) The subgroup B or

$$B(R_i, \rho_i)_{i=0}^{i=2} = \langle R_0, R_1, R_2, \rho_0, \rho_1, \rho_2 \rangle$$

of the automorphism group $\underline{\underline{S}}(Q, Q)^*$ of the underlying S-module Q of the linear quantum T-quasigroup $Q(R_i, \rho_i)_{i=0}^{i=2}$ is defined as its *bimultiplication group*. The subgroup $\langle R_0, R_1, R_2 \rangle$ is the *comultiplication group* or *Latin group*, while the subgroup $\langle \rho_0, \rho_1, \rho_2 \rangle$ is the *multiplication group* or *Greek group*.

(b) The group algebra SB of the bimultiplication group B is described as the *bimultiplication algebra* of the linear quantum T-quasigroup $Q(R_i, \rho_i)_{i=0}^{i=2}$. Its subalgebras $S\langle R_0, R_1, R_2\rangle$ and $S\langle \rho_0, \rho_1, \rho_2\rangle$ are the respective *multiplication algebra* and *comultiplication algebra*.

4.2. **Specification of linear quantum T-quasigroups.** This section will examine alternative approaches to specifying a linear quantum T-quasigroup. The original Definition 4.1 was based on the three combinatorial quantum quasigroups which constitute the underlying structure. Subsequently, the triality notation of Definition 4.3 specified a linear quantum T-quasigroup $Q(R_i,\rho_i)_{i=0}^{i=2}$ by the automorphisms $R_0, R_1, R_2, \rho_0, \rho_1, \rho_2$ of the S-module Q. The following lemma will eventually lead to a less redundant description (§4.3). Its hypotheses are satisfied by a linear quantum T-quasigroup as given by Definition 4.3.

Lemma 4.6. *Consider an S-module Q. Suppose that the automorphism group $\underline{\underline{S}}(Q,Q)^*$ of the S-module Q has commuting subgroups* (4.14).

(a) *The nine equations*

$$R_{i-1}R_iR_{i+1}\rho_{j+1}\rho_j\rho_{j-1} = -1 \tag{4.16}$$

for $i, j \in \mathbb{Z}/_3$ are equivalent.

(b) *If (any one of) the equivalent equations of* (a) *hold, then there is a central element Ω in the group*

$$B = \langle R_0, R_1, R_2, \rho_0, \rho_1, \rho_2\rangle \tag{4.17}$$

of automorphisms of the S-module Q such that the equations

$$\Omega = R_{i-1}R_iR_{i+1} = (-\rho_{j-1}^{-1})(-\rho_j^{-1})(-\rho_{j+1}^{-1}) \tag{4.18}$$

hold for each $i, j \in \mathbb{Z}/_3$.

(c) *If the chiral shift equations* (4.7)–(4.8) *hold, then*

$$\Omega = L_{i-1}^{-1}L_i^{-1}L_{i+1}^{-1} = (-\lambda_{j-1})(-\lambda_j)(-\lambda_{j+1})$$

for each $i, j \in \mathbb{Z}/_3$, and the equations (4.9)–(4.10) *are satisfied.*

Proof. (a): Note that

$$\begin{aligned} & R_{i-1}R_iR_{i+1}\rho_{j+1}\rho_j\rho_{j-1} = -1 \quad \Leftrightarrow \quad R_{i-1}R_iR_{i+1}\rho_{j+1}\rho_j = -\rho_{j-1}^{-1} \\ \Leftrightarrow \quad & \rho_{j-1}R_{i-1}R_iR_{i+1}\rho_{j+1}\rho_j = -1 \quad \Leftrightarrow \quad R_{i-1}R_iR_{i+1}\rho_{j-1}\rho_{j+1}\rho_j = -1 \end{aligned}$$

for each $i \in \mathbb{Z}/_3$, and similarly

$$R_{i-1}R_iR_{i+1}\rho_{j+1}\rho_j\rho_{j-1} = -1 \quad \Leftrightarrow \quad R_iR_{i+1}R_{i-1}\rho_{j+1}\rho_j\rho_{j-1} = -1$$

for each $i \in \mathbb{Z}/_3$.

