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Comtrans algebras, Thomas sums, and bilinear forms

By

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Abstract. A comtrans algebra is said to decompose as the Thomas sum of two subalgebras if it is a direct sum at the module level, and if its algebra structure is obtained from the subalgebras and their mutual interactions as a sum of the corresponding split extensions. In this paper, we investigate Thomas sums of comtrans algebras of bilinear forms. General necessary and sufficient conditions are given for the decomposition of the comtrans algebra of a bilinear form as a Thomas sum. Over rings in which 2 is not a zero divisor, comtrans algebras of symmetric bilinear forms are identified as Thomas summands of algebras of infinitesimal isometries of extended spaces, the complementary Thomas summand being the algebra of infinitesimal isometries of the original space. The corresponding Thomas duals are also identified. These results represent generalizations of earlier results concerning the comtrans algebras of finite-dimensional Euclidean spaces, which were obtained using known properties of symmetric spaces. By contrast, the methods of the current paper involve only the theory of comtrans algebras.

1. Introduction. Comtrans algebras are unital modules over a commutative ring R, equipped with two basic trilinear operations: a *commutator* $[x, y, z]$ satisfying the *left alternative identity*

(1.1) $$[x, x, y] = 0,$$

and a *translator* $\langle x, y, z\rangle$ satisfying the *Jacobi identity*

(1.2) $$\langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0,$$

such that together the commutator and translator satisfy the *comtrans identity*

(1.3) $$[x, y, x] = \langle x, y, x\rangle.$$

Comtrans algebras were originally introduced [13] in answer to a problem from differential geometry, asking for the algebraic structure in the tangent bundle corresponding to the coordinate n-ary loop of an $(n + 1)$-web (cf. [3]). The role played by comtrans algebras is analogous to the role played by the Lie algebra of a Lie group. In particular, given

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a Lie algebra L with bilinear commutator $[x, y]$ over R, one obtains a comtrans algebra $CT(L)$ by setting

$$[x, y, z] = \langle x, y, z\rangle = [[x, y], z] \quad (1.4)$$

for x, y, z in L [11]. A comtrans algebra is said to be *monic* if its commutator and translator agree. Thus for any Lie algebra L, the comtrans algebra $CT(L)$ is monic. Generalizing terminology from Lie algebras, a comtrans algebra is said to be *abelian* if its commutator and translator are zero. Thus abelian comtrans algebras are essentially just R-modules.

Amongst the diverse ways in which comtrans algebras arise [5], [6], [8], [10], [11], [12], [14], this paper focuses on the comtrans algebra $CT(E, \beta)$ furnished by a *formed module* (E, β), an R-module E equipped with a bilinear form β. The comtrans algebra $CT(E, \beta)$ has underlying module E. Its algebra structure is defined by

$$[x, y, z] = y\beta(x, z) - x\beta(y, z) \quad (1.5)$$

and

$$\langle x, y, z\rangle = y\beta(z, x) - x\beta(y, z) \quad (1.6)$$

[10]. (Note that $CT(E, \beta)$ is monic if β is symmetric.) In this context, we investigate the concept of a *Thomas sum*

$$G = E \circledT F \quad (1.7)$$

of comtrans algebras (see [5] and Section 3 below, comparing with "Thomas decompositions" as identified in [2, p. 502]). In (1.7), the comtrans algebra G is a module direct sum of subalgebras E and F, each of which acts on the other (as outlined in Section 2). The algebra structure of G is recovered from the subalgebras and their mutual interactions. Thus (1.7) does not just give a decomposition of a known algebra G, but may also be used to construct an algebra G from smaller algebras E and F. Going beyond the initial treatment of Thomas sums in [5], Section 3 of the current paper relates the Thomas sum construction to the split extensions constructed in [9] as part of the representation theory of comtrans algebras.

