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CONFORMAL ALGEBRAS, VERTEX ALGEBRAS, AND THE LOGIC OF LOCALITY

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Dedicated to Professor Anatolij Dvurečenskij on the occasion of his 65th birthday

(*Communicated by Vincenzo Marra*)

ABSTRACT. In a new algebraic approach, conformal algebras and vertex algebras are extended to two-sorted structures, with an additional component encoding the logical properties of locality. Within these algebras, locality is expressed as an identity, without the need for existential quantifiers. Two-sorted conformal algebras form a variety of two-sorted algebras, an equationally-defined class, and free conformal algebras are given by standard universal algebraic constructions. The variety of two-sorted conformal algebras is equivalent to a Mal'tsev variety of single-sorted algebras. Motivated by a question of Griess, subalgebras of reducts of conformal algebras are shown to satisfy a set of quasi-identities. The class of two-sorted vertex algebras does not form a variety, so open problems concerning the nature of that class are posed.

1. Introduction

In a quantum field theory, the fields Φ_a (with a as an index) are described as operator-valued distributions on the underlying space-time M. Wightman's *locality axiom* states that if two test functions f, h on the space-time M have supports that are space-like separated, then $\Phi_a(f)\Phi_b(h) = \Phi_b(h)\Phi_a(f)$ for fields Φ_a, Φ_b [14: §1.1]. Note that the logic underlying this axiom is "crisp" — a classical two-valued logic: It is either true or false that the given test functions are space-like separated.

The right chiral part of a two-dimensional *conformal* [14: (1.1.5)] quantum field theory[1] is modeled by a *vertex algebra* V (§3), equipped with binary multiplications $\underline{n}$ (for $n \in \mathbb{Z}$). In this algebra, the nonnegativity of the conformal weights implies that Wightman's locality axiom is expressed as the *locality condition*

$$(\forall\, a, b \in V)(\exists\, N \in \mathbb{N})(\forall\, n \in \mathbb{N})[n \geq N \implies a\,\underline{n}\,b = 0] \tag{1.1}$$

on the multiplications [14: §1.2]. A similar condition appears in the *conformal algebras* (§2) that model a key algebraic fragment of vertex algebras. As noted early in the study of vertex algebras by Borcherds [6: §4], the existential quantifier that appears in (1.1) is an obstacle to the construction of free vertex and conformal algebras. By combinatorial methods, Borcherds [*op. cit.*] and Roitman [18, 19] considered vertex and conformal algebras that were universal subject to certain choices of the integer N in (1.1).

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[1] Note that the term "conformal" is often given a different meaning in the physics literature — compare [10: §3.6].

The intention of the present paper is to establish a new, more strictly algebraic approach to conformal (§4) and vertex algebras (§6). In this approach, the algebras are reformulated as two-sorted or heterogeneous algebras. (Compare [3, 13, 15]. In the operadic incarnation of universal algebra, heterogeneous algebras are described as *colored* [4]). The first sort of the two-sorted algebras corresponds respectively to conformal or vertex algebras as normally understood, while the second sort is a lattice-ordered, commutative and unital ring known as the *locality ring*. If this ring is the ring of integers, then the two-sorted conformal or vertex algebras correspond respectively to usual conformal (Theorem 4.3) or vertex (Theorem 6.7) algebras. In a general two-sorted conformal or vertex algebra, the locality ring is an f-ring or *function ring* as defined by Birkhoff [2: § XVII.5]. These rings form a variety or equationally-defined class, and two-sorted conformal algebras form a variety of two-sorted algebras that includes the variety of commutative, unital f-rings. Thus the construction of free conformal algebras is reduced to a standard issue of universal algebra (Theorem 4.4, Corollary 4.5). Moreover, the variety of two-sorted conformal algebras is equivalent to a Mal'tsev variety of single-sorted algebras, the *equational conformal algebras* (Theorem 5.3, Proposition 5.4). For these algebras, standard module, structure and cohomology theories are available [22, 23]. Motivated by a question of Griess, Theorem 5.6 shows that subreducts of conformal algebras satisfy certain quasi-identities.

For two-sorted vertex algebras, the situation is subtler than for two-sorted conformal algebras. The locality ring of a two-sorted vertex algebra has the multiplicative identity 1 as a strong (order) unit (§6). Now the class of such commutative f-rings does not form a variety, and as a consequence, the class of two-sorted vertex algebras does not form a variety either (Theorem 6.8). Nevertheless, the category of commutative, unital f-rings with 1 as a strong unit is equivalent to a variety of algebras, the so-called *PMV-algebras*, that embody the truth values of certain "fuzzy" or multiple-valued logics (Theorem 7.4). A PMV-algebra appears as the unit interval in the locality ring of a two-sorted vertex algebra. In the ring of integers, the unit interval just contains 0 and 1 or "false" and "true" — the truth values of the classical logic of Wightman's locality axiom. The algebraic considerations of this paper thus suggest the relevance of a general two-sorted vertex algebra in a quantum field theory with exotic space-time geometry, for which the logic of space-like separation would be nonclassical, "fuzzy" or multiple-valued.

Algebraic notation and concepts that are not explicitly described or referenced in the paper will follow the usage of [21, 24].

2. Conformal algebras

The definition of a conformal algebra given below essentially follows Roitman [18: Defn. 1.1], with a minor distinction that is discussed in Remark 2.3.

Definition 2.1. Let C be an abelian group that is equipped with an endomorphism

$$C \to C; \quad a \mapsto a' \tag{2.1}$$

and a bilinear operation or *multiplication*

$$C^2 \to C; \quad (a, b) \mapsto a \,\underline{n}\, b \tag{2.2}$$

for each integer n.

