COMBINATORIAL CHARACTERS OF QUASIGROUPS

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CONTENTS

1. Introduction. Over a century ago, when the character theory of finite abelian groups had become established, Dedekind began the programme of extending the theory to finite non–abelian groups. Having made little headway a decade later, he proposed the task to Frobenius. Developments progressed rapidly in Frobenius' hands, along both Dedekind's original group determinant approach and Frobenius' new approach that is now considered part of the theory of association schemes. Shortly afterwards, representation theory methods using matrices took over, and have dominated ever since.

The motivation behind the present survey is a continuation of Dedekind's programme, passing from abelian groups beyond non–commutative groups to "non–associative groups" or quasigroups. Since quasigroups (in the form of Latin squares) pre–date groups by several decades, Dedekind might conceivably have formulated his programme in these terms (but did not, as far as the records seem to indicate). In particular, his group determinant may be equally well considered as a quasigroup determinant. The three approaches–quasigroup determinants, association schemes, and representation theory–turn out to give three distinct theories when applied to quasigroups. The current survey focuses on the combinatorial character theory of quasigroups, which results from the association scheme approach.

It is possible to take a narrow view of the theory, regarding it merely as an example in or application of the theory of association schemes. From that point of view the fourth section presents the example, replacing the standard association scheme notation (i.e. that of [BI], [De]) with notation better adapted to the example. As usual when a mathematical theory finds an application, though, the application begins to suggest developments of the theory itself. This is illustrated here by the

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concepts of induction and superschemes. (See also [So] for new primitive schemes discovered by analogy with simple quasigroup schemes.)

The main aim of this survey, however, is to go beyond the narrow view and to present the theory in its proper context. This context comprises the historical background of Dedekind's programme and its continuation, presented in the second section, and the general algebraic theory of quasigroups, sketched in the third section. It is this background which motivates the formulations of the fourth section, and the developments of the theory discussed in the fifth section. Space considerations have precluded giving full details or proofs of theorems. The historical background in Section 2 makes use of Hawkins' excellent guides [H1–2]. References for quasigroup theory include Bruck's works such as [B1–3], the forthcoming [Ch], and parts of [S3–4]. The emphasis in the survey is on the various underlying ideas, following the quirks of the subject as it vacillates from direct generalisation given correct definitions to a perverse confounding of naive intuition.

2. Dedekind's programme–extending abelian characters. A *quasigroup* $(Q,.)$ is a set Q equipped with a binary operation $Q \times Q \longrightarrow Q; (x,y) \longmapsto x.y = xy$ called *multiplication*, such that in the equation

$$x.y = z \,, \tag{2.1}$$

knowledge of any two of x, y, z in Q specifies the third uniquely. Immediate examples are provided by groups, for which the multiplication satisfies the associative law $xy.z = x.yz$ (written here using the convention that multiplications denoted by juxtaposition are to be carried out before multiplications denoted by a dot). Thus quasigroups may be considered as non–associative (i.e. not necessarily associative) generalisations of groups. Groups in turn may be considered as non–commutative (i.e. not necessarily commutative) generalisations of abelian groups.

Character theory originated in Fourier analysis and Gauss' "Disquisitiones Arithmeticae" (cf. [Ga, §230] for introduction of the term "character"). It was developed for finite abelian groups by Dedekind in the late 1870's (published in his supplement to [Di]) and in further detail by Weber shortly after ([W1]–W4]). The methods of finite Fourier transforms belong to this stage of the theory. In the language of 20-th century algebra, the essence of the theory may be summarised as follows. Let Q be an abelian group of finite order s. Then the complex group algebra $\mathbf{C}Q$, being semisimple and commutative, decomposes as the (internal) direct sum

$$\mathbf{C}Q = \bigoplus_{i=1}^{s} \mathbf{C}\epsilon_i \tag{2.2}$$

of 1–dimensional subalgebras $\mathbf{C}\epsilon_i$ that consist of all scalar multiples of an idempotent ϵ_i. Conventionally, $\epsilon_1 = \frac{1}{|Q|} \sum_{q \in Q} q$. Under (2.2), each element q of Q is written as a linear combination

$$q = \sum_{i=1}^{s} \chi_i(q)\epsilon_i \,. \tag{2.3}$$

The coefficients in these linear combinations yield functions

$$\chi_i : Q \longrightarrow \mathbb{C}; q \longmapsto \chi_i(q) \tag{2.4}$$

that are the *irreducible characters* $\chi_1 = 1, \chi_2, \ldots, \chi_s$ of the abelian group Q. These functions satisfy

PROPOSITION 2.5. *The set $X = \{\chi_1 = 1, \chi_2, \ldots, \chi_s\}$ under multiplication is the group $Hom(Q, \mathbb{C}^*)$ of homomorphisms of Q into the multiplicative group $\mathbb{C}^*$ of non–zero complex numbers.* □

At Dedekind's instigation [Fr, pp. 2,38], Frobenius worked in the mid–1890's to extend this elegant and useful theory from abelian groups to general finite groups. His initial approach considered Dedekind's concept of a "group determinant". In the current context it is perhaps more appropriate to use the term "(associative) quasigroup determinant". Consider the (unbordered) multiplication table of a finite group $Q = \{q_1 = 1, q_2, \ldots, q_n\}$ as an $(n \times n)$–matrix. For each $i = 1, \ldots, n$, replace each matrix entry q_i by a variable X_i. The determinant of the new matrix obtained thus is an element $\Delta(X_1, \ldots, X_n)$ of the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$. This homogeneous polynomial is a product

$$\Delta(X_1, \ldots, X_n) = \pm \prod_{i=1}^{s} p_i(X_1, \ldots, X_n)^{d_i} \tag{2.6}$$

of irreducible factors $p_i(X_1, \ldots, X_n)$, monic as elements of $(\mathbb{C}[X_2, \ldots, X_n])\ [X_1]$, whose degrees are equal to the powers d_i to which they appear in the factorisation. In modern language, each irreducible factor p_i corresponds to an irreducible character χ_i, and the degree $d_i = \deg p_i$ of the irreducible polynomial p_i is equal to the degree $\chi_i(1)$ of the corresponding irreducible character χ_i. (See [H1, Th. 7.1] for the precise correspondence, and [H1, §§1–5] [H2] for a detailed history.) If Q is abelian, then

$$p_i(X_1, \ldots, X_n) = \sum_{j=1}^{n} \chi_i(q_j) X_j\ . \tag{2.7}$$

In general, however, the quasigroup determinant proved extremely intractable, even to an algebraist as skilled as Frobenius [H2, §4]. What emerged as a more fruitful way of extending the abelian theory was consideration of the centre $Z\mathbb{C}Q$ of the group algebra $\mathbb{C}Q$. This centre is a commutative, associative semisimple complex algebra spanned by the sums $c_i = \sum_{q \in C_i(1)} q$ of the elements in the various group conjugacy classes $C_1(1) = \{1\}, C_2(1), \ldots, C_s(1)$. It has a direct decomposition

$$Z\mathbb{C}Q = \bigoplus_{i=1}^{s} \mathbb{C}\epsilon_i \tag{2.8}$$

into a sum of sets $\mathbb{C}\epsilon_i$ of scalar multiples of idempotents ϵ_i, directly generalising (2.2). The corresponding analogue of (2.3) is the expression

$$c_j = \sum_{i=1}^{s} \frac{1}{d_i} \left(\sum_{q \in C_j(1)} \chi_i(q) \right) \epsilon_i \quad . \tag{2.9}$$

Up to this point, everything carries over nicely from the abelian case. Proposition 2.5, however, breaks down drastically. All that remains may be summarised in Proposition 2.10 below. Recall that a *complex-valued group class function* f on Q is a function $f : Q \longrightarrow \mathbb{C}$ whose restriction to each group conjugacy class $C_i(1)$ is constant. A homomorphism $f \in Hom(Q, \mathbb{C}^*)$ is a class function, but the converse is false. Denote the set of all complex-valued group class functions on Q by $\mathbb{C}gc(Q)$. The set $\mathbb{C}gc(Q)$ of functions inherits a $\mathbb{C}$-algebra structure from the domain $\mathbb{C}$ of its elements.