There are three main results in the paper. Theorem 4.1 specifies the exact conditions under which the comtrans algebra of a formed module decomposes as a Thomas sum. If the summands are free modules, then the conditions reduce to the mutual orthogonality of the summands (Corollary 4.2), but an example shows that orthogonality is too strong in the general case. The second main result, Theorem 6.4, generalizes work in [5] exhibiting the comtrans algebra of a finite-dimensional Euclidean space as a Thomas summand of a comtrans algebra of skew-symmetric matrices. For general commutative rings R, the analytic method of passing from orthogonal transformations to infinitesimal isometries represented by the skew-symmetric matrices is no longer available. Thus Section 3 recalls the purely algebraic analogue of this passage provided by use of the dual numbers over R. The infinitesimal isometries of a formed module (E, β) constitute a Lie algebra, whose comtrans algebra obtained via (1.4) is called the *isometry algebra* $\mathrm{O}(\beta)$ of the formed module. Under a mild assumption (namely that 2 is not a zero divisor in R), Theorem 6.4 then exhibits the comtrans algebra of a formed module (E, β) with symmetric form β as a

Thomas summand in a larger isometry algebra $\mathrm{O}(\beta \oplus 1)$, the other Thomas summand being the isometry algebra $\mathrm{O}(\beta)$. Now in the context of Thomas sums of comtrans algebras, there is a notion of duality called *Thomas duality* which serves as an algebraic generalization of aspects of symmetric space duality and Weyl's unitary trick (see [5] and the beginning of Section 7). The final result of the paper, Theorem 7.2, then identifies an isometry algebra $\mathrm{O}(\beta \oplus -1)$ as the Thomas dual of the Thomas sum furnished by Theorem 6.4. It is worth noting that the results on finite-dimensional Euclidean spaces in [5] (cf. Example 7.3 for the physically most relevant case) were proved using known properties of Lie algebras and symmetric spaces, while the proofs of the much more general theorems of the current paper lie entirely within the theory of comtrans algebras.

For concepts and conventions of algebra that are not otherwise explained here, readers are referred to [15].

2. Enveloping algebras. The class $\mathfrak{CT}_R$ of all comtrans algebras over a ring R forms a variety in the sense of universal algebra. This variety becomes (the class of objects of) a bicomplete category whose morphisms are the homomorphisms between comtrans algebras (cf. Theorems IV 2.1.3 and 2.2.3 of [15]). For a member E of $\mathfrak{CT}_R$, let $E[X]$ denote the coproduct of E in $\mathfrak{CT}_R$ with the free $\mathfrak{CT}_R$-algebra on a singleton $\{X\}$. For x, y in E, there are R-module homomorphisms

$$K(x, y) : E[X] \to E[X]; z \mapsto [z, x, y], \tag{2.1}$$

$$R(x, y) : E[X] \to E[X]; z \mapsto \langle z, x, y\rangle, \tag{2.2}$$

and

$$L(x, y) : E[X] \to E[X]; z \mapsto \langle y, x, z\rangle. \tag{2.3}$$

The *universal enveloping algebra* $U(E)$ of E is the R-subalgebra of the endomorphism ring of the R-module $E[X]$ generated by [9].

$$\{K(x, y), R(x, y), L(x, y) \mid x, y \in E\}$$

A comtrans algebra E is said *to act* on another comtrans algebra F if the R-module F is a module over the enveloping algebra $U(E)$ of E. The action is *trivial* if

$$fK(e, e') = fR(e, e') = fL(e, e') = 0$$

for all f in F and e, e' in E. The algebras E and F are said to *interact mutually* if each acts on the other.

3. Thomas sums

Definition 3.1. A comtrans algebra G is said to be the *internal Thomas sum* of subalgebras E and F if:

(1) as a module, G is the internal direct sum of its submodules E and F; and
(2) the containments

$$[E, F, F] \subseteqq E, \quad [F, E, F] \subseteqq E, \quad [F, F, E] \subseteqq E, \tag{3.1}$$

$$\langle E, F, F\rangle \subseteqq E, \quad \langle F, E, F\rangle \subseteqq E, \quad \langle F, F, E\rangle \subseteqq E, \tag{3.2}$$

$$[F, E, E] \subseteqq F, \quad [E, F, E] \subseteqq F, \quad [E, E, F] \subseteqq F, \tag{3.3}$$

$$\langle F, E, E\rangle \subseteqq F, \quad \langle E, F, E\rangle \subseteqq F, \quad \langle E, E, F\rangle \subseteqq F \tag{3.4}$$

are satisfied.