(a) Suppose that the endomorphism (2.1) of the group C is a derivation

$$\big(a \,\underline{n}\, b\big)' = a' \,\underline{n}\, b + a \,\underline{n}\, b' \tag{2.3}$$

for each multiplication $\underline{n}$, while successive multiplications are connected by the identity

$$a' \,\underline{n}\, b = (-na) \,\underline{n-1}\, b\,. \tag{2.4}$$

Then C is said to be a *pre-conformal algebra* $(C, +, \underline{\mathbb{Z}}, ')$.

(b) If the products (2.2) are *local* in the sense that

$$(\forall\, a,b\in C)(\exists\, N\in\mathbb{N})(\forall\, n\in\mathbb{N})[n\geq N \implies a\,\underline{n}\,b=0], \tag{2.5}$$

then C is said to be a *conformal algebra*.

Remark 2.2. Note that pre-conformal algebras, being entirely defined by (2.3), (2.4), the identities for abelian groups, the endomorphism property of $'$, and the bilinearity of the multiplications, do form a variety of universal algebras.

Remark 2.3. Roitman [18: Defn. 1.1] only requires the multiplications of a conformal algebra to be defined for natural numbers. A conformal algebra in that sense yields a conformal algebra in the current sense of Definition 2.1 on defining $a\,\underline{n}\,b=0$ for negative integers n. Note that (2.3) and (2.4) are then trivially satisfied for negative n, while (2.4) still holds for $n=0$.

***Example* 2.4.** Let A be an *algebra* or *bilinear algebra*,[2] an abelian group equipped with a bilinear multiplication $A^2\to A$; $(x,y)\mapsto xy$. Let $A^{\mathbb{Z}}$ denote the set of functions $a\colon \mathbb{Z}\to A$; $n\mapsto a_n$ from $\mathbb{Z}$ to A. The functions are often encoded as formal power series $a(z)=\sum_{n\in\mathbb{Z}} a_n z^{-n-1}$. Define the formal derivative

$$'\colon A^{\mathbb{Z}}\to A^{\mathbb{Z}};\quad a(z)\mapsto a'(z)$$

and the formal residue (coefficient of w^{-1})

$$(a\,\underline{m}\,b)(z)=\mathrm{Res}_w\,\{a(w)b(z)(z-w)^m\}$$

for each natural number m. In other words, for $a,b\in A^{\mathbb{Z}}$, $m\in\mathbb{N}$, and $n\in\mathbb{Z}$, one defines

$$a'_n=-na_{n-1}$$

and

$$(a\,\underline{m}\,b)_n=\sum_{s=0}^{m}(-1)^s\binom{m}{s}a_{m-s}b_{n+s}\,.$$

Then $A^{\mathbb{Z}}$ is a pre-conformal algebra, the *power-series* (*pre-conformal*) *algebra* $\left(A^{\mathbb{Z}},+,\underline{\mathbb{Z}},'\right)$ of the algebra A [18: §1.2.1].

Let C be a conformal algebra. For each integer n, take a copy C_n of the abelian group C given by the isomorphism $C\to C_n$; $a\mapsto a_n$. Given the direct sum $\bigoplus_{n\in\mathbb{Z}} C_n$, consider the subgroup E generated by

$$\{a'_n+na_{n-1}\mid a\in C,\;\; n\in\mathbb{Z}\}\,,$$

the quotient

$$\widehat{C}=\bigoplus_{n\in\mathbb{Z}}C_n\Big/E\,,$$

and the subgroup

$$\widehat{C}_+=\Big(E+\bigoplus_{n\in\mathbb{N}}C_n\Big)\Big/E$$

of the quotient.

Proposition 2.5. ([18: §1.3]) *Let C be a conformal algebra. For $a,b\in C$ and $m,n\in\mathbb{Z}$, the formula*

$$(a_m+E)\cdot(b_n+E)=E+\sum_{s=0}^{N(a,b)-1}\binom{m}{s}(a\,\underline{s}\,b)_{m-s+n} \tag{2.6}$$

induces a well-defined bilinear algebra structure $\big(\widehat{C},+,\cdot\big)$. The subgroup $\widehat{C}_+$ is a subalgebra of $\big(\widehat{C},+,\cdot\big)$.

[2] To distinguish from universal algebras.

The (possibly non-associative) algebra $\big(\widehat{C}, +, \cdot\big)$ of Proposition 2.5 is known as the *coefficient algebra* of the conformal algebra C. Let $\underline{\underline{V}}$ be a variety of bilinear algebras (for example, associative or Lie algebras). Define $\underline{\underline{\widehat{V}}}$ to be the class of all conformal algebras C such that the coefficient algebra $\widehat{C}$ lies in $\underline{\underline{V}}$. Elements of $\underline{\underline{\widehat{V}}}$ are described as $\underline{\underline{V}}$*-conformal algebras* [18: §1.6].

3. Vertex algebras

Many different, more or less equivalent axiomatizations of vertex algebras have appeared in the literature. For current purposes, it is best to take the following reformulation of the axioms used by Roitman [20: §2.1], avoiding questions of grading (over $\mathbb{Z}$, $\mathbb{Z}/_2$, or otherwise), Virasoro structure, or negative locality:

Definition 3.1. Let V be an abelian group equipped with a bilinear operation or *multiplication*

$$V^2 \to V; \quad (a,b) \mapsto a\,\underline{n}\,b \tag{3.1}$$

for each integer n. The group V is known as the *state space*, and its elements are known as *states*. The products (3.1) are *local* in the sense that

$$(\forall\, a,b \in V)(\exists\, N \in \mathbb{N})(\forall\, n \in \mathbb{N})[n \geq N \implies a\,\underline{n}\,b = 0] \tag{3.2}$$