PROPOSITION 2.10. *(i) The set $X = \{\chi_1 = 1, \chi_2, \ldots, \chi_s\}$ is a basis for the underlying $\mathbb{C}$-vector space of $\mathbb{C}gc(Q)$.*
(ii). The set $\mathbb{N}X$ of sums of irreducible characters is a submonoid of the monoid $(\mathbb{C}gc(Q), .)$.
(iii) The set $\{\chi \in X | \chi(1) = 1\}$ is the subgroup $Hom(Q, \mathbb{C}^*)$ of the monoid $(\mathbb{C}gc(Q), .)$. □

With his understanding of the centre of the group algebra, Frobenius was able to prove parts (i) [Fr, p. 8, (8.)], [H2, §4] and (iii) [Fr, pp. 42–4], [H1; Th. 4.1, (7.4)] of Proposition 2.10. Part (ii), however, did not lie within the scope of either of his two approaches to the characters of general finite groups. It needed the third approach, which subsumed character theory under representation theory by regarding characters from $\mathbb{N}X$ as traces of complex matrix representations. Inspired by [Mo], Frobenius proved (ii) [Fr, p. 119] by observing that the product of a pair of characters is the trace of the tensor product of the representations of which the factors are traces [H2, §6]. Since the beginning of the twentieth century, this third, representation theory approach has so dominated the field that it has often been difficult to perceive character theory as a separate topic. The one area in which the second approach has continued to remain viable has been the study of the characters of the symmetric and related linear groups [Ma]. Subsequently, at the latest since the publication of Delsarte's thesis [De], the second approach has become part of the theory of association schemes [Bi, §2.2, Example 2.1 (2)]. The three approaches that Frobenius adopted to the problem of extending character theory from finite abelian groups to general finite groups may thus be characterised as

$$\begin{cases} (i) & \text{quasigroup determinants,} \\ (ii) & \text{association schemes,} \\ (iii) & \text{ordinary representation theory.} \end{cases} \tag{2.11}$$

Given the extremely successful generalisation from commutative groups to non-commutative groups, i.e. to associative quasigroups, it is now natural to turn one's attention to the question of further generalisation from associative quasigroups to non–associative quasigroups. Since multiplication of finite matrices is irredeemably associative, a naive attempt to apply the standard modern approach (2.11) (iii) fails. A more sophisticated attempt may be described as follows. (If the jargon is too daunting, skip to the end of the paragraph.) Observe that a complex vector space M on which a group Q acts as a group of automorphisms furnishes a split extension $M]Q$ having a projection $\pi : M]Q \to Q$. This projection $\pi : M]Q \longrightarrow Q$ is a complex vector space object in the comma category of groups over Q. Conversely, any such object $\pi : E \longrightarrow Q$ gives a Q–module $\pi^{-1}(1)$. The ordinary representation theory of a finite, non–empty quasigroup Q lying in a given variety $\mathfrak{V}$ of quasigroups may thus be construed as the study of the so–called *complex Q–modules in $\mathfrak{V}$*, the complex vector space objects in the comma category of $\mathfrak{V}$–quasigroups over Q. For appropriate varieties $\mathfrak{V}$, these objects turn out to be equivalent to complex representations of a group $U(Q;\mathfrak{V})$ known as the *universal multiplication group* of Q in $\mathfrak{V}$. However, $U(Q;\mathfrak{V})$ may turn out to be infinite. As an extreme example, if $\mathfrak{V}$ is just the variety of all quasigroups, then $U(Q;\mathfrak{V})$ is the free group on the disjoint union $Q\dot{\cup}Q$. The Q–modules are characterised by almost periodic functions on a subgroup of this free group, leading to what is called the *analytic character theory* of the quasigroup Q. The current, nascent state of the theory is described in [S4] [S5]. For present purposes it suffices to summarise by saying that the ordinary representation theory approach (2.11) (iii) leads to analytic character theory.

Despite its apparent intractability, the quasigroup determinant approach (2.11) (i) now begins to gain points in its favour. To start with, presence or lack of associativity in Q has little direct effect on the formulation of the basic concept or the factorisation problem (2.6). This is why Dedekind's term "group determinant" was replaced by "quasigroup determinant" above. Secondly, the availability of symbolic computation packages has facilitated the study of small examples (i.e. $|Q|$ up to the order of twenty at present). Indeed, an optimist might hope that ninety years of explosive development of mathematics would have provided more powerful analytic tools than those that Frobenius had at his disposal. (To which a pessimist might reply that there has not been a comparable development in the most vital analytic tool of all, the one between the ears.) The quasigroup determinant does have one significant feature. Two quasigroups $(Q,.)$ and $(P,*)$ are said to be *isotopic*, written $(Q,.) \sim (P,*)$, if there is an *isotopy*, an ordered triple (α,β,γ) of bijections $Q \longrightarrow P$, such that

$$x^{\alpha} * y^{\beta} = (x.y)^{\gamma} \tag{2.12}$$

for all x, y in Q. The relation of isotopy is an equivalence relation. For certain purposes, such as the coordinatisation of nets [BS] [B3] [Pi], it is the isotopy classes of quasigroups, rather than individual quasigroups themselves, that are important. And, to within sign, quasigroup determinants are invariants of isotopy classes. They may thus prove to give the most appropriate version of character theory in such contexts. These factors have led K.W. Johnson to begin a new study of quasigroup determinants ([J2] and work in progress).

It is the association scheme approach (2.11) (ii) which is the main topic of the current survey. This approach leads to what is now referred to as the *combinatorial character theory* of quasigroups (to distinguish it from the analytic character theory that results from the representation theory approach (2.11) (iii)). The initial inspiration came from the examples in Delsarte's thesis [De], and from the work on S–rings done by Tamaschke (e.g [Ta]) and Wielandt (e.g. [Wi]). After preparatory papers showing how S–rings (or "Gel'fand pairs" or "(commutative) association schemes") arose from loops and quasigroups [J1] [S2], the basic theory was started in [J3] (written in 1982 at the Technische Hochschule Darmstadt). Over the following six years the theory has been developed in a series of papers [J4], [J6]–[J8], [S6]. Some important examples have been studied by Song in his Ph.D. thesis [So] at Ohio State University under Bannai's direction. A detailed introduction to the earlier parts of the theory, with full proofs virtually from first principles, is given in [S4, Ch. 5]. References [S4, Ch. 6] and [S5] discuss the few tenuous connections between the analytic and combinatorial character theories that have so far been established.

3. Quasigroups: examples and some theory. This section discusses a number of important examples of quasigroups and classes of quasigroups together with certain aspects of their general theory that impinge directly on the combinatorial character theory. Bruck's work (e.g. [B1]–[B3]) gives a good guide to quasigroup theory (particularly loop theory) up to the 1950's. Subsequent developments up to the 1980's, including an intriguing range of applications, are presented in [Ch].

By the defining property (2.1) for a quasigroup Q, it follows that for each x in Q, the *right multiplication*

(3.1) $$R(x) : Q \longmapsto Q;\ y \longmapsto yx$$

and *left multiplication*

(3.2) $$L(x) : Q \longrightarrow Q;\ y \longmapsto xy$$

are permutations of Q. The subgroup of the group $Q!$ of all permutations on Q generated by $\{R(x), L(x) | x \in Q\}$ is called the *multiplication group* of Q, denoted by $Mlt(Q, .), MltQ$, or generically by G. Much of the structure of a quasigroup Q is embodied in the transitive permutation group action of G on Q. For an element q of Q, the stabiliser of q in G will be denoted by G_q.

Example 3.3 (groups). If Q is a group, then its multiplication group G is given by the exact sequence of group homomorphisms

(3.4) $$1 \longrightarrow Z(Q) \xrightarrow{\Delta} Q \times Q \xrightarrow{T} G \longrightarrow 1\ ,$$

where the diagonal $\Delta : Z(Q) \longrightarrow Q \times Q; z \longmapsto (z, z)$ embeds the centre of Q in its direct square, and where $T : Q \times Q \longrightarrow G; (x, y) \longmapsto L(x)^{-1}R(y)$. The stabiliser $G_1 = \{T(x, x) | x \in Q\}$ is the group $InnQ$ of inner automorphisms of Q. For general quasigroups Q, it is often helpful to think of the stabilisers G_q as generalisations of the inner automorphism group of a group. These stabilisers G_q need not consist of automorphisms of Q: this is true even for a group Q if $q \neq 1$. □

Example 3.5 (Latin squares). By the defining property (2.1), the (unbordered) multiplication table of a finite quasigroup Q is a Latin square. (Recall that a ***Latin square of size*** n has each element of a fixed set of n elements appearing (precisely once) in each row and each column of the square.) Conversely, a Latin square becomes the (bordered) multiplication table of a quasigroup $(Q, .)$ on its set Q of entries when the columns and the rows are each headed by the distinct elements of Q. For example, the quasigroup Q given thus

(3.6)

Q	1	2	3	4	5	6	7
1	1	3	2	5	6	7	4
2	3	2	1	6	7	4	5
3	2	1	3	7	4	5	6
4	5	6	7	4	3	2	1
5	6	7	4	3	5	1	2
6	7	4	5	2	1	6	3
7	4	5	6	1	2	3	7

,

together with its subquasigroup $P = \{1, 2, 3\}$, will prove useful below in the study of induced characters. Note that the vast number of Latin squares with a given set of entries produces an even vaster number of quasigroup structures on that set, scarcely diminished by the usual algebraic tricks such as identifying members of the same isomorphism or isotopy classes [B3]. Combinatorial character theory is useful in helping to come to terms with this plethora of quasigroups. In particular, it provides generally (although not universally) valid criteria for judging which are "uninteresting", singling out many of those which are worth examining. □