(b): Each of the second equations of (4.18) follows immediately from the corresponding version of (4.16) upon right multiplication by $\rho_{j-1}^{-1}\rho_j^{-1}\rho_{j+1}^{-1}$. The expression $\Omega = (-\rho_0^{-1})(-\rho_1^{-1})(-\rho_2^{-1})$ shows that Ω, as an element of the group algebra $S\langle \rho_0, \rho_1, \rho_2\rangle$, commutes with each of the generators

R_0, R_1, R_2. Similarly, the expression $\Omega = R_0R_1R_2$ shows that Ω commutes with each of the remaining generators ρ_0, ρ_1, ρ_2.

(c): Here, the first part is straightforward. The second part follows by the previously noted equivalence between (4.11) and (4.9)–(4.10) when the chiral shift equations hold. □

Theorem 4.7. *Suppose that Q is a module over a commutative, unital ring S. Consider the following structures on Q as an object of the Cartesian symmetric monoidal category $\big(\underline{\underline{S}}, \oplus, \{\,0\,\}\big)$ of S-modules:*

- (A) *A linear quantum T-quasigroup structure on Q given as $Q(R_i, \rho_i)_{i=0}^{i=2}$ in the triality notation of Definition 4.3;*
- (B) *A subset $\mathcal{S} = \{\,R_0, R_1, R_2, \rho_0, \rho_1, \rho_2\,\}$ of the automorphism group $\underline{\underline{S}}(Q,Q)^*$ of the S-module Q such that the following relations hold:*
  - (a) *The elements of $\{\,R_0, R_1, R_2\,\}$ commute with the elements of $\{\,\rho_0, \rho_1, \rho_2\,\}$;*
  - (b) *Any one of the equivalent relations* (4.16) *holds in the group algebra $S\,\langle R_0, R_1, R_2, \rho_0, \rho_1, \rho_2\rangle$.*

Then the two structures (A) *and* (B) *on Q are equivalent.*

Proof. Since (4.11) and the hypotheses of Lemma 4.6 are satisifed by a linear quantum T-quasigroup, the implication (A) $\Rightarrow$ (B) is immediate.

Conversely, suppose that (B) holds. By (B)(a), the initial hypothesis of Lemma 4.6 holds. By (B)(b), the equivalent conditions of Lemma 4.6(a) hold. Thus, a central element Ω of the group B of (4.17) is defined by (4.18). The chiral shift equations (4.7)–(4.8) define S-automorphisms L_i, λ_i of Q for $i \in \mathbb{Z}/_3$. Multiplications (4.1) and comultiplications (4.2) are now defined on Q, with corresponding composites (4.5) and (4.6). In order to complete the derivation of (A), it suffices (by Lemma 4.2) to establish the validity of the dual pair of equations (4.9)–(4.10). Since the chiral shift equations hold, this now follows by Lemma 4.6(c). □

Remark 4.8. (a) Theorem 4.7(B)(a) may be rewritten as saying that there is a bimodule

$$S\,\langle R_0, R_1, R_2\rangle^{\mathsf{op}} \xrightarrow{\;Q\;} S\,\langle \rho_0, \rho_1, \rho_2\rangle \tag{4.19}$$

over the opposite group algebra of the comultiplication group and the group algebra of the multiplication group — compare (4.4).

(b) Theorem 4.7(B)(b) eliminates one of the six apparent degrees of freedom in filling the set $\mathcal{S} = \{\,R_0, R_1, R_2, \rho_0, \rho_1, \rho_2\,\}$.

Corollary 4.9 (The Six-Parameter or Triality Representation)**.** *The three bimagmas that constitute the linear quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$*

are

$$(Q,\nabla_0,\Delta_0)=Q(\rho_0,\rho_2^{-1},R_2^{-1},R_0)\,, \tag{4.20}$$

$$(Q,\nabla_1,\Delta_1)=Q(\rho_1,\rho_0^{-1},R_0^{-1},R_1)\,, \tag{4.21}$$

$$(Q,\nabla_2,\Delta_2)=Q(\rho_2,\rho_1^{-1},R_1^{-1},R_2) \tag{4.22}$$

in the notation of (4.3).