A constant member v_0 of V, the *vacuum* (*state*), is selected. The vacuum is a unit for the -1 multiplication,

$$v_0\,\underline{-1}\,a = a = a\,\underline{-1}\,v_0\,, \tag{3.3}$$

and a left zero for the other multiplications:

$$v_0\,\underline{n}\,a = 0 \tag{3.4}$$

for $n \neq -1$ and for each state a. The vacuum is a right zero,

$$a\,\underline{n}\,v_0 = 0\,, \tag{3.5}$$

for $n \geq 0$. Consider right multiplication

$$R_{\underline{-2}}(v_0)\colon V \to V; \quad a \mapsto a\,\underline{-2}\,v_0$$

by the vacuum under the product $\underline{-2}$. The endomorphism $R_{\underline{-2}}(v_0)$ of the group V is a derivation for each multiplication:

$$\big(a\,\underline{n}\,b\big)\,\underline{-2}\,v_0 = \big(a\,\underline{-2}\,v_0\big)\,\underline{n}\,b + a\,\underline{n}\,\big(b\,\underline{-2}\,v_0\big)\,. \tag{3.6}$$

Moreover, successive multiplications are connected by the identity

$$\big(a\,\underline{-2}\,v_0\big)\,\underline{n}\,b = (-na)\,\underline{n-1}\,b\,. \tag{3.7}$$

Finally, suppose that

$$\big(a\,\underline{m}\,b\big)\,\underline{n}\,c - b\,\underline{n}\,\big(a\,\underline{m}\,c\big) = \sum_{s\in\mathbb{N}} \binom{m}{s} \big(a\,\underline{s}\,b\big)\,\underline{m-s+n}\,c \tag{3.8}$$

for integers m and n. Then V is said to be a *vertex algebra*.

Remark 3.2. Roitman also invokes an identity [20: §2.1, (V2)] which may be written in abelian group terms as

$$n!\big(a\,\underline{-n-1}\,v_0\big) = aR_{\underline{-2}}(v_0)^n \tag{3.9}$$

for natural numbers n. However, this identity follows by induction on n using (3.3) and (3.7) — compare [14: Remark 1.3].

Remark 3.3. The sum on the right hand side of (3.8) is finite, since the products $a\,\underline{s}\,b$ are zero for large s by virtue of the locality condition.

Remark 3.4. Comparing (2.2)–(2.5) with (3.1)–(3.2) and (2.3)–(2.4) with (3.6)–(3.7), it is apparent that each vertex algebra forms a conformal algebra with $a' = a\,\underline{-2}\,v_0$.

4. Two-sorted conformal algebras

In a unital ring R with multiplicative identity 1, write $n = n1$ for each integer n, and refer to these elements $n1$ of R as *integers*. Define a *commutative f-ring* to be a commutative lattice-ordered ring satisfying the identity

$$(k \vee 0) \wedge \big((-k \vee 0)(l \vee 0)\big) = 0$$

— compare [2: Lemma XVII.5.3]. Examples of such rings are given by the ring of integers, and by rings $C(X)$ of continuous, real-valued functions on a topological space X.

Definition 4.1. Let (C, L) be a pair consisting of an abelian group C and a commutative, unital f-ring L. The elements of C are described as *states*, while the ring L is known as the *locality ring*. Suppose that C is equipped with an abelian group endomorphism

$$C \to C; \quad a \mapsto a' \tag{4.1}$$

and a bilinear operation or *multiplication*

$$C^2 \to C; \quad (a, b) \mapsto a\,\underline{l}\,b \tag{4.2}$$

for each element l of L. There is a *locality operation*

$$N\colon C^2 \to L; \quad (a, b) \mapsto N(a, b) \tag{4.3}$$

with

$$N(a, 0) = N(0, b) = 0 \tag{4.4}$$

for $a, b \in C$ such that the *nonnegativity identity*

$$0 \vee N(a, b) = N(a, b) \tag{4.5}$$

is satisfied in the lattice $(L, \vee, \wedge)$. The multiplications are required to satisfy the *locality identity*

$$a\,\underline{k \vee N(a, b)}\,b = 0 \tag{4.6}$$

for k in L. The endomorphism (4.1) of the group C is a derivation for each integral multiplication:

$$\big(a\,\underline{n}\,b\big)' = a'\,\underline{n}\,b + a\,\underline{n}\,b'\,, \tag{4.7}$$

while successive integral multiplications are connected by the identity

$$a'\,\underline{n}\,b = (-na)\,\underline{n-1}\,b \tag{4.8}$$

for $n \in \mathbb{Z}1$. Then (C, L) is said to be a *two-sorted conformal algebra*.

By [2: Th. XVII.8], the class of f-rings forms a variety or equationally-defined class. Since the conditions (4.4)–(4.8) of Definition 4.1 are all identities, the class of two-sorted conformal algebras forms a variety of two-sorted algebras. This class includes all commutative unital f-rings:

Proposition 4.2. *If $(\{0\}, L)$ is a two-sorted conformal algebra, then L is a commutative unital f-ring. Conversely, each commutative unital f-ring L may be construed as a two-sorted conformal algebra $(\{0\}, L)$.*

Conformal algebras are most directly related to two-sorted conformal algebras as follows:

Theorem 4.3. *If $(C, \mathbb{Z})$ is a two-sorted conformal algebra, then C is a conformal algebra. Conversely, each conformal algebra C may be construed as a two-sorted conformal algebra $(C, \mathbb{Z})$.*

P r o o f. First note that $\mathbb{Z}$ is a commutative f-ring. Then if $(C, \mathbb{Z})$ is a two-sorted conformal algebra, the locality condition (2.5) must be verified. But given states a and b, and an integer n with $n \geq N(a,b)$, one has $n = n \vee N(a,b)$, so $a\,\underline{n}\,b = 0$ follows from (4.6). For the converse direction, consider a pair of states a and b. Using (2.5), define the value $N(a,b)$ of the locality operation to be the least nonnegative integer N for which $a\,\underline{N}\,b = 0$. Then for each integer k, one has $k \vee N(a,b) \geq N$, so the locality identity (4.5) follows from the locality condition (2.5). $\square$

The final part of this section is concerned with free algebras. Let X be a set. For each function $N\colon X \times X \to \mathbb{N}$, let $\mathcal{C}(N)$ denote the class of conformal algebras generated by X such that the restricted locality condition

$$(\forall\, x_1, x_2 \in X)(\forall\, n \in \mathbb{N})[n \geq N(x_1, x_2) \implies x_1\,\underline{n}\,x_2 = 0]$$

is satisfied. Consider $\mathcal{C}(N)$ as a category in which the morphisms are all the homomorphisms between algebras from the class $\mathcal{C}(N)$ that fix the generating set X. Roitman defines the initial object $C(N)$ of $\mathcal{C}(N)$ as the *free conformal algebra* determined by the locality function N, and gives a combinatorial construction for it [18: §3].