In the guise of Latin squares, the concept of quasigroup dates back at least to 1782 and Euler's famous problem of the 36 officers. This raises the question as to why the theory of quasigroups was not developed at least in parallel with its core, the theory of groups. Why, for example, did Dedekind not formulate the problem of studying quasigroup determinants rather than group determinants? One answer may be that suitable algebraic techniques did not appear until the middle of the twentieth century, when universal algebra developed as an outgrowth of the study of groups with operators. Even then, this (now "classical") universal algebra studies sets with operations satisfying certain identities. Thus it does not apply directly to quasigroups as defined by (2.1), since (2.1) is not an identity. A *quasigroup* $(Q, ., /, \backslash)$ has to be redefined as a set Q equipped with three binary operations: *multiplication.* (or juxtaposition), *right division* $/$ and *left division* $\backslash$ satisfying the identities

$$(3.7) \qquad \begin{cases} (x/y).y &= x\ ; \\ (x \cdot y)/y &= x\ ; \\ x \cdot (x\backslash y) &= y\ ; \\ x\backslash(x \cdot y) &= y\ . \end{cases}$$

For the equivalence of the two definitions, see [S4, 117]. Definition (3.7) enables one to apply universal algebraic ideas such as homomorphism, subalgebra (subquasigroup), congruence, free algebra, and variety to quasigroups. A quick introduction to these rudiments of classical universal algebra is given in [RS, Ch. 1]. Some of the algebraic difficulties inherent in the original definition (2.1) may be appreciated on noting that although $(\mathbb{N}, +)$ forms a subalgebra of $(\mathbb{Z}, +)$, and although $(\mathbb{Z}, +)$ is a quasigroup, it does not follow that $(\mathbb{N}, +)$ is a quasigroup.

Example 3.8 (conjugates). If $(Q, .)$, i.e. $(Q, ., /, \backslash)$, is a quasigroup, then so are its *conjugates* $(Q, /), (Q, \backslash), (Q, .), (Q, (x, y) \longmapsto yx), (Q, (x, y) \longmapsto y/x), (Q, (x, y) \longmapsto y\backslash x)$. Taking conjugates of a known quasigroup is a useful way of generating new ones. The most familiar not–associative quasigroup, namely the integers under subtraction, is obtained in this way. □

Example 3.9 (Steiner triple systems). A element x of a quasigroup Q is said to be *idempotent* if $x.x = x$. The quasigroup Q is itself said to be *idempotent* if each of its elements is idempotent. (The quasigroup Q of (3.6) is idempotent.) A quasigroup is called *totally symmetric* if it coincides with each of its conjugates. A (finite, non–empty,) totally symmetric idempotent quasigroup Q forms a Steiner triple system with set $\{\{x, y, xy\} | x \neq y \in Q\}$ of blocks. Conversely, a Steiner triple system Q forms a totally symmetric idempotent quasigroup if the product xy of two distinct elements x, y is defined to be the unique third element of the unique block containing them. The representation theory of Steiner triple systems is discussed in [S4, 4.3]. □

Both within combinatorial character theory and in many other parts of the theory of quasigroups, the mapping

$$(3.10) \qquad \rho : Q \times Q \longrightarrow G;\ (x, y) \longmapsto \rho(x, y) = R(x\backslash x)^{-1} R(x\backslash y)$$

from the direct square of a quasigroup Q to its multiplication group G plays an important role. For given x in Q, the set $\{\rho(x, y) | y \in Q\}$ is a transversal from G

to the stabiliser G_x. Moreover, $\rho(x,x) = 1$ and $x\rho(x,y) = y$. This means that the ternary operation

$$P : Q^3 \longrightarrow Q; (x,y,z) \longmapsto x\rho(y,z) \tag{3.11}$$

is a *Mal'cev parallelogram operation* [Ml] [S3, 1.4]: it satisfies the identities

$$P(x,x,y) = y = P(y,x,x) \ . \tag{3.12}$$

The existence of such an operation makes quasigroups very well behaved from the universal–algebraic point of view. They are "Mal'cev algebras", to which the centrality theory of [S1] presented in [S3] applies. A subquasigroup N of a quasigroup Q is said to be a *normal subquasigroup*, notation $N \triangleleft Q$, if there is a congruence α on Q such that N is an α–class. In this case the quotient Q^α is denoted by Q/N. Now a congruence α on Q is a subquasigroup of $Q \times Q$ containing the diagonal $\widehat{Q} = \{(q,q)|q \in Q\}$ as a subquasigroup. (One of the pleasant consequences following from the Mal'cev property of quasigroups is that any quasigroup α with $\widehat{Q} \leq \alpha \leq Q \times Q$ is a congruence on Q [S3,143][S4, 135].) The congruence α is said to be *central* if the diagonal $\widehat{Q}$ is a normal subquasigroup of α. If Q is a group, then α is central if and only if the congruence class 1^α containing 1 is contained in the centre $Z(Q)$ of Q. There is a unique maximal central congruence $\zeta(Q)$ on a quasigroup Q, known as the *centre congruence* [Ch, Th. III.3.10] [S3, 2.2–3]. Furthermore, there is a congruence $(Q^2|\zeta)$ on the (congruence considered as a) quasigroup $\zeta(Q)$ such that

$$\forall\, (x_1,x_2) \in \zeta(Q),\ \pi_1 : (x_1,x_2)^{(Q^2|\zeta)} \longrightarrow Q; (y_1,y_2) \longmapsto y_1 \quad \text{bijects} \tag{3.13}$$

[Ch, Prop. III. 3.5] [S3, 2.1–2]. Two quasigroups $(Q,.,/,\backslash)$ and $(P,.,/,\backslash)$ are said to be *centrally isotopic*, written $(Q,.,/,\backslash) \approx (P,.,/,\backslash)$, if there is a bijection $\theta : P \longrightarrow Q$ such that for each (postfix) operation $\omega = .,/,\backslash$, there is an element $(q_\omega, \overline{q}_\omega)$ of $\zeta(Q)$ such that for all p_1, p_2 in P,

$$(p_1^\theta p_2^\theta \omega, p_1 p_2 \omega^\theta)(Q^2|\zeta)(q_\omega, \overline{q}_\omega) \tag{3.14}$$

[Ch, Defn. III.4.1] [S3, 4.1]. Central isotopy is an equivalence relation [Ch, Th. III.4.5] [S3, 412], stronger than isotopy but weaker than isomorphism. As with isotopy, there are occasions when central isotopy classes are more important than individual quasigroups or isomorphism classes. One of the most striking illustrations is that finite quasigroups P and Q are centrally isotopic if and only if there is a finite quasigroup Z such that the direct products $Z \times P$ and $Z \times Q$ are isomorphic [S3, 4.2]. Such non–cancellation phenomena cannot be found amongst finite groups.

Example 3.15 (loops and piques). A *loop* $(Q,.,1)$ is a quasigroup $(Q,.)$ equipped with an *identity element* 1 such that

$$1.x = x = x.1 \tag{3.16}$$

for all x in Q. In universal–algebraic terms, loops are best considered as algebras $(Q, ., /, \backslash, 1)$ having the three binary operations forming a quasigroup, and then a nullary operation $1 : Q^0 = \{1\} \longrightarrow Q; 1 \longmapsto 1$ selecting the identity element. All groups are loops, but there are also many non–associative loops. For example, given an element q of a quasigroup $(Q, .)$, there is a loop $(Q, (x, y) \longmapsto P(x, q, y), q)$ with identity q constructed using the Mal'cev parallelogram (3.12). The quasigroup and loop are isotopic via the isotopy $(R(q\backslash q), L(q), 1)$. Sometimes the full strength of the loop property (3.16) is not required, merely the existence of the special subquasigroup $\{1\}$. A ***pique*** is thus defined as an algebra $(Q, ., /, \backslash, e)$ such that $(Q, ., /, \backslash)$ is a quasigroup with a nullary operation selecting an idempotent e of $(Q, .)$. The abbreviated notations $(Q, ., e)$ or Q_e are often used. The name "pique" is an acronym for *Pointed Idempotent QU*asigroup*E*. Motivated by the discussion at the end of Example 3.3, the inner multiplication group $Inn(Q_e)$ of a pique or loop $(Q, ., e)$ is the stabiliser G_e of e in the multiplication group G. □

There is a hierarchy

(3.17) abelian groups ⟶ groups ⟶ loops ⟶ piques ⟶ quasigroups

of increasingly general classes of quasigroups. The classes of this hierarchy provide natural stages at which to test a proposition about quasigroups.

The ***stability congruence*** $\sigma(Q)$ on a quasigroup Q with multiplication group G is the congruence

(3.18) $$\sigma(Q) = \{(x, y) \in Q \times Q | G_x = G_y\}$$

[Ch, §III.6]. If Q is a loop, then the stability and centre congruences coincide. In general, however, the stability congruence may be a proper subcongruence of the centre congruence.