Definition 3.2. Let E and F be two mutually interacting comtrans algebras over a ring R. Their *external Thomas sum* is defined to be the external direct sum $G = E \oplus F$ of the R-modules E and F, equipped with a commutator

$$\begin{aligned}[][e_1 + f_1, e_2 + f_2, e_3 + f_3] &= [e_1, e_2, e_3] + [f_1, f_2, f_3] + e_1 K(f_2, f_3) \\ &\quad - e_2 K(f_1, f_3) + e_3\{L(f_2, f_1) + R(f_2, f_1) - K(f_2, f_1)\} \\ &\quad + f_1 K(e_2, e_3) - f_2 K(e_1, e_3) + f_3\{L(e_2, e_1) + R(e_2, e_1) - K(e_2, e_1)\}\end{aligned}$$

and a translator

$$\begin{aligned}\langle e_1 + f_1, e_2 + f_2, e_3 + f_3\rangle &= \langle e_1, e_2, e_3\rangle + \langle f_1, f_2, f_3\rangle \\ &\quad + e_1 R(f_2, f_3) - e_2\{R(f_3, f_1) + L(f_1, f_3)\} + e_3 L(f_2, f_1) \\ &\quad + f_1 R(e_2, e_3) - f_2\{R(e_3, e_1) + L(e_1, e_3)\} + f_3 L(e_2, e_1)\end{aligned}$$

defined using elements e_i of E and f_i of F.

Proposition 3.3 [5, Prop. 4.2]**.** *The external Thomas sum G of two mutually interacting comtrans algebras E and F over R is a comtrans algebra. This comtrans algebra G is the internal Thomas sum of its subalgebras $E \oplus \{0\}$ and $\{0\} \oplus F$. Conversely, suppose that a comtrans algebra G is the internal Thomas sum of subalgebras E and F. Then G is isomorphic to the external Thomas sum of E and F.*

On the strength of Proposition 3.3, one abuses language and suppresses the distinction between internal and external Thomas sums, speaking simply of the *Thomas sum* of two mutually interacting comtrans algebras. This usage is similar to that for direct sums. Indeed, Thomas sums specialize to direct sums in the case where the mutual interaction is trivial. The notation $G = E \textcircled{\scriptsize T} F$ is used to record that a comtrans algebra G is a Thomas sum of E and F, although it does not indicate the specific mutual interaction of the subalgebras.

For a comtrans algebra E, Theorem 3.10 of [9] exhibited an equivalence between the category of $U(E)$-modules and the category of abelian groups in the slice category $\mathfrak{CT}_R/E$. (The latter objects are modules in the sense of Beck [1].) In particular, it was shown that a $U(E)$-module V furnishes a *split extension* comtrans algebra structure $V \rtimes E$ on the R-module $V \oplus E$. Comparison between Proposition 3.4 of [9] and Definition 3.2 above shows that this split extension may be described as the Thomas sum $V \textcircled{\scriptsize T} E$ of the abelian algebra V and E, with the given action of E on V and the trivial action of V on E. Conversely, Thomas sums may be described in terms of split extensions. To this end, it is helpful to have the option of putting the split extension structure furnished by the $U(E)$-module V on the underlying module $E \oplus V$ rather than on $V \oplus E$. In such format the split extension is written as $E \ltimes V$. Now recall that the collection of all comtrans algebra structures on a given R-module G itself has an R-module structure. Specifically, given scalars $r_1, r_2 \in R$ and comtrans algebras $G_1 = (G, [x, y, z]_1, \langle x, y, z\rangle_1)$, $G_2 = (G, [x, y, z]_2, \langle x, y, z\rangle_2)$, there is a comtrans algebra $r_1G_1 + r_2G_2$ given by

$$(G, r_1[x, y, z]_1 + r_2[x, y, z]_2, r_1\langle x, y, z\rangle_1 + r_2\langle x, y, z\rangle_2). \tag{3.5}$$

Examination of Definition 3.2 then shows that

$$E \mathbin{\textcircled{\scriptsize T}} F = E \rtimes F + E \ltimes F$$

specifies the external Thomas sum of two mutually interacting comtrans algebras E and F.