By Theorem 4.3, the conformal algebra $C(N)$ corresponds to a two-sorted conformal algebra $\big(C(N), \mathbb{Z}\big)$. Consider the free two-sorted conformal algebra (C_X, L_X) over the pair (X, X^2) [15: II.§2.7, (9)b]. The freeness property of (C_X, L_X) then yields the following result:

Theorem 4.4. *For each set X and function $N\colon X \times X \to \mathbb{N}$, there is a unique homomorphism*

$$(\varepsilon, \nu)\colon (C_X, L_X) \to \big(C(N), \mathbb{Z}\big) \tag{4.9}$$

of two-sorted conformal algebras such that ε restricts to the identity on X and ν restricts to N on X^2.

In view of the universality property in Theorem 4.4, (C_X, L_X) may be described as the *free two-sorted conformal algebra* over the set X. Roitman's free conformal algebras $C(N)$ are then given as follows:

Corollary 4.5. *The algebra $\big(C(N), \mathbb{Z}\big)$ is obtained as the quotient of the free algebra (C_X, L_X) by the least congruence containing the pairs*

$$\Big((x_1, x_1), \big((x_1, x_2), N(x_1, x_2)\big)\Big)$$

for $x_1, x_2 \in X$.

P r o o f. Apply the First Isomorphism Theorem [15: II.2.4(5)] to (ε, ν) in (4.9), using the fact that both ε and ν surject. $\square$

5. Equational conformal algebras

By a result of Barr [1: Th. 5], reformulated by Goguen and Meseguer [11: p. 331], the category of two-sorted conformal algebras is equivalent to a variety of single-sorted algebras. (Such an equivalence holds for any variety of heterogeneous algebras with the property that, whenever one sort is empty, then all the sorts are empty.) The single-sorted equivalents of two-sorted conformal algebras may be defined explicitly as follows.

Definition 5.1. Let E or $(E,+,\circ,\cdot,\vee,\wedge,1)$ be an abelian group $(E,+)$ equipped with a constant element 1, an endomorphism

$$E \to E; \quad x \mapsto x' \tag{5.1}$$

of $(E,+,\circ)$, and a bilinear operation or *multiplication*

$$E^2 \to E; \quad (x,y) \mapsto x\,\underline{z}\,y \tag{5.2}$$

for each z in E. Suppose that there is a binary *locality operation*

$$\nu\colon E^2 \to E; \quad (x,y) \mapsto xy\nu$$

with

$$x0\nu = 0y\nu = 0$$

satisfying the *nonnegativity identity*

$$xy\nu \vee 0 = xy\nu$$

and the *locality identity*

$$x\,\underline{xy\nu \vee z}\,y = 0\,.$$

For $n \in \mathbb{Z}$, write $n = n1$, and refer to these elements of E as *integers*. Suppose that the endomorphism (5.1) is a derivation for each integral multiplication:

$$\big(x\,\underline{n}\,y\big)' = x'\,\underline{n}\,y + x\,\underline{n}\,y'\,, \tag{5.3}$$

and that the integral multiplications are connected by the identity

$$x'\,\underline{n}\,y = (-nx)\,\underline{n-1}\,y \tag{5.4}$$

for $n \in \mathbb{Z}1$. Suppose that $(E,+,\circ,\vee,\wedge)$ and $(E,+,\cdot,\vee,\wedge)$ are both nonunital lattice-ordered rings, for which the *entropic identities*

$$(x\circ y)\vee(z\circ t) = (x\vee z)\circ(y\vee t) \quad \text{and} \quad (x\circ y)\cdot(z\circ t) = (x\cdot z)\circ(y\cdot t)$$

and the respective *rectangular band* and *commutative* identities

$$x\circ z\circ y = x\circ y \quad \text{and} \quad x\cdot y = y\cdot x$$

are satisfied. For each element x of E, define the *projections*

$$x^\kappa = x\circ 0 \quad \text{and} \quad x^\lambda = 0\circ x\,.$$

Suppose that

$$\big(x\,\underline{z}\,y\big)^\kappa = x^\kappa\,\underline{z^\lambda}\,y^\kappa \quad \text{and} \quad xy\nu^\lambda = x^\kappa y^\kappa \nu^\lambda \tag{5.5}$$

for $x,y,z \in E$, while $\big(E^\lambda,+,\cdot,\vee,\wedge\big)$ is an f-ring in which 1^λ is a multiplicative unit. (Note that these requirements are equational.) Then E is said to be an *equational conformal algebra*.

Remark 5.2. Suppose that E is an equational conformal algebra. Then the two distributive laws in the ring $(E,+,\circ)$ imply that the projection mappings $\kappa\colon E\to E$ and $\lambda\colon E\to E$ are both abelian group homomorphisms, so their respective images E^κ and E^λ inherit an abelian group structure from E. Similarly, the first of the entropic identities makes $\lambda\colon (E,\vee)\to(E,\vee)$ a semilattice homomorphism.

The actual equivalence between two-sorted and equational conformal algebras may be summarized by the following theorem.