Example 3.19 (abelian quasigroups and $\mathfrak{Z}$–quasigroups). A quasigroup Q is said to be ***abelian*** if it is commutative and associative, i.e. is empty or an abelian group. In other words, $\sigma(Q) = Q \times Q$. A quasigroup Q is said to be a ***$\mathfrak{Z}$–quasigroup*** if $\widehat{Q} \lhd Q \times Q$. In other words, $\zeta(Q) = Q \times Q$. This is the origin of the designation: such quasigroups are "all centre" (**Zentrum**). If a $\mathfrak{Z}$–quasigroup Q is a group or a loop, then it is abelian. Every $\mathfrak{Z}$–quasigroup is centrally isotopic to a $\mathfrak{Z}$–pique, namely $(Q^2/\widehat{Q}, ., \widehat{Q})$ [Ch, Prop. III.5.5] [S3,417]. Given a $\mathfrak{Z}$–pique $(Q, ., e)$, the corresponding loop $(Q, (x, y) \longmapsto P(x, e, y), e)$ of Example 3.15 is an abelian group $(Q, +, e)$. The pique may be recovered from knowledge of the automorphisms $R = R(e)$ and $L = L(e)$ of $(Q, +, e)$ via

(3.20) $$x.y = xR + yL\ .$$

Indeed, (3.20) may be used to construct a $\mathfrak{Z}$–pique $(A, ., 0)$ from any pair (R, L) of automorphisms of an abelian group $(A, +, 0)$. (The $\mathfrak{Z}$–pique $(\mathbb{Z}, -, 0)$ mentioned in Example 3.8 is constructed in this way.) The inner multiplication group $Inn(A, ., 0)$ is the subgroup $\langle R, L\rangle$ of $Aut(A, +, 0)$ generated by $\{R, L\}$, and the full multiplication group $Mlt(A, .)$ is the split extension $(A, +, 0)]\langle R, L\rangle$. For example, $\mathrm{Mlt}(\mathbb{Z}_n, -) \cong D_n$, the dihedral group. □

4. Quasigroup conjugacy classes and character tables. Throughout this section, Q will denote a quasigroup of positive integral order n with multiplication group G. The combinatorial character theory arises from the transitive permutation group action of G on Q, and investigates the extent to which such actions govern and reflect the algebraic structure of Q. If Q is a group, then its group conjugacy classes are subsets of Q : the orbits on Q of the inner multiplication group Inn Q, the stabiliser G_1 of the pointed idempotent 1. Working across the hierarchy (3.17), analogous decompositions make natural sense as far as piques, but not at the general quasigroup level. A similar problem arises with kernels of homomorphisms. In group theory, kernels of homomorphisms from a group Q are taken to be subsets of Q, namely normal subgroups. For general quasigroups Q, universal algebra suggests the definition of kernels of homomorphisms from Q as relations on Q (subsets of $Q \times Q$), namely congruences. One is thus led to a definition of quasigroup conjugacy classes as relations. The multiplication group G has a *diagonal action* on $Q \times Q$ given by

$$g : Q \times Q \longrightarrow Q \times Q; (x, y) \longmapsto (xg, yg) \tag{4.1}$$

for g in G. The *(quasigroup) conjugacy classes* of Q are then defined to be the orbits under the diagonal action of G on $Q \times Q$. They provide the *conjugacy class partition*

$$\Gamma = \{C_1 = \widehat{Q}, C_2, \ldots, C_s\} \tag{4.2}$$

of $Q \times Q$. The incidence matrices of the conjugacy classes $C_1 = \widehat{Q}, C_2, \ldots, C_s$ are the matrices with respect to Q of complex vector space endomorphisms $a_1 = 1, a_2, \ldots, a_s$ of the complex vector space $\mathbf{C}Q$. The action of G on Q extends by linearity to make $\mathbf{C}Q$ a right module for the complex group algebra $\mathbf{C}G$. Denote the $\mathbf{C}$–algebra of $\mathbf{C}G$–module endomorphisms of $\mathbf{C}Q$ by $\mathrm{End}_{\mathbf{C}G}\, \mathbf{C}Q$. In Wielandt's language [Wi], this algebra is the *Vertauschungsring* or *centraliser ring* $V(G, Q)$ of G on Q. The fundamental theorem of the combinatorial character theory of quasigroups then has a number of essentially equivalent formulations in various terms as follows:

THEOREM 4.3. *Let G be the multiplication group of a finite non–empty quasigroup Q.*

(i) *The action of G on Q is multiplicity–free.*

(ii) *(Q, Γ) is a (commutative) association scheme.*

(iii) *For any q in Q, the pair (G, G_q) is a Gel'fand pair.*

(iv) $\mathrm{End}_{\mathbf{C}G}\, \mathbf{C}Q = V(G, Q)$ *is commutative, with vector space basis $\{a_1, \ldots, a_s\}$.* □

Theorem 4.3 (ii) means that

$$
(4.4)\quad \begin{cases} \text{(A1)} & C_1 = \widehat{Q}\ ; \\ \text{(A2)} & \text{for each } C_i \text{ in } \Gamma, \text{ the converse } C_i^{-1} \\ & \text{is an element } C_{i'} \text{ of } \Gamma\ ; \\ \text{(A3)} & \forall\, C_i \in \Gamma,\ \forall C_j \in \Gamma,\ \forall C_k \in \Gamma,\ \exists c_{ijk} \in \mathbb{N}\ . \\ & \forall (x,y) \in C_k, |\{z \in Q | (x,z) \in C_i, (z,y) \in C_j\}| = c_{ijk}\ ; \\ \text{(A4)} & \forall\, 1 \le i,j,k \le s, \quad c_{ijk} = c_{jik}\ . \end{cases}
$$

The critical (A4) follows from Theorem 4.3 (iv) on noting that $a_i a_j = \sum_{k=1}^{s} c_{ijk} a_k$. The algebra $\mathbb{C}\Gamma$ of complex linear combinations of the a_i, with this product, is called the *Bose–Mesner algebra* of (Q, Γ). It is isomorphic to $V(G,Q)$. Theorem 4.3(i) means that each irreducible representation of G appears at most once in the permutation representation of G on Q. It follows from (A4) via [BI, Ch. 2, Th. 1.4]. For Theorem 4.3 (iii) and the terminology of Gel'fand pairs, see [Dg, Ch. 3] [He, Defn. 4.1]. For current purposes Theorem 4.3 (iv) is the most convenient formulation. An explicit proof is given in [S4, 523]. The crucial step [S4, 522] goes back to [J1] [S2]. The proof is computational rather than conceptual, involving the mapping ρ of (3.10). A better understanding of the proof would be helpful in the continuing search for new Gel'fand pairs, and in recognising new areas in which something like the current combinatorial character theory holds. The best that can be said at present, vague though it is, is that the proof is using a sort of linearised version of the centrality theory of [S1] [S3]. To work properly this theory appears to need categories, such as the category of quasigroup homomorphisms or the category of topological groups, in which group objects are necessarily abelian.

Theorem 4.3 (iv) achieves the desired continuation of Dedekind's programme extending character theory from abelian groups beyond groups to quasigroups. In analogy with (2.2) and (2.8), the semisimple commutative algebra $\mathrm{End}_{\mathbb{C}G}\, \mathbb{C}Q$ has the direct decomposition

$$
(4.5)\qquad \mathrm{End}_{\mathbb{C}G}\ \mathbb{C}Q = \bigoplus_{i=1}^{s} \mathbb{C}e_i
$$

into a sum of sets $\mathbb{C}e_i$ of scalar multiples of idempotents e_i. The analogue of (2.3) and (2.9) is the expression

$$
(4.6)\qquad a_i = \sum_{j=1}^{s} \xi_{ij} e_j\ .
$$

Setting $\Xi = (\xi_{ij})$, one has $\Xi^{-1} = H = (\eta_{ij})$ with

$$
(4.7)\qquad e_i = \sum_{j=1}^{s} \eta_{ij}\ a_j\ .
$$

Corresponding to the special numbering $a_1 = 1$, the idempotent e_1 is chosen to be the projection onto the 1–dimensional submodule $\mathbb{C}\sum_{q\in Q} q$. Set $f_i = tre_i$ and $|C_i| = nn_i$. Then $f_i\xi_{ji} = nn_j\overline{\eta}_{ij}$ [J3, p. 47] [S4, p. 94]. This leads to the following

Definition 4.8. The *character table* Ψ of the quasigroup Q is the $(s \times s)$–matrix $\Psi = (\psi_{ij})$ with entries

$$\psi_{ij} = \frac{\sqrt{f_i}}{n_j}\,\xi_{ji} = \frac{n}{\sqrt{f_i}}\,\overline{\eta}_{ij}\,. \tag{4.9}$$

The character table of a quasigroup Q is completely determined by the action of the multiplication group G on Q. By [Ch, Prop. III.4.6], centrally isotopic quasigroups have similar multiplication group actions, whence the same character table. One may thus compare the approaches (2.11) (i) and (ii) to the continuation of Dedekind's programme by noting that (2.11)(i) leads to quasigroup determinants, which are isotopy invariants, whereas (2.11)(ii) leads to character tables, which are central isotopy invariants.