4.3. **Five-parameter representations.** Corollary 4.9 presents a linear quantum T-quasigroup in terms of the six generators $R_0, R_1, R_2, \rho_0, \rho_1, \rho_2$ of the bimultiplication group. It was mentioned in the introduction to the previous section that Lemma 4.6 would give a less redundant specification of a linear quantum T-quasigroup. This approach forms the topic of the current section. The penalty to be paid for the reduction in the number of parameters is a breaking of the triality symmetry of the six-parameter representations.

Consider a quantum T-quasigroup $(Q,\nabla_i,\Delta_i)_{i=0}^{i=2}$ in $\big(\underline{\underline{S}},\oplus,\{\,0\,\}\big)$, as given in the original Definition 4.1. The basis for its five-parameter representation is to consider

$$(\rho,\lambda,L,R)=(\rho_0,\lambda_0,L_0,R_0) \tag{4.23}$$

from (4.3), together with Ω from Lemma 4.6. We then have the following.

Theorem 4.10 (The Five-Parameter Representation)**.** *The three bimagmas that constitute the linear quantum T-quasigroup* $(Q,\nabla_i,\Delta_i)_{i=0}^{i=2}$ *are*

$$\begin{aligned}(Q,&\nabla_0,\Delta_0)\\ &=Q(\rho_0,\lambda_0,L_0,R_0)=Q(\rho,\lambda,L,R)\,,\end{aligned} \tag{4.24}$$

$$\begin{aligned}(Q,&\nabla_1,\Delta_1)\\ &=Q(\rho_1,\lambda_1,L_1,R_1)=Q(-\rho^{-1}\lambda\Omega^{-1},\rho^{-1},R^{-1},R^{-1}L\Omega)\,,\quad \textit{and}\end{aligned} \tag{4.25}$$

$$\begin{aligned}(Q,&\nabla_2,\Delta_2)\\ &=Q(\rho_2,\lambda_2,L_2,R_2)=Q(\lambda^{-1},-\lambda^{-1}\rho\Omega,L^{-1}R\Omega^{-1},L^{-1})\end{aligned} \tag{4.26}$$

in the notation of (4.3).

Proof. The first equation (4.24) is immediate from (4.23). The entries in the remaining equations are then obtained as follows. The chiral shift equations supply

$$\rho_2=\lambda_0^{-1}=\lambda^{-1}\ \text{ and }\ \lambda_1=\rho_0^{-1}=\rho^{-1} \tag{4.27}$$

along with

$$R_2=L_0^{-1}=L^{-1}\ \text{ and }\ L_1=R_0^{-1}=R^{-1}\,. \tag{4.28}$$

Next, (4.18) from Lemma 4.6(b) with $i = 0$ gives $\Omega = R_2 R_0 R_1$, whence

$$R_1 = R_0^{-1} R_2^{-1} \Omega = R^{-1} L \Omega \,, \tag{4.29}$$

while (4.16) from Lemma 4.6(a) with $i = j = 0$ gives

$$R_2 R_0 R_1 \rho_2 \rho_0 \rho_1 = \Omega \rho_2 \rho_0 \rho_1 = -1 \,,$$

whence

$$\rho_1 = -\rho_0^{-1} \rho_2^{-1} \Omega^{-1} = -\rho^{-1} \lambda \Omega^{-1} \,. \tag{4.30}$$

From these, the chiral shift equations and the centrality of Ω yield

$$\lambda_2 = \rho_1^{-1} = -\lambda^{-1} \rho \Omega \quad \text{and} \quad L_2 = R_1^{-1} = L^{-1} R \Omega^{-1} \,, \tag{4.31}$$

completing the proof of the theorem. □

For the formulation of Corollary 4.12 below, it is useful to recall a group-theoretical definition.