Theorem 5.3. *Let (C, L) be a two-sorted conformal algebra. On the abelian group $C \oplus L$, define the endomorphism $(c, l)' = (c', 0)$, the constant $1 = (0, 1)$, the products*

$$(c_1, l_1) \circ (c_2, l_2) = (c_1, l_2) \quad \text{and} \quad (c_1, l_1) \cdot (c_2, l_2) = (0, l_1 l_2)\,,$$

the multiplications

$$(c_1, l_1)\,\underline{(c_3, l_3)}\,(c_2, l_2) = \big(c_1\,\underline{l_3}\,c_2, 0\big)\,,$$

the lattice operations

$$(c_1, l_1) \vee (c_2, l_2) = (0, l_1 \vee l_2) \quad \text{and} \quad (c_1, l_1) \wedge (c_2, l_2) = (0, l_1 \wedge l_2)\,,$$

and the locality operation

$$(c_1, l_1)(c_2, l_2)\nu = \big(0, N(c_1, c_2)\big)\,.$$

Then $C \oplus L$ is an equational conformal algebra with

$$(C, L) \cong \big((C \oplus L)^\kappa, (C \oplus L)^\lambda\big)\,. \tag{5.6}$$

Conversely, if E is an equational conformal algebra, then $\big(E^\kappa, E^\lambda\big)$ is a two-sorted conformal algebra with $E \cong E^\kappa \oplus E^\lambda$.

P r o o f. In the forward direction, the proof merely consists of routine computations. As a sample, note the verification of the lattice ring identity [2: Th. XVII.2] for $(E, +, \circ, \vee, \wedge)$:

$$\begin{aligned}\Big\{\big[(c_1, l_1) \vee (0, 0)\big] \circ \big[(c_2, l_2) \vee (0, 0)\big]\Big\} \wedge (0, 0) &= \big[(0, l_1 \vee 0) \circ (0, l_2 \vee 0)\big] \wedge (0, 0)\\ &= (0, l_2 \vee 0) \wedge (0, 0) = \big(0, (l_2 \vee 0) \wedge 0\big) = (0, 0)\,,\end{aligned}$$

where the final equality holds by absorption in the lattice $(L, \vee, \wedge)$.

For the converse direction, let E be an equational conformal algebra. Definition 5.1 directly implies that $L = E^\lambda$ is an f-ring with 1^λ as a multiplicative unit. Define $C = E^\kappa$. Since (5.1) is an endomorphism of the ring $(E, +, \circ)$, it commutes with κ and restricts to an endomorphism of the abelian group $(C, +)$. The first identity of (5.5) implies that for each element l of L, there is a well-defined binary operation

$$\underline{l}\colon C^2 \to C; \quad (a, b) \mapsto a\,\underline{l}\,b\,.$$

The bilinearity of the multiplications (5.2), together with the fact that κ is a group homomorphism, imply that the multiplications $\underline{l}$ are bilinear on C. For $n \in \mathbb{Z}$, (5.5) and the fact that λ is a group homomorphism imply

$$x^\kappa\,\underline{n}\,y^\kappa = x^\kappa\,\underline{n1^\lambda}\,y^\kappa = x^\kappa\,\underline{(n1)^\lambda}\,y^\kappa = \big(x\,\underline{n1}\,y\big)^\kappa = \big(x\,\underline{n}\,y\big)^\kappa \tag{5.7}$$

for $x, y \in E$, so the integral multiplications in C correspond to the integral multiplications in E. Furthermore, one has $\big(x^\kappa\,\underline{n}\,y^\kappa\big)' = \big(x\,\underline{n}\,y\big)^{\kappa\,\prime} = \big(x\,\underline{n}\,y\big)^{\prime\,\kappa} = \big[(x'\,\underline{n}\,y) + (x\,\underline{n}\,y')\big]^\kappa = (x'\,\underline{n}\,y)^\kappa + (x\,\underline{n}\,y')^\kappa = x'^\kappa\,\underline{n}\,y^\kappa + x^\kappa\,\underline{n}\,y'^\kappa = x^{\kappa\,\prime}\,\underline{n}\,y^\kappa + x^\kappa\,\underline{n}\,y^{\kappa\,\prime}$ using (5.7), (5.3), and the endomorphism property of (5.1), so that C satisfies (4.7). Similarly, (5.4) implies $x^{\kappa\,\prime}\,\underline{n}\,y^\kappa = (x'\,\underline{n}\,y)^\kappa = \big[(-nx)\,\underline{n-1}\,y\big]^\kappa = (-nx)^\kappa\,\underline{n-1}\,y^\kappa = (-nx^\kappa)\,\underline{n-1}\,y^\kappa$, so C satisfies (4.8).

The nest step is to verify the locality properties of (C, L). The second identity of (5.5) implies that there is a well-defined binary operation

$$N\colon C^2 \to L; \quad (x^\kappa, y^\kappa) \mapsto xy\nu^\lambda\,.$$

Note that $N(x^\kappa, 0) = x0\nu^\lambda = 0 = 0y\nu^\lambda = N(0, y^\kappa)$. Then $0 \vee N(x^\kappa, y^\kappa) = 0^\lambda \vee xy\nu^\lambda = (xy\nu \vee 0)^\lambda = xy\nu^\lambda = N(x^\kappa, y^\kappa)$ for $x, y \in E$, verifying the nonnegativity identity (4.5). Similarly, $x^\kappa\,\underline{z^\lambda \vee N(x^\kappa, y^\kappa)}\,y^\kappa = x^\kappa\,\underline{(z \vee xy\nu)^\lambda}\,y^\kappa = \big(x\,\underline{xy\nu \vee z}\,y\big)^\kappa = 0^\kappa = 0$ for $x, y, z \in E$, verifying the locality identity (4.6). Thus (C, L) is a two-sorted conformal algebra.