In the course of generalising character theory from abelian groups to non–commutative groups, Proposition 2.5 was weakened to Proposition 2.10. In the combinatorial character theory of quasigroups, one of the first concerns is to see how much of Proposition 2.10 carries over, suitably rephrased to reflect the change from subsets of Q to relations on Q Let $\mathbb{C}(Q \times Q)$ denote the set of all complex–valued functions on $Q \times Q$. This set carries a lot of algebraic structure. To begin with, it has the *pointwise* or *Hadamard* involutive $\mathbb{C}$–algebra structure induced from $\mathbb{C}$ with complex conjugation. Secondly, it has a right $\mathbb{C}G$–module structure given by

$$g : \mathbb{C}(Q \times Q) \longrightarrow \mathbb{C}(Q \times Q); \theta \longmapsto (\theta^g : (x, y) \longmapsto \theta(xg^{-1}, yg^{-1})) \tag{4.10}$$

for g in G. Thirdly, it has a bilinear, associative *convolution* $*$ given by

$$\theta * \varphi(x, y) = \sum_{z\in Q} \theta(x, z)\varphi(z, y) \tag{4.11}$$

for x, y in Q. The group G is a group of automorphisms of both the Hadamard and convolution algebra structures on $\mathbb{C}(Q \times Q)$. It follows that the set $\mathbb{C}Cl(Q)$ of G–invariant functions forms a Hadamard and convolution subalgebra. These G–invariant functions are (*quasigroup*) *class functions* on Q. Their restrictions to conjugacy classes are constant. The isomorphism

$$\mathbb{C}(Q \times Q) \longrightarrow \mathrm{End}_{\mathbb{C}}\, \mathbb{C}Q; \theta \longmapsto (\widetilde{\theta} : x \longmapsto \sum_{y\in Q} \theta(x, y)y) \tag{4.12}$$

preserves all the algebraic structure on $\mathbb{C}(Q \times Q)$. It may be used to give a non–degenerate, associative bilinear form

$$\langle \theta, \varphi\rangle = |Q \times Q|^{-1}\, tr(\widetilde{\theta}\widetilde{\varphi}) \tag{4.13}$$

making $\mathbb{C}(Q \times Q)$ a Frobenius algebra [CR, 9.5]. Each row ψ_i of the character table Ψ gives a class function ψ_i whose restriction to C_j is ψ_{ij}. These functions $\psi_1 = 1, \psi_2 \ldots, \psi_s$ are known as the ***basic (combinatorial) characters*** of Q. By mild abuse of notation, the set $\{\psi_1 = 1, \psi_2, \ldots, \psi_s\}$ is also labelled Ψ. If Q is a group, then

$$\psi_i(x, y) = \chi_i(x^{-1}y) \tag{4.14}$$

with appropriate numbering. The analogue of Proposition 2.10(i) for general quasigroups Q is

PROPOSITION 4.15. *The set Ψ forms an orthonormal basis for $\mathbb{C}Cl(Q)$ under the inner product (4.13)* [J3, Th. 3.4] [S4, 541]. □

Along with the one orthogonality relation

$$\sum_{k=1}^{s} \psi_{ik}\overline{\psi}_{jk} n_k = n\delta_{ij} \tag{4.15}$$

embodied in Proposition 4.15, the character table also satisfies the other orthogonality relation

$$\sum_{k=1}^{s} \psi_{ki}\overline{\psi}_{kj} = n\delta_{ij}/n_i \tag{4.17}$$

[J3;(3.3), (3.4)] [S4;541(a),(b)]. The table contains enough information to specify n, s, the $n_i, f_i, \eta_{ij}, \xi_{ij}$, and c_{ijk}[J3,Cor. 3.5][S4,542–3]. It does determine the stability congruence $\sigma(Q)$ [J8, Prop. 5.1], but not the centre congruence $\zeta(Q)$, as will become apparent below. Further:

THEOREM 4.18. *The character table of Q specifies the congruence lattice of Q.* □

The proof of Theorem 4.18 is easy in the group case: the normal subgroups are just the kernels of the characters. The quasigroup case [J3, Th. 3.6] [S4, 545] is less immediate. It uses the idea of [CG, Prop. 3.1].

A quasigroup Q is said to be a ***rank*** 2 ***quasigroup*** if it has only 2 conjugacy classes, namely $C_1 = \widehat{Q}$ and $C_2 = Q^2 - \widehat{Q}$. Using the orthogonality relations (4.16), (4.17), it follows that the character table of Q is

(4.19)

Q	C_1	C_2
ψ_1	1	1
ψ_2	$\sqrt{n-1}$	$-1\big/\sqrt{n-1}$

From the standpoint of combinatorial character theory, such quasigroups are essentially "uninteresting". This is one of the criteria discussed at the end of Example 3.8. Note, however, that the 7–element Steiner triple system coming from the Fano plane $PG(2,2)$ gives a rank 2 quasigroup according to Example 3.9, and this structure hardly qualifies as "uninteresting". It is conjectured that almost all quasigroups are rank 2 quasigroups. More precisely [J4, Conj. 5.2]:

CONJECTURE 4.20. *Let $l(n)$ denote the number of Latin squares of of order n. Let $r(n)$ denote the number of Latin squares of order n that are multiplication tables of rank 2 quasigroups. Then* $\lim_{n\to\infty} \frac{r(n)}{l(n)} = 1.$ □

Rank 2 quasigroups Q (with $n > 2$) do serve to show that the analogue of Proposition 2.10 (ii) breaks down for quasigroups. Indeed,

$$\psi_2\psi_2 = \psi_1 + (n-2)(n-1)^{-1/2}\psi_2 \qquad (4.21)$$

from (4.19). The breakdown may be attributed to the divergence of the combinatorial character theory from the representation theory of quasigroups. Some of its implications are investigated in the final section of this survey.

Now suppose that $\theta : Q \longrightarrow P$ is a quasigroup epimorphism. If Q is a group, and $D : P \longrightarrow \mathrm{Aut}_{\mathbf{C}} V$ represents P as a group of automorphisms of a complex vector space V, then the composite $\theta D : Q \longrightarrow \mathrm{Aut}_{\mathbf{C}} V$ represents Q. Composing with the trace map $tr : \mathrm{Aut}_{\mathbf{C}} V \longrightarrow \mathbf{C}$, it is a trivial matter to lift characters from the quotient P up to Q. For general quasigroups Q, a comparable lifting result, the Quotient Theorem, holds. The proof [J6, Th.2.1] is much less direct.

THEOREM 4.22. *Let $\theta : Q \longrightarrow P$ be a quasigroup epimorphism, with corresponding diagonal $\widehat{\theta} : Q \times Q \longrightarrow P \times P; (x,y) \longmapsto (x\theta, y\theta)$. Then for each basic character $\varphi_k : P \times P \longrightarrow \mathbf{C}$ of P, the lift $\widehat{\theta}\varphi_k = Q \times Q \longrightarrow \mathbf{C}$ is a basic character of Q.* □

Lifting characters from a quotient quasigroup is one way of finding characters of a given quasigroup, helping to complete its character table. Another method is to induce characters from known characters of a subquasigroup. If P is a subgroup of a group Q, then the induction map $\uparrow_P^Q \colon \mathbb{C}gc(P) \longrightarrow \mathbb{C}gc(Q); f \longmapsto f^Q$ is defined by the Frobenius formula

$$f^Q(s) = \frac{1}{|P|} \sum_{\substack{u \in Q \\ u^{-1}su \in P}} f(u^{-1}su) \tag{4.23}$$

[Se, 7.2]. In the combinatorial character theory of quasigroups, the corresponding formula (4.24) below becomes much simpler. Let P be a non–empty subquasigroup of Q, with conjugacy class partition $\{B_{ij} | 1 \leq i \leq s, 1 \leq j \leq r_s\}$ such that each B_{ij} is a subset of the corresponding C_i. If $P \times P$ does not intersect C_i, then $r_i = 0$. Set $B_i = \cup\{B_{ij} | 1 \leq j \leq r_i\}$. Then the *induction* map $\uparrow_P^Q \colon \mathbb{C}Cl(P) \longrightarrow \mathbb{C}Cl(Q); f \longmapsto f^Q$ is given by

$$|P \times P| \sum_{(x,y) \in C_i} f^Q(x,y) = |Q \times Q| \sum_{(x,y) \in B_i} f(x,y) \tag{4.24}$$

[J4, (2.1)] [S4, (551)]. If Q is a group, the isomorphisms $\mathbb{C}Cl(P) \cong \mathbb{C}gc(P)$ and $\mathbb{C}Cl(Q) \cong \mathbb{C}gc(Q)$ given by (4.14) commute with the induction maps $\uparrow_P^Q$, so that the simple formula (4.24) subsumes the more complicated (4.23) [J4, §3][S4, 5.5]. The definition (4.24) works for general association schemes [J5]. This illustrates one of the side benefits of the development of the combinatorial character theory of quasigroups: it may suggest new developments in the general theory of association schemes. In [S7], induced class functions of association schemes defined according to (4.24) are interpreted as conditional expectations in the sense of J. Doob [Do, Ch. 1].