Definition 4.11. [12, A(19.3)] Consider a group G with subgroups H, K, such that $[H, K] = \{ 1 \}$ and $G = HK$. Then G is described as the (*internal*) *central product* of its subgroups H and K, with the central subgroup $H \cap K$ *amalganated.*

Corollary 4.12. *Take the context of the Five-Parameter Representation Theorem 4.10.*

(a) *The group $\langle \rho, \lambda, \Omega \rangle$ is the multiplication group of the linear quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$.*
(b) *The group $\langle L, R, \Omega \rangle$ is the comultiplication group of $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$.*
(c) *The group $\langle \rho, \lambda, \Omega, L, R \rangle$ is the bimultiplication group of the linear quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$.*
(d) *The bimultiplication group is the central product of the multiplication group and the comultiplication group.*

Proof. It suffices to note that the commutator condition of Definition 4.11 follows from Lemma 4.4. □

4.4. **Construction of linear quantum T-quasigroups.** Corollary 4.12 exhibits a central product group as part of the structure of a linear quantum T-quasigroup. The aim of this section is to examine a converse, showing how certain central product groups yield a linear quantum T-quasigroup. We begin by codifying the group-theoretical content of Corollary 4.12.

Definition 4.13. A group G is described as a (2+1+2)-*central product* if it is the product MW of mutually commuting subgroups $M = \langle \rho, \lambda, \Omega \rangle$ and $W = \langle L, R, \Omega \rangle$.

Theorem 4.14. *Let S be a commutative, unital ring. Let G be a (2+1+2)-central product group, with subgroups M and W as in Definition 4.13.*

(a) *Bimagmas are defined by* (4.24)–(4.26) *on the group algebra* SG.
(b) *The bimagmas of* (a) *combine to yield a linear quantum* T*-quasigroup structure* $(SG, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *on the group algebra* SG.
(c) *The group algebra* SG *is the bimultiplication algebra of the linear quantum* T*-quasigroup structure* $(SG, \nabla_i, \Delta_i)_{i=0}^{i=2}$. *The subalgebras* SM *and* SW *are the respective multiplication and comultiplication algebras.*

Proof. (a): The equations (4.27)–(4.31), together with (4.23), provide the individual module automorphisms needed for the multiplications (4.1) and comultiplications (4.2) of Definition 4.1. The mutual commutativity of M and W then suffices for (4.24)–(4.26) to yield bimagmas [13, Prop. 3.7][31, Prop. 3.39].

(b): The six chiral shift equations follow immediately from the equations (4.24)–(4.26). To complete the proof of (b), it will then suffice to verify the equations (4.11). Indeed,

$$R_2R_0R_1\rho_2\rho_0\rho_1 = L^{-1} \cdot R \cdot (R^{-1}L\Omega) \cdot \lambda^{-1} \cdot \rho \cdot (-\rho^{-1}\lambda\Omega^{-1}) = -1\,.$$

The remaining versions of (4.11) follow by Lemma 4.6(a).

(c): Note that $SM = S\langle\rho, \lambda, -\Omega\rangle = S\langle\rho_0, \rho_1, \rho_2\rangle$ and $W = \langle L, R, \Omega\rangle = \langle R_0, R_1, R_2\rangle$. The result follows from Definition 4.5(b). □

Corollary 4.15. *The respective multiplication and comultiplication groups of the linear quantum* T*-quasigroup structure* $(SG, \nabla_i, \Delta_i)_{i=0}^{i=2}$. *are* $\langle\rho, \lambda, -\Omega\rangle$ *and* $\langle L, R, \Omega\rangle$.

References

[1] A. Almeida Costa, Cours d'Algèbre Générale, Vol. I, Fundação Calouste Gulbenkian, Lisboa, 1969.
[2] N. Andruskiewitsch, M. Graña, From racks to pointed Hopf algebras, Adv. Math. 178, 177–243.
[3] R.R. Baldwin, W.E. Cantey, H. Maisel, J.P. McDermott, The optimum strategy in blackjack, J. Amer, Statist. Assoc. 51 (1956), 429–439. DOI:10.2307/2281431
[4] G.E. Bates, F. Kiokemeister, A note on homomorphic mappings of quasigroups into multiplicative systems, Bull. Amer. Math. Soc. 54 (1948), 1180–1185.
[5] G. Benkart, S. Madariaga, J.M. Pérez-Izquierdo, Hopf algebras with triality, Trans. Amer. Math. Soc. 365 (2012), 1001–1023.
[6] J. Bichon, Hopf-Galois objects and cogroupoids, Rev. Un. Mat. Argentina 55 (2014), 11–69.
[7] R.H. Bruck, A Survey of Binary Systems, Springer, Berlin, 1958.
[8] T. Brzeziński, Hopf modules and the fundamental theorem for Hopf (co)quasigroups, Int. Electron. J. Algebra 8 (2010), 114–128.
[9] A. Bundy, L. Wallen, Skolemization, p.123 in Catalogue of Artifical Intelligence Tools, Springer, Berlin, 1984. DOI: 10.1007/978-3-642-96868-6_235