In conclusion, note that the isomorphism (5.6) is given (naturally) by

$$(C,L) \to \big((C \oplus L)^{\kappa}, (C \oplus L)^{\lambda}\big); \quad (c,l) \mapsto \big((c,0),(0,l)\big)\,,$$

while the isomorphism $E \cong E^{\kappa} \oplus E^{\lambda}$ is given by $x \mapsto (x \circ 0, 0 \circ x)$. □

The next result witnesses that the variety of equational conformal algebras belongs to a class of varieties, the Mal'tsev varieties, which have very well-behaved properties, including readily available module, structure and cohomology theories [22, 23].

Proposition 5.4. *The class of equational conformal algebras forms a Mal'tsev variety.*

Proof. It suffices to take the Mal'tsev operation

$$(x, y, z)P = x - y + z$$

in the abelian group reduct of an equational conformal algebra. □

A *Mal'tsev category* is a finitely complete category in which each internal reflexive relation is an equivalence relation (compare [5, 7]).[3]

Corollary 5.5. *The category of two-sorted conformal algebras is a Mal'tsev category.*

Proof. By Theorem 5.3, the category of two-sorted conformal algebras is equivalent to the category of equational conformal algebras. By Proposition 5.4, this latter category is a Mal'tsev category. □

The final result of this section has its roots in an issue raised by Griess [12: **8**(3,4)]. Let $\mathcal{A}$ be the class of commutative algebras A that appear as subalgebras $(A, \underline{1}\,)$ of the $\underline{1}$-reduct $(V, \underline{1}\,)$ of a vertex algebra V, and thus as $\underline{1}$-subreducts of a conformal algebra V. Griess' questions may be loosely paraphrased as follows:

(1) Are there any special identities satisfied by algebras in $\mathcal{A}$?

(2) More generally, what special properties are possessed by the algebras in $\mathcal{A}$?

Roitman essentially gave a negative answer [20: Th. 1.1] to Griess' first question. The theorem below is wider in its scope, since it concerns general subreducts (subalgebras of reducts) of conformal algebras. Addressing Griess' second question in this broader context, it promises quasi-identities rather than identities.

Theorem 5.6. *Let $\mathcal{Q}$ be the class of algebras that are isomorphic to conformal algebra subreducts of a given type. Then algebras from $\mathcal{Q}$ satisfy a particular set of quasi-identities.*

Proof. Let A be an algebra from $\mathcal{Q}$. Then there is a conformal algebra reduct A' such that A is a subalgebra of A'. By Theorem 4.3, the full conformal algebra A' appears as the first sort of a two-sorted conformal algebra $(A', \mathbb{Z})$. According to Theorem 5.3, $A' \oplus \mathbb{Z}$ forms an equational conformal algebra. Since $A \cong A \oplus \{0\} \leq A' \oplus \mathbb{Z}$ (where the isomorphism and subalgebra relationship apply to the reduced type in question), the algebra A belongs to a class $\mathcal{Q}'$ of (algebras isomorphic to) equational conformal algebra subreducts of a given type. Equational conformal algebras form a variety. A result of Mal'tsev — [16: Cor. 11.1.5], compare [21: Cor. 3.7.16] — then shows that the algebras from $\mathcal{Q}'$, such as A, satisfy a particular set of quasi-identities. □

[3]Note the changing conventions for transliteration from Cyrillic.

6. Two-sorted vertex algebras

Definition 6.1. ([2: p. 300]) Let L be an ordered group. An element u of L is said to be a *strong* (*order*) *unit* if $(\forall\, l \in L)(\exists\, k \in \mathbb{N})[ku \geq l]$.

Definition 6.2. Suppose that (V, L) is a pair consisting of an abelian group V and a commutative, unital lattice-ordered ring L where the multiplicative identity 1 is a strong unit. The group V is known as the *state space*, and its elements are known as *states*. The ring L is known as the *locality ring*. For each element l of L, a bilinear operation of *multiplication*

$$V^2 \to V; \quad (a, b) \mapsto a\, \underline{l}\, b \tag{6.1}$$

is defined. There is a *locality operation*

$$N \colon V^2 \to L; \quad (a, b) \mapsto N(a, b) \tag{6.2}$$

with

$$N(a, 0) = N(0, b) = 0 \tag{6.3}$$

for $a, b \in V$ such that the nonnegativity identity

$$0 \vee N(a, b) = N(a, b) \tag{6.4}$$

is satisfied in the lattice group L. The multiplications are required to satisfy the *locality identity*

$$a\, \underline{k \vee N(a, b)}\, b = 0 \tag{6.5}$$

for k in L. A constant member v_0 of V, the *vacuum* (*state*), is selected. The vacuum is a unit for the -1 multiplication,

$$v_0\, \underline{-1}\, a = a = a\, \underline{-1}\, v_0\,, \tag{6.6}$$

and a left zero for the other integral multiplications:

$$v_0\, \underline{n}\, a = 0 \tag{6.7}$$

for $-1 \neq n \in \mathbb{Z}.1$ and for each state a. The vacuum is a right zero,

$$a\, \underline{n}\, v_0 = 0\,, \tag{6.8}$$

for $n \in \mathbb{N}.1$. The endomorphism $R_{\underline{-2}}(v_0)$ of the group V is a derivation for each integral multiplication:

$$\big(a\, \underline{n}\, b\big)\, \underline{-2}\, v_0 = \big(a\, \underline{-2}\, v_0\big)\, \underline{n}\, b + a\, \underline{n}\, \big(b\, \underline{-2}\, v_0\big)\,. \tag{6.9}$$

Successive integral multiplications are connected by the identity

$$\big(a\, \underline{-2}\, v_0\big)\, \underline{n}\, b = (-na)\, \underline{n-1}\, b\,. \tag{6.10}$$

Finally, suppose that

$$\big(a\, \underline{m}\, b\big)\, \underline{n}\, c - b\, \underline{n}\, \big(a\, \underline{m}\, c\big) = \sum_{s \in \mathbb{N}} \binom{m}{s} (a\, \underline{s}\, b)\ \underline{m - s + n}\, c \tag{6.11}$$

for integers m and n. Then the pair (V, L) is said to be a *two-sorted vertex algebra*.