Along with the induction map $\uparrow_P^Q \colon \mathbb{C}Cl(P) \longrightarrow \mathbb{C}Cl(Q)$ given by (4.24), there is also a *restriction* map $\downarrow_P^Q \colon \mathbb{C}Cl(Q) \longrightarrow \mathbb{C}Cl(P); f \longmapsto f|_{P \times P}$. Under the inner products (4.13), namely $\langle , \rangle_P$ on $\mathbb{C}Cl(P)$ and $\langle , \rangle_Q$ on $\mathbb{C}Cl(Q)$, these linear mappings are mutually adjoint:

$$\langle f, g \uparrow_P^Q \rangle_Q = \langle f \downarrow_P^Q, g \rangle_P \tag{4.25}$$

for f in $\mathbb{C}Cl(Q)$ and g in $\mathbb{C}Cl(P)$ [J4, Th. 4.3] [S4, 557]. This relationship (4.25) is the so–called *Frobenius reciprocity*. Easily derived from (4.24), it subsumes Frobenius reciprocity for groups via (4.14).

Abelian quasigroups form a variety, so each quasigroup Q has an abelian "replica" [RS, p. 17], a maximal abelian quotient $Q^{\gamma(Q)}$. The congruence $\gamma(Q)$ is called the *abelian replica congruence* of Q. If Q is non–empty, the quotient $Q^{\gamma(Q)}$ has a unique idempotent, whose preimage is a normal subquasigroup of Q called the *derived subquasigroup* Q' of Q. The abelian replica may then be written as Q/Q'. Of course, if Q is a group, then Q' is the commutator subgroup, generated by all the commutators $(xy)/(yx)$. If Q is a loop, then Q' is the normal subloop generated by all the commutators $(xy)/(yx)$ and associators $(x.yz)/(xy.z)$. For the general quasigroup Q of positive integral order n, the order m of Q' divides n. By the Quotient Theorem 4.22, the n/m basic characters of the abelian group Q/Q' lift to n/m basic characters $\psi_1, \ldots, \psi_{n/m}$ of Q. A basic character of Q is called *linear* if it restricts to 1 on $\widehat{Q}$. Certainly the characters $\psi_1, \ldots, \psi_{n/m}$ are linear. The first part of the following theorem is the analogue of Proposition 2.10 (iii). The other two parts show that the linear characters retrieve some of the content of the failed analogue of Proposition 2.10 (ii).

THEOREM 4.26. *(i)* [J8, Th. 3.1] *The basic characters $\psi_1, \ldots, \psi_{n/m}$ form the complete set Λ of linear basic characters of Q. Under Hadamard multiplication, Λ forms an abelian group isomorphic to Hom $(Q/Q', \mathbf{C}^*)$.*
(ii)[J8, Prop. 4.2] *For a linear basic character λ of Q, Hadamard multiplication by λ gives an isometry of $\mathbf{C}Cl(Q)$.*
(iii) [J8, Th. 4.3 (i)] *The abelian group Λ of linear basic characters acts by Hadamard multiplication on the full set Ψ of basic characters.* □

5. Induction, products, and $\mathfrak{Z}$–quasigroups. The previous section covered the fundamentals of the combinatorial character theory of quasigroups. Once the right definitions and notations had been set up, these fundamental parts provided direct generalisations and analogues of the character theory of finite groups. The only point of divergence was the breakdown (4.21) of the analogue of Proposition 2.10(ii). This section, by contrast, concentrates on those aspects of the theory where new phenomena begin to emerge. Their origin is usually related to the "misbehaviour" of products, and to the evolution of the abelian/non–abelian dichotomy for groups into an abelian/$\mathfrak{Z}$/ non–$\mathfrak{Z}$ trichotomy for quasigroups. Unless explicitly stated otherwise, the notation of the previous section is used. Thus again Q is normally a quasigroup of positive integral order n, etc.

By Proposition 2.10 (ii) and (4.14), the set of integral combinations of basic characters of a group Q forms a ring under Hadamard multiplication. For a general quasigroup Q, the *coefficient ring* $\mathbf{Z}[Q]$ is defined to be the ring $\mathbf{Z}[\langle\psi_i\psi_j, \psi_k\rangle_Q | 1 \leq i, j, k \leq s]$. The *character ring* $R[Q]$ is then defined to be the ring of $\mathbf{Z}[Q]$–linear combinations of basic characters under Hadamard multiplication. If Q is associative, this agrees with the usual group–theoretic definition [Se, 9.1]. If Q is a rank 2 quasigroup, (4.21) shows that the coefficient ring $\mathbf{Z}[Q]$ is $\mathbf{Z}[(n-1)^{-1/2}] = \mathbf{Z}[X]/\langle(n-1)X^2 - 1\rangle$. Now let P be a non–empty subquasigroup of a quasigroup Q. If Q is associative, the induction map $\uparrow_P^Q$: $\mathbf{C}Cl(P) \longrightarrow \mathbf{C}Cl(Q)$ restricts to an abelian group homomorphism $\uparrow_P^Q$: $(R[P], +) \longrightarrow (R[Q], +)$. If Q is not associative, this need no longer be true. Consider the two rank 2 quasigroups P and Q of (3.6), with character tables

	P	B_1	B_2
(5.1)	φ_1	1	1
	φ_2	$\sqrt{2}$	$-1/\sqrt{2}$

Q	C_1	C_2
ψ_1	1	1
ψ_2	$\sqrt{6}$	$-1/\sqrt{6}$

Then $\varphi_2^Q = (7\sqrt{3}/9)\psi_2$, although $7\sqrt{3}/9$ does not lie in the coefficient ring $\mathbb{Z}[Q] = \mathbb{Z}[6^{-1/2}]$. To study rings of quasigroup characters under inducing, it appears to be necessary to admit at least the ring $\mathbb{A}$ of algebraic numbers as coefficients. Let $\mathbb{A}Cl(Q)$ denote the ring of class functions on Q taking values in $\mathbb{A}$. The set Ψ of basic characters of Q forms an $\mathbb{A}$-basis for $\mathbb{A}Cl(Q)$, and the induction map $\uparrow_P^Q\colon \mathbb{C}Cl(P) \longrightarrow \mathbb{C}Cl(Q)$ restricts to $\uparrow_P^Q\colon \mathbb{A}Cl(P) \longrightarrow \mathbb{A}Cl(Q)$. Moreover,

$$f\uparrow_P^Q \cdot g = (f\cdot g\downarrow_P^Q)\uparrow_P^Q \tag{5.2}$$

for f in $\mathbb{C}Cl(P)$ and g in $\mathbb{C}Cl(Q)$, so that $\mathbb{A}Cl(P)\uparrow_P^Q$ is an ideal of $\mathbb{A}Cl(Q)$ and $\mathbb{C}Cl(P)\uparrow_P^Q$ is an ideal of $\mathbb{C}Cl(Q)$ [J4, §6] [S4, 562]

A set $\{P_j | 1 \le j \le N\}$ of non–empty subquasigroups of a quasigroup Q is said to be ***protrusive*** if $\cup\{P_j \times P_j | 1 \le j \le N\}$ contains a member of each conjugacy class C_i of Q (so that some $P_j \times P_j$ "protrudes" into each C_i). For example, the set of cyclic subgroups of a group is protrusive, since an element $(1, x)$ of C_i is contained in $\langle x\rangle \times \langle x\rangle$. If Q is associative, Artin's Theorem [Se, 9.2] shows that each character of Q is a rational linear combination of characters induced by characters of members of any protrusive set of subquasigroups of Q. These results do not apply verbatim to non–associative quasigroups. In the quasigroup Q of (3.6), the set $\{\langle x\rangle | x \in Q\} = \{\{x\} | x \in Q\}$ of "cyclic" or singly generated subquasigroups is not protrusive, since $\cup\{\{x\} \times \{x\} | x \in Q\} = \widehat{Q}$ does not intersect the conjugacy class C_2. As for Artin's Theorem, the singleton $\{P\}$ is protrusive, but $\psi_1 = \varphi_1^Q - (2\sqrt{2}/7)\varphi_2^Q$ and $\psi_2 = (3\sqrt{3}/7)\varphi_2^Q$, so the characters of Q are not obtained as rational linear combinations of characters induced by characters of P. The closest analogue of Artin's Theorem holding for general quasigroups Q appears to be the following

THEOREM 5.3. *Let $\{P_j | 1 \le j \le N\}$ be a protrusive set of subquasigroups of Q. Then the direct sum maps $\bigoplus_{j=1}^N \uparrow_{P_j}^Q\colon \bigoplus_{j=1}^N \mathbb{A}Cl(P_j) \longrightarrow \mathbb{A}Cl(Q)$ and $\bigoplus_{j=1}^N \uparrow_{P_j}^Q\colon \bigoplus_{j=1}^N \mathbb{C}Cl(P_j) \longrightarrow \mathbb{C}Cl(Q)$ surject* [J4, Th. 7.1] [S4, 565] . □

One of the long–term programmes of the combinatorial character theory of quasigroups is to investigate the dependence and influence of the coefficient ring

$\mathbb{Z}[Q]$ on the algebraic structure of Q. Quasigroups with a unique non-linear basic character form a suitable nursery for the earliest stage of the programme. The most elementary quasigroups of this type are described by the following theorem [J8, Th. 5.2]. The notation is that of Theorem 4.26, with $\Psi(P)$ denoting the character table of a quasigroup P.