4. Formed modules.

Theorem 4.1. *Let (G, β) be a formed module, with submodules E and F. Then the comtrans algebra $CT(G, \beta)$ decomposes as the Thomas sum $CT(E, \beta) \mathbin{\textcircled{\scriptsize T}} CT(F, \beta)$ if and only if the following three conditions are satisfied:*

(a) *The module G is the direct sum of the submodules E and F;*
(b) $\forall e \in E,\ \forall f \in F,\ \beta(e, f) \in \operatorname{An} E \cap \operatorname{An} F$;
(c) $\forall e \in E,\ \forall f \in F,\ \beta(f, e) \in \operatorname{An} E \cap \operatorname{An} F$.

Proof. Note that the conditions (a)–(c) are symmetric in E and F. First suppose that the conditions are satisfied. Consider elements e, e' of E and f, f' of F. Then by (1.5),

$$[e, f, f'] = f\beta(e, f') - e\beta(f, f') \in E,$$

the first term of the difference vanishing since $\beta(e, f') \in \operatorname{An} F$ by (b). This verifies the first containment of (3.1). By (1.6),

$$\langle f, e, e'\rangle = e\beta(e', f) - f\beta(e, e') \in F,$$

the first term of the difference vanishing since $\beta(e', f) \in \operatorname{An} E$ by (b). This verifies the first containment of (3.4). Verification of the other containments of (3.1)–(3.4) is similar, producing $CT(G, \beta)$ as the (internal) Thomas sum $CT(E, \beta) \mathbin{\textcircled{\scriptsize T}} CT(F, \beta)$.

Conversely, suppose that $CT(G, \beta)$ decomposes as the (internal) Thomas sum $CT(E, \beta) \mathbin{\textcircled{\scriptsize T}} CT(F, \beta)$. Condition (a) is just part of this assumption. Consider elements e, e' of E and f, f' of F. Then

$$f\beta(e, f') = e\beta(f, f') + [e, f, f'] \in E \cap F = \{0\},$$

the membership in E resulting from the first containment of (3.1), and the final equality holding by (a). Thus

(4.1) $\qquad \forall e \in E,\ \forall f \in F,\ \beta(e, f) \in \operatorname{An} F.$

Similarly,

$$e\beta(e', f) = f\beta(e, e') + \langle f, e, e'\rangle \in F \cap E = \{0\},$$

the membership in F resulting from the first containment of (3.4), and the final equality holding by (a). Thus

(4.2) $\qquad \forall e \in E,\ \forall f \in F,\ \beta(e, f) \in \operatorname{An} E.$

Together, (4.1) and (4.2) yield condition (b). Condition (c) follows immediately by the symmetry between E and F. □

Recall that a subset E of a formed module (G, β) is *orthogonal* to a subset F if

(4.3) $\qquad \forall e \in E,\ \forall f \in F,\ \beta(e, f) = 0$

[4, §5.2]. The subsets are said to be *mutually orthogonal* if E is orthogonal to F and F is orthogonal to E.

Corollary 4.2. *Let (G, β) be a formed module whose underlying module is a direct sum of free submodules E and F. Then the comtrans algebra $CT(G, \beta)$ decomposes as the Thomas sum $CT(E, \beta)$ Ⓣ $CT(F, \beta)$ if and only if the submodules E and F are mutually orthogonal.*

P r o o f. If the submodules E and F are mutually orthogonal, then the conditions (b) and (c) of Theorem 4.1 are certainly satisfied, so that $CT(G, \beta)$ decomposes as the Thomas sum $CT(E, \beta)$ Ⓣ $CT(F, \beta)$. Conversely, suppose that $CT(G, \beta) = CT(E, \beta)$ Ⓣ $CT(F, \beta)$. Since the submodules E and F are free, their annihilators are trivial. The conditions (b) and (c) of Theorem 4.1 thus show that E and F are mutually orthogonal. □