Remark 6.3. By [2: Lemma XVII.5.2], a lattice-ordered unital ring is an f-ring if the multiplicative identity 1 is a strong unit. Thus by analogy with Remark 3.4, each two-sorted vertex algebra forms a two-sorted conformal algebra, with $a' = a\, \underline{-2}\, v_0$.

For two-sorted vertex algebras, the analogue of Proposition 4.2 is as follows:

Proposition 6.4. *Suppose that $(\{0\}, L)$ is a two-sorted vertex algebra. Then L is a commutative, unital lattice-ordered ring L in which the multiplicative identity 1 is a strong unit. Conversely, each such ring L may be construed as a two-sorted vertex algebra $(\{0\}, L)$.*

Lemma 6.5. *Given two states a, b in a two-sorted vertex algebra (V, L), one has $a\,\underline{s}\,b = 0$ for large positive integers s.*

Proof. Since 1 is a strong unit in the lattice-ordered group L, there is an integer $k = k1$ such that $k1 \geq N(a, b) \geq 0$. Then $s = s \vee N(a, b)$ for $s \geq k$. The locality identity (6.5) gives $a\,\underline{s}\,b = 0$ for such s. □

Corollary 6.6. *The sum on the right hand side of* (6.11) *is finite.*

Theorem 6.7. *If $(V, \mathbb{Z})$ is a two-sorted vertex algebra, then V is a vertex algebra. Conversely, each vertex algebra V may be construed as a two-sorted vertex algebra $(V, \mathbb{Z})$.*

Proof. First note that 1 is a strong unit in $\mathbb{Z}$. Then the forward statement is a direct consequence of Lemma 6.5. The converse direction follows as in the proof of Theorem 4.3, using (3.2) in place of (2.5). □

The class of commutative, unital f-rings, appearing as the second sort of two-sorted conformal algebras, forms a variety or equationally-defined class. By contrast, the class of commutative, unital f-rings with 1 as a strong unit is not a variety. For example, note that while $\mathbb{Z}$ has 1 as a strong unit, the power $\mathbb{Z}^{\mathbb{N}}$ does not: The multiplicative identity of the power is the constant function $u \colon \mathbb{N} \to \mathbb{Z};\ n \mapsto 1$, and no integer multiple of u can dominate the embedding function $j \colon \mathbb{N} \hookrightarrow \mathbb{Z}$. As a consequence of Proposition 6.4, one concludes:

Theorem 6.8. *The class of two-sorted vertex algebras does not form a variety of two-sorted algebras.*

Proof. The pair $(\{0\}, \mathbb{Z})$ forms a two-sorted vertex algebra. Suppose that the class of two-sorted vertex algebras did form a variety of two-sorted algebras. Then the power

$$\big(\{0\}, \mathbb{Z}\big)^{\mathbb{N}} = \big(\{0\}^{\mathbb{N}}, \mathbb{Z}^{\mathbb{N}}\big) = \big(\{0\}, \mathbb{Z}^{\mathbb{N}}\big)$$

would lie in that variety, and its second component $\mathbb{Z}^{\mathbb{N}}$ would have its multiplicative identity u as a strong unit. □

Despite the negative implication of Theorem 6.8, some construction methods are available.

Proposition 6.9. *The class of two-sorted vertex algebras is closed under the taking of subalgebras, homomorphic images, and finite products.*

Proof. The first two closure properties are straightforward (compare [15: §II.2: 1(3), 2(3)]). For the third, it suffices to consider the product of two two-sorted vertex algebras, say (V, L) and (W, M) (compare [15: §II.2.1(7)]). The issue then is to note that the ring $L \times M$ has $(1_L, 1_M)$ as a strong unit. Indeed, for given $(l, m) \in L \times M$, consider natural numbers h, k with $hl \geq 1_L$ and $km \geq 1_M$. Then for the maximum s of h and k, one has $s(l, m) = (sl, sm) \geq (hl, km) \geq (1_L, 1_M)$. □

Corollary 6.10. *Let (V, L) be a two-sorted vertex algebra. For a finite set τ, the power (V^τ, L^τ) is a two-sorted vertex algebra.*

Corollary 6.10 and Theorem 6.7 provide non-trivial examples of two-sorted vertex algebras which are not directly related to conventional vertex algebras.

Corollary 6.11. *Let V be a vertex algebra. For a finite set τ, the power $(V^\tau, \mathbb{Z}^\tau)$ is a two-sorted vertex algebra.*

Remark 6.12. In Corollary 6.11, suppose that the vertex algebra V represents a physical object at a given time. If the finite set τ, say the finite ordinal $\tau = \{t \in \mathbb{N} \mid t < |\tau|\}$, is construed as a series of discrete time steps, then the power two-sorted vertex algebra $(V^\tau, \mathbb{Z}^\tau)$ represents the evolution of the object over the time interval τ.[4]

7. The logic of locality

As presented in Definition 6.2, the locality rings of two-sorted vertex algebras are commutative unital f-rings, in which the multiplicative identity 1 is a strong unit. Let $\mathcal{C}$ be the category of all commutative unital f-rings with 1 as a strong unit (with all the homomorphisms as the morphisms). Now although the object class of $\mathcal{C}$ does not form a variety, the category $\mathcal{C}$ is known to be equivalent to a variety of algebras (again with homomorphisms as morphisms). The variety is specified using definitions of various kinds of algebras whose elements are typically interpreted as truth values from appropriate multiple-valued logics.

Definition 7.1. A *hoop* is an algebra $(H, \star, \to, 1)$ such that $(H, \star, 1)$ is a commutative monoid, and the *implication* $\to$ satisfies the following identities:

(H1) $x \to x = 1$;

(H2) $x \to (y \to z) = (x \star y) \to z$;

(H3) $x \star (x \to y) = y \star (y \to x)$.