THEOREM 5.4. *For $|Q| > 2$, the following are equivalent:*

(a) *Q has a unique non-linear basic character, whose square is the sum of all the linear basic characters of Q;*

(b) *The character table $\Psi(Q)$ of Q has the form*

Q	C_1	C_2	C_3	$\cdots$	C_s
ψ_1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$		$\Psi(Q/Q')$		
$\psi_{n/2}$	1	1			
ψ_s	$\sqrt{n/2}$	$-\sqrt{n/2}$	0		0

;

(c) *the congruences $\gamma(Q)$ and $\sigma(Q)$ coincide, each having order $2n$.* □

Examples of quasigroups described by Theorem 5.4 are the symmetric group S_3, the dihedral group D_4, the quaternion group Q_8, and the octonion loop K_{16} [J4, §8][S4,558] which does for the octonion algebra what the quaternion group does for the quaternion algebra. Another example is Parker's Moufang loop P used in Conway's construction of the Fischer–Griess monster group [Co]. One curious feature of this loop is that all its character table entries are integral [J8, Prop. 5.3]. Each quasigroup Q described by Theorem 5.4 has the integers $\mathbb{Z}$ as its coefficient ring $\mathbb{Z}[Q]$. In fact [J8, Th. 5.4], THEOREM 5.5. *Suppose $|Q| = n > 2$, $|Q'| = m$, and that Q has a unique non-linear basic character. Then Q has integral coefficient ring if and only if one of the following two (mutually exclusive) conditions is satisfied:*

(a) *Q satisfies the conditions of Theorem 5.4;*

(b) *the Diophantine equation $p^2m(m-1) = n(m-2)^2$ has a positive integral solution p.* □

An immediate topic for subsequent research is to study quasigroups with a unique non-linear basic character from the standpoint of the cohomology theory for quasigroups given in [S3, Ch. 6]. Indeed, examination of the relationship between the character ring $R[Q]$ and the cohomology of Q is one of the major tasks facing the combinatorial character theory. There are many mysteries, even in the associative case.

The Quotient Theorem 4.22 relates the character tables $\Psi(Q)$ and $\Psi(P)$ when there is a quasigroup epimorphism $Q \twoheadrightarrow P$. Such an epimorphism induces an epimorphism $MltQ \twoheadrightarrow MltP$ of the corresponding multiplication groups. A dual situation arises when there are two quasigroup structures $(Q,+)$ and $(Q,.)$ on the same underlying set Q, such that $Mlt(Q,+)$ embeds as a subgroup of $Mlt(Q,.)$ with the monomorphism $Mlt(Q,+) \rightarrowtail Mlt(Q,.)$. The construction (3.20) of 3-piques provides many natural examples of this. Set $H = Mlt(Q,+)$. Suppose that $\{D_{ij} | 1 \le i \le s, 1 \le j \le r_i\}$ is the conjugacy class partition of $(Q,+)$, with $C_i = \bigcup_{j=1}^{r_i} D_{ij}$ for $i = 1, \dots, s$. The partition $\{\{D_{ij}|1 \le j \le r_i\}|1 \le i \le s\}$ and the corresponding partition of the columns of $\Psi(Q,+)$ are called the *G–fusion of H–classes* or the *$(Q,.)$–fusion of $(Q,+)$–classes*. Since $V(G,Q)$ is a subring of $V(H,Q)$, each idempotent e_i from (4.5) is a sum $e_i = \sum_{j=1}^{t_i} f_{ij}$ of idempotents of $V(H,Q) = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{t_i} \mathbb{C} f_{ij}$. Set $\Psi(Q,+) = (\psi^H_{ij,kl})_{1\le i\le s, 1\le j\le t_i; 1\le k\le s, 1\le l\le r_k}$. The partition $\{\{f_{ij}|1 \le j \le t_i\}|1 \le i \le s\}$ and the corresponding partition of the rows of $\Psi(Q,+)$ are called the *G–fusion of H–characters* or the *$(Q,.)$–fusion of $(Q,+)$–characters*. The dual of the Quotient Theorem 4.22, the Fusion Theorem [J6, Th.3.1] may then be formulated as follows.

THEOREM 5.6. *The character table $\Psi(Q,.)$ is determined by the character table $\Psi(Q,+)$ along with the $(Q,.)$–fusion of $(Q,+)$–classes and $(Q,+)$–characters.* □

The exact form of the determination is best stated geometrically. Consider the specification of the i-th basic character ψ^G_i of $(Q,.)$. In the fusion data, this character corresponds to the fusion of the t_i basic characters $\psi^H_{i1}, \dots, \psi^H_{it_i}$ of $(Q,+)$. For each conjugacy class D_{kl} of $(Q,+)$, there is a t_i–dimensional complex vector $\underline{w}_{kl} = [\psi^H_{i1,kl}, \dots, \psi_{it_k,kl}]$. These vectors may be taken to lie in $W = \mathbb{C}^{t_i}$. The subspace $W_0 = \mathbb{C}\underline{w}_{11}$ is called the *principal subspace*. An inner product $(\underline{x}|\underline{y}) = \underline{x}\ \underline{y}^*$ is given on W, where * denotes the conjugate transpose. The corresponding norm is $\|x\| = (\underline{x}|\underline{x})$. The character value ψ_{ik}^G is then given as the unique scalar λ minimising the expression $\|\underline{w}_{kl} - \lambda(\underline{w}_{11}/\|\underline{w}_{11}\|)\|$ for each $l = 1, \ldots, r_k$, namely [J6, Th. 4.1]

$$\psi_{ik}^G = (\underline{w}_{kl}|\underline{w}_{11})/\|\underline{w}_{11}\| \; . \tag{5.6}$$

Thus $\lambda \underline{w}_{11}/\|\underline{w}_{11}\|$ is the unique best approximation, in the principal subspace, to the vectors $\underline{w}_{kl}$. Another condition, very useful for the completion of partial fusion data, is the *magic rectangle condition* [J6, Th. 6.1]. This says that for fixed $i, k \in \{1, \ldots, s\}$, and then for each $l \in \{1, \ldots, t_i\}$, $l' \in \{1, \ldots, r_k\}$, the relation

$$\frac{\sum_{j=1}^{r_k} |D_{kj}| \psi_{il,kj}}{\sum_{j=1}^{r_k} |D_{kj}| \psi_{il,11}^H} = \frac{\sum_{j'=1}^{t_i} \psi_{ij',11} \; \psi_{ij',kl'}}{\sum_{j'=1}^{t_i} \psi_{ij',11}^H \; \psi_{ij',11}^H} \tag{5.7}$$

holds. As an example of its use, consider the problem of constructing the character table of the $\mathfrak{Z}$-pique $(\mathbb{Z}_5, -)$ from the character table of $(\mathbb{Z}_5, +)$. The $(\mathbb{Z}_5, -)$-fusion of $(\mathbb{Z}_5, +)$-classes is $\{\{0\}, \{\pm 1\}, \{\pm 2\}\}$. The condition (5.7) then fixes the $(\mathbb{Z}_5, -)$-fusion of $(\mathbb{Z}_5, +)$-characters, giving the character table of $(\mathbb{Z}_5, -)$ as

$$\begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & \sqrt{2} \cos \frac{2\pi}{5} & -\sqrt{2} \cos \frac{\pi}{5} \\ \sqrt{2} & -\sqrt{2} \cos \frac{\pi}{5} & \sqrt{2} \cos \frac{2\pi}{5} \end{bmatrix} \tag{5.8}$$

[J6, §7]. By [Ch,Th, III.5.9], the $\mathfrak{Z}$-quasigroups of prime order are the quasigroups of prime order whose multiplication groups are soluble. By Burnside's Theorem [Hu, Satz V.21.3] that a transitive permutation group of prime degree is either soluble or doubly transitive, all the prime order quasigroups that are not $\mathfrak{Z}$-quasigroups are rank 2 quasigroups. Since loops that are $\mathfrak{Z}$-quasigroups are abelian, loops of prime order are either abelian or rank 2. Thus character tables such as (5.8) with $2 < s < n$ for prime n cannot be loop character tables.