E x a m p l e 4.3. Let S be a non-zero unital, commutative ring. Let $R = S[X]/X^3S[X]$ be the quotient of the polynomial ring $S[X]$ by the ideal generated by X^3. Let $E = F = XR$ as R-modules. Note that $\operatorname{An} E = \operatorname{An} F = X^2R$. Consider the direct sum $G = E \oplus F$. Define a symmetric bilinear form β on G by $\beta((Xp_1, Xq_1), (Xp_2, Xq_2)) = X^2(p_1q_2 + p_2q_1)$ for $p_i, q_i \in R$. Note that $\beta((X, 0), (0, X)) = X^2 \neq 0$, so the submodules E and F are not mutually orthogonal. On the other hand, the conditions of Theorem 4.1 are satisfied, so that $CT(G, \beta)$ does decompose as the Thomas sum $CT(E, \beta)$ Ⓣ $CT(F, \beta)$.

5. Infinitesimal isometries. For a commutative, unital ring R, the ring $R[\varepsilon]$ of *dual numbers* over R is the quotient $R[X]/\langle X^2\rangle$, the element ε being identified as the coset of X [7, 16, 17]. The forgetful functor from the category of $R[\varepsilon]$-modules to the category of

R-modules has a left adjoint given by tensoring with $R[\varepsilon]$ over R. Usually the canonical extension of an R-morphism $\theta : E \to F$ to an $R[\varepsilon]$-morphism $R[\varepsilon] \otimes_R E \to R[\varepsilon] \otimes_R F$ is denoted by the same symbol θ. Then a formed module (E, β) over R, described by the linear map $\beta : E \otimes_R E \to R$, yields a formed module over $R[\varepsilon]$ described by the image of the morphism β under the left adjoint. Thus

$$\beta(e_1 + \varepsilon f_1, e_2 + \varepsilon f_2) = \beta(e_1, e_2) + \varepsilon\,[\beta(e_1, f_2) + \beta(f_1, e_2)]$$

for e_i, f_j in E.

Let φ be an endomorphism of an R-module E, extending canonically to an $R[\varepsilon]$-endomorphism φ of $R[\varepsilon] \otimes_R E$. Then the *exponential* is the map

$$\exp\varphi : R[\varepsilon] \otimes_R E \to R[\varepsilon] \otimes_R E;\, x \mapsto x + \varepsilon x\varphi.$$

Note that for endomorphisms θ, φ of E, one has

$$\exp\theta \exp\varphi = \exp(\theta + \varphi).$$

Definition 5.1. An *infinitesimal isometry* of a formed module (E, β) is an endomorphism of E whose exponential is an isometry of the corresponding formed module over $R[\varepsilon]$.

Proposition 5.2. *An endomorphism φ of the underlying module E of a formed module (E, β) is an infinitesimal isometry if and only if*

$$\forall\, x, y \in E,\ \beta(x\varphi, y) + \beta(x, y\varphi) = 0. \tag{5.1}$$

Proof. Consider the equation

$$\begin{aligned} \beta(x \exp\varphi, y \exp\varphi) &= \beta(x + \varepsilon x\varphi, y + \varepsilon y\varphi) \\ &= \beta(x, y) + \varepsilon\,[\beta(x\varphi, y) + \beta(x, y\varphi)]\,. \end{aligned} \tag{5.2}$$

If $\exp\varphi$ is an isometry, then (5.2) holds for x, y in E. Thus the coefficient of ε there, namely the predicate of (5.1), vanishes. Conversely, if (5.1) holds, then $\varepsilon\,[\beta(x\varphi, y) + \beta(x, y\varphi)]$ vanishes for all x, y in $R[\varepsilon] \otimes_R E$, so that $\exp\varphi$ is an isometry. □

Corollary 5.3. *The set of infinitesimal isometries of a formed module (E, β) forms a Lie algebra $\mathfrak{o}(\beta)$ under the binary commutator $[\varphi_1, \varphi_2] = \varphi_1\varphi_2 - \varphi_2\varphi_1$.*