Definition 7.2. A *Wajsberg algebra* is a hoop with a constant 0 that satisfies the identities:

(W1) $(x \to y) \to y = (y \to x) \to x$;

(W2) $0 \to x = 1$.

Definition 7.3. [17: Defn. 2.2] A *PMV-algebra* is an algebra

$$(H, \star, \to, \cdot, 0, 1)$$

such that:

(P1) $(H, \star, \to, 0, 1)$ is a Wajsberg algebra;

(P2) $(H, \cdot, 1)$ is a commutative monoid;

(P3) $x \cdot \big((z \to y) \to 0\big) = \big((x \cdot z) \to (x \cdot y)\big) \to 0$.

Let $\mathcal{P}$ be the variety of PMV-algebras, considered as a category with homomorphisms of PMV-algebras as its morphisms. The following result of Montagna has a precursor in work of Di Nola and Dvurečenskij [9: Th. 3.2], following a line of research going back to C. C. Chang [8].

Theorem 7.4. ([17: Th. 2.8])

(a) *For an object L of $\mathcal{C}$, the unit interval $L^I = \{l \in L \mid 0 \le l \le 1\}$ forms a PMV-algebra.*

(b) *For a morphism $f \colon L_1 \to L_2$ of $\mathcal{C}$, there is a well-defined restriction $f^I \colon L_1^I \to L_2^I$.*

(c) *The unit interval assignment $I \colon \mathcal{C} \to \mathcal{P}$ is a functor that forms part of an equivalence between the categories $\mathcal{C}$ and $\mathcal{P}$.*

[4] The discrete time parameter indexed by τ may be regarded as macroscopic, to distinguish it from any microscopic, quantum-scale time parameter incorporated within the physical description encoded by V.

***Example* 7.5.** Theorem 7.4 may be applied to the locality ring $\mathbb{Z}^\tau$ of the two-sorted vertex algebra $(V^\tau, \mathbb{Z}^\tau)$ discussed in Corollary 6.11. Remark 6.12 interpreted the finite index set τ, namely $\{0, 1, \dots, |\tau| - 1\}$, as a series of discrete time steps. According to Theorem 7.4(a), the unit interval $(\mathbb{Z}^\tau)^I$ of the locality ring $\mathbb{Z}^\tau$ is a PMV-algebra. This algebra is the $|\tau|$-th power of the two-element PMV-algebra $\mathbb{Z}^I = \{0, 1\}$ with the usual Boolean implication, and the usual multiplication $x \cdot y = x \star y$ of integers. The algebra $(\mathbb{Z}^\tau)^I$ represents a logic whose truth values vary over time, while the usual vertex algebra power V^τ corresponds (via Theorems 6.7 and 7.4) to the two-element Boolean subalgebra $\{(0, \dots, 0), (1, \dots, 1)\}$ of $(\mathbb{Z}^\tau)^I$, "crisp" truth values that are constant over time.

The category equivalence appearing in Theorem 7.4(c) is only an equivalence of abstract categories, and not an equivalence of concrete categories. Against the background of §5, it does raise the following question:

Problem 7.6. Is the abstract category of two-sorted vertex algebras equivalent to a variety of algebras?

Comparison with Corollary 5.5 also suggests the following problem:

Problem 7.7. Is the category of two-sorted vertex algebras a Mal'tsev category?

A positive answer to Problem 7.7 would make two-sorted vertex algebras amenable to some of the techniques mentioned in connection with Proposition 5.4.

8. Bilinear algebra identities

Up to this point, the conformal and vertex algebras under consideration have been completely general. While this has served to establish a basic framework for the methods being introduced, it does raise the question of what happens when further conditions are to be imposed, such as associativity or Lie algebra identities. That question is the topic of this brief, final section of the paper.

Suppose that (C, L) is a two-sorted conformal algebra for which the multiplicative identity 1 of the ring L is a strong unit. In particular, one may consider a two-sorted vertex algebra (C, L) (compare Remark 6.3). Reprising the construction of Proposition 2.5, take a copy C_n of the abelian group C given by the isomorphism $C \to C_n$; $a \mapsto a_n$ for each integer n. Given the direct sum $\bigoplus_{n\in\mathbb{Z}} C_n$, consider the subgroup E generated by $\{a'_n + na_{n-1} \mid a \in C,\ n \in \mathbb{Z}\}$, the quotient $\widehat{C} = \bigoplus_{n\in\mathbb{Z}} C_n \Big/ E$, and the subgroup $\widehat{C}_+ = \left(E + \bigoplus_{n\in\mathbb{N}} C_n\right) \Big/ E$ of the quotient. Then for $a, b \in C$ and $m, n \in \mathbb{Z}$, the analogue

$$(a_m + E) \cdot (b_n + E) = E + \sum_{s\in\mathbb{N}} \binom{m}{s} (a \,\underline{s}\, b)_{m-s+n}$$

of the product (2.6) is well-defined, by the argument of Lemma 6.5. Bilinearity extends the product definition, to yield an algebra $(\widehat{C}, +, \cdot)$.

For a variety $\underline{\underline{V}}$ of bilinear algebras (e.g., associative or Lie algebras), one may now impose a condition on (C, L) requiring that the bilinear algebra $(\widehat{C}, +, \cdot)$ lies in $\underline{\underline{V}}$. In this case, (C, L) is called a *(two-sorted)* $\underline{\underline{V}}$*-conformal algebra*. Theorem 4.3 is refined as follows.

THEOREM 8.1. *Let $\underline{\underline{V}}$ be a variety of bilinear algebras. If $(C,\mathbb{Z})$ is a two-sorted $\underline{\underline{V}}$-conformal algebra, then the conformal algebra C lies in $\widehat{\underline{\underline{V}}}$. Conversely, each conformal algebra C in the class $\widehat{\underline{\underline{V}}}$ may be construed as a two-sorted $\underline{\underline{V}}$-conformal algebra $(C,\mathbb{Z})$.*

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