[10] N. Combe, Yu.I. Manin, M. Marcolli, Quantum operads, pp. 113–145. in M.-L. Ge, Y.-H. He (eds.), Dialogues Between Physics and Mathematics, Springer, Cham, 2022. DOI: 10.1007/978-3-031-17523-7
[11] J.H. Conway, D.A. Smith, On Quaternions and Octonions, Peters, Natick, MA, 2003.
[12] K. Doerk, T. Hawkes, Finite Soluble Groups, de Gruyter, Berlin, 1992.
[13] B. Im, A. Nowak, J.D.H. Smith, Algebraic properties of quantum quasigroups, J. Pure Appl. Alg. 225 (2021), 106539. DOI: 10.1016/j.jpaa.2020.106539
[14] B. Im, A. Nowak, J.D.H. Smith, Symmetry classes of quantum quasigroups, J. Pure Appl. Alg. 228 (2024), 107722. DOI: 10.1016/j.jpaa.2024.107722
[15] I.M. James, Quasigroups and topology, Math. Zeitschr. 84 (1964), 329–342.
[16] C.B. Jay, Languages for monoidal categories, J. Pure Appl. Algebra 59 (1989), 61–85.
[17] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Alg. 23 (1982), 37–65.
[18] A.D. Keedwell, J. Dénes, Latin Squares and Their Applications, North-Holland, Amsterdam, 2015.
[19] Yu.A. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Physics 303 (2003), 2–30.
[20] J. Klim, S. Majid, Hopf quasigroups and the algebraic 7-sphere, J. Algebra 323 (2010), 3067–3110.
[21] J. Klim, S. Majid, Bicrossproduct Hopf quasigroups, Comment. Math. Univ. Carolin. 51 (2010), 287–304.
[22] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[23] J. Mostovoy, J.M. Pérez-Izquierdo, I.P. Shestakov, Hopf algebras in non-associative Lie theory, Bull. Math. Sci. 4 (2014), 129–173.
[24] J.M. Pérez-Izquierdo, Algebras, hyperalgebras, nonassociative bialgebras and loops, Adv. Math. 208 (2007), 834–876.
[25] D.E. Radford, Hopf Algebras, World Scientific, Singapore, 2012.
[26] V. Shcherbacov, Elements of Quasigroup Theory and Applications, CRC Press, Boca Raton, FL, 2017.
[27] J.-P. Serre, Lie Algebras and Lie Groups, Benjamin, New York, NY, 1965.
[28] T. Skolem, Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theorem über dichte Mengen, Videnskapsselskapet Skrifter, I. Matematisk-naturvidenskabelig Klasse 4 (1920), 1–36.
[29] J.D.H. Smith, An Introduction to Quasigroups and Their Representations, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[30] J.D.H. Smith, One-sided quantum quasigroups and loops, Demonstr. Math. 48 (2015), 620–636.
[31] J.D.H. Smith, Quantum quasigroups and loops, J. Algebra 456 (2016), 46–75. DOI: 10.1016/j.jalgebra.2016.02.014
[32] J.D.H. Smith, A.B. Romanowska, Post-Modern Algebra, Wiley, New York, NY, 1999.
[33] J.D.H. Smith, P. Vojtěchovský, Okubo quasigroups, Doc. Math. 27 (2022), 535–580.
[34] S.J. Witherspoon, The representation ring of the quantum double of a finite group, J. Algebra 179 (1996), 305–329.

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