It is easy to recognise abelian quasigroups Q from their character tables, e.g. via $n = s$ or $\sigma(Q) = Q \times Q$ or $\forall \; 1 \leq i \leq s, \; n_i = 1$. On the other hand, the character table $\Psi(Q)$ of a quasigroup Q does not determine the centre congruence $\zeta(Q)$, or even whether Q is a $\mathfrak{Z}$-quasigroup or not. For example, a non-abelian loop of order 5 and the $\mathfrak{Z}$-pique $(\mathbb{Z}_5, ., 0)$ with $x.y = 2x + y$ share the same character table, both having rank 2. However, the character table $\Psi(Q^2)$ of the direct Q^2 of a quasigroup Q does determine whether Q is a $\mathfrak{Z}$-quasigroup or not [J7, Th. 3.1]. One implication is that the character table $\Psi(Q)$ of a quasigroup Q does not determine the character table $\Psi(Q^2)$. By contrast, the character table of a direct product $P \times Q$ of two loops P and Q is the tensor product $\Psi(P) \otimes \Psi(Q)$ [J7, Th. 2.1]. The problem of interpreting the tensor square $\Psi(Q)\otimes\Psi(Q)$ of a quasigroup character table $\Psi(Q)$ remains. One solution is provided by the concept of a "superscheme" [J7, §4].

An association scheme (Q,Γ) on an underlying finite non-empty set Q is a partition Γ of Q^2 satisfying (4.4). A *superscheme* (Q,Γ^*) on an underlying finite non-empty set Q is a partition $\Gamma^n = \{C_1^n,\dots,C_{s_n}^n\}$ of Q^{n+2}, for each natural number n, such that

$$
(5.9)\quad \begin{cases}
(S1) & C_1^0 = \hat{Q}\ ; \\
(S2) & \forall f:\{1,\dots,m+2\}\to\{1,\dots,n+2\},\ \forall\, C_j^n\in\Gamma^n\ , f^*(C_j^n) = \\
 & \{(x_1,\dots,x_{m+2})\,|\,\exists(y_1,\dots,y_{n+2})\in C_j^n.\ \forall\, 1\le i\le m+2, x_i=y_{if}\} \\
(S3) & \forall\, m\in\mathbb{N},\ \forall\, n\in\mathbb{N},\ \forall\, C_i^m\in\Gamma^m,\ \forall\, C_j^n\in\Gamma^n,\ \forall\, C_k^{m+n}\in\Gamma^{m+n}\ , \\
 & \exists c(i,j,k;m,n)\in\mathbb{N}.\ \forall(x_0,\dots,x_m,y_0,\dots,y_n)\in C_k^{m+n}\ , \\
 & \left|\{z\in Q\,|\,(x_0,\dots,x_m,z)\in C_i^m,(z,y_0,\dots,y_n)\in C_j^n\}\right| \\
 & = c(i,j,k;m,n),\ \text{and} \\
(S4) & \forall\, 1\le i,j,k\le s_0,\quad c(i,j,k;0,0)=c(j,i,k;0,0)\ .
\end{cases}
$$

In particular, (5.9) implies that (Q,Γ^0) is an association scheme, the *associated scheme* of the superscheme. Note how (5.9)(S2) for the bijection $f : 1\longmapsto 2$, $2\longmapsto 1$ reduces to (4.4) (A2). The exponential generating function

$$
(5.10)\qquad f(x) = 1 + x + \sum_{n=0}^{\infty} s_n \frac{x^{n+2}}{(n+2)!}
$$

is called the *augmented Poincaré series* of (Q,Γ). If G is a multiplicity–free transitive permutation group with permutation character π acting on Q (e.g. the multiplication group of a quasigroup Q), then taking Γ^n to be the set of orbits of G in its diagonal action on Q^{n+2} makes (Q,Γ^*) a superscheme, with augmented Poincaré series given by

$$
(5.11)\qquad f(x) = \frac{1}{|G|}\sum_{g\in G} e^{x\pi(g)}\ .
$$

For each natural number n, take a complex vector space $\mathbb{C}\Gamma^n$ with basis Γ^n. Set $\mathbb{C}\Gamma = \bigoplus_{n\in\mathbb{N}}\mathbb{C}\Gamma^n$, the complex vector space direct sum of the $\mathbb{C}\Gamma^n$. Then $\mathbb{C}\Gamma$ carries an algebra structure defined by

$$
(5.12)\qquad C_i^m\, C_j^n = \sum_{k=1}^{s_{m+n}} c(i,j,k;m,n)C_k^{m+n}\ ,
$$

called the *Bose–Mesner superalgebra* of the superscheme (Q,Γ). This algebra is associative [J7, Th. 5.1] and graded: $\mathbb{C}\Gamma^m.C\Gamma^n \le C\Gamma^{m+n}$. The subspace $\mathbb{C}\Gamma^n$ is called the *homogeneous component of degree n*. The homogeneous component of degree 0 is the Bose–Mesner algebra of the associated scheme (Q,Γ°). Each homogeneous component $\mathbf{C}\Gamma^n$ is a bimodule for this Bose–Mesner algebra, under left multiplication

$$L : \mathbf{C}\Gamma^\circ \longrightarrow \mathrm{End}_{\mathbf{C}}\,\mathbf{C}\Gamma^n ; x \longmapsto (y \longmapsto xy) \tag{5.13}$$

and right multiplication

$$R : \mathbf{C}\Gamma^\circ \longrightarrow \mathrm{End}_{\mathbf{C}}\,\mathbf{C}\Gamma^n ; x \longmapsto (y \longmapsto yx)\ . \tag{5.14}$$

The crucial result [J7, Th. 6.2] is

THEOREM 5.15. *The algebra homomorphism* $L \otimes R : \mathbf{C}\Gamma^\circ \otimes \mathbf{C}\Gamma^\circ \longrightarrow \mathrm{End}_{\mathbf{C}}\,\mathbf{C}\Gamma^1$ *embeds the tensor square* $\mathbf{C}\Gamma^\circ \otimes \mathbf{C}\Gamma^\circ$ *as a commutative subalgebra of the endomorphism ring* $\mathrm{End}_{\mathbf{C}}\,\mathbf{C}\Gamma^1$. □

Considering the case of the multiplication group G acting on the quasigroup Q, the character table $\Psi(Q)$ is obtained via (4.9) from the relationships (4.5)–(4.7) in the commutative algebra $\mathbf{C}\Gamma^\circ = V(G, Q)$. Theorem 5.15 enables one to obtain the tensor square $\Psi(Q) \otimes \Psi(Q)$ in a similar way from the commutative subalgebra $\mathbf{C}\Gamma^\circ \otimes \mathbf{C}\Gamma^\circ$ of $\mathrm{End}_{\mathbf{C}}\,\mathbf{C}\Gamma'$. In other words [J7, Th. 7.1],

THEOREM 5.16. *The tensor square* $\Psi(Q)\otimes\Psi(Q)$ *is determined by the two-sided action of* $V(G, Q)$ *on the orbits of* G *on* Q^3. □

A recent approach [S6] to the problem of locating quasigroups within the abelian/𝔖/non-𝔖 trichotomy has used two numerical invariants, the "entropy" and "asymptotic entropy". The *entropy* $H(Q)$ of the quasigroup Q is defined to be

$$H(Q) = \sum_{i=1}^{s} \frac{n_i}{n} \log \frac{n}{n_i} \quad , \tag{5.17}$$

logarithms being taken to a fixed base. This base is usually 2, in which case entropy has the units of *bits*.

PROPOSITION 5.18 [S6, PROP. 1.4]. *The entropy* $H(Q)$ *of a quasigroup* Q *of finite positive order* n *satisfies*

$$\log n - (1 - n^{-1}) \log(n - 1) \leq H(Q) \leq \log n\ . \tag{5.19}$$

Equality obtains on the left if and only if Q *has rank 2. Equality obtains on the right if and only if* Q *is abelian.* □

The *asymptotic entropy* $h(Q)$ is defined to be

$$h(Q) = \limsup_{m \to \infty} \frac{1}{m} H(Q^m)\ . \tag{5.20}$$

THEOREM 5.21 [S6; (1.5), TH. 3.1]. *The asymptotic entropy* $h(Q)$ *of a quasigroup* Q *of finite order* n *satisfies*

$$0 \leq h(Q) \leq \log n\ . \tag{5.21}$$

Equality obtains on the right if and only if Q *is a 𝔖–quasigroup.* □

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