Proof. Suppose $\varphi_i \in \mathfrak{o}(\beta)$ for $i = 1, 2$, so that $\beta(x\varphi_i, y) = -\beta(x, y\varphi_i)$ for $x, y \in E$. Then

$$\begin{aligned} \beta(x[\varphi_1, \varphi_2], y) &= \beta(x(\varphi_1\varphi_2 - \varphi_2\varphi_1), y) \\ &= \beta(x\varphi_1\varphi_2, y) - \beta(x\varphi_2\varphi_1, y) \\ &= -\beta(x\varphi_1, y\varphi_2) + \beta(x\varphi_2, y\varphi_1) \\ &= \beta(x, y\varphi_2\varphi_1) - \beta(x, y\varphi_1\varphi_2) \\ &= \beta(x, y(\varphi_2\varphi_1 - \varphi_1\varphi_2)) = -\beta(x, y[\varphi_1, \varphi_2]) \end{aligned}$$

for $x, y \in E$, as required. □

For a formed module (E, β), the monic comtrans algebra furnished according to (1.4) by the Lie algebra $\mathfrak{o}(\beta)$ is denoted by $\mathrm{O}(\beta)$. It is called the *isometry algebra* of the formed module (E, β).

6. Formed modules in isometry algebras. Let (E, β) be a formed module over a commutative unital ring R, with corresponding isometry algebra $\mathrm{O}(\beta)$. Let b be a scalar. Consider the formed module $(E \oplus R, \beta \oplus b)$ with

$$(\beta \oplus b)((x, r), (y, s)) = \beta(x, y) + brs$$

for x, y in E and scalars r, s in R. Let $\mathrm{O}(\beta \oplus b)$ be the isometry algebra of $(E \oplus R, \beta \oplus b)$. Note that an endomorphism Φ of $E \oplus R$ may be decomposed in the general form

$$\Phi : E \oplus R \to E \oplus R; (e, r) \mapsto (e\varphi + rx, e^\eta + rm) \tag{6.1}$$

for a scalar m from R, an element x of E, a functional η on E, and an endomorphism φ of the R-module E.

Proposition 6.1. *The endomorphism* (6.1) *is an infinitesimal isometry of* $\beta \oplus b$ *if and only if the following conditions are satisfied:*

(a) $2bm = 0$*;*
(b) $\forall e \in E,\ be^\eta = -\beta(e, x) = -\beta(x, e)$*;*
(c) $\varphi \in \mathrm{O}(\beta)$.

Proof. The condition (5.1) on Φ reduces to

$$\begin{aligned} 0 &= (\beta \oplus b)((e, r)\Phi, (f, s)) + (\beta \oplus b)((e, r), (f, s)\Phi) \\ &= \beta(e\varphi, f) + \beta(e, f\varphi) + r[\beta(x, f) + bf^\eta] \\ &\quad + s[\beta(e, x) + be^\eta] + 2bmrs \end{aligned} \tag{6.2}$$

for elements (e, r) and (f, s) of $E \oplus R$. If the conditions (a)–(c) of the proposition hold, then it is clear that Φ is an infinitesimal isometry of $\beta \oplus b$. Conversely, suppose that Φ is such an infinitesimal isometry. Setting $e = f = 0$, $r = s = 1$ in (6.2) yields (a). Setting $f = 0$, $s = 1$ and $e = 0$, $r = 1$ respectively then yields (b). If the conditions (a) and (b) hold, then (6.2) reduces to condition (c). □

Corollary 6.2. *There is a comtrans algebra embedding*

$$\mathrm{O}(\beta) \to \mathrm{O}(\beta \oplus b); \varphi \mapsto ((e, r) \mapsto (e\varphi, 0)).$$

On the strength of Corollary 6.2, the algebra $\mathrm{O}(\beta)$ is identified with its image inside $\mathrm{O}(\beta \oplus b)$. Consider the complement $C(\beta, b)$ of this image consisting of those endomorphisms (6.1) with $\varphi = 0$ that satisfy the conditions (a) and (b) of Proposition 6.1. In general, this complement need not form a subalgebra of $\mathrm{O}(\beta \oplus b)$.

Example 6.3. Let the formed module (E, β) be the real line with $\beta = 0$, and take $b = 0$. Represent endomorphisms of $E \oplus \mathbb{R}$ by their matrices with respect to $\{(1, 0), (0, 1)\}$. Then the conditions (a) and (b) of Proposition 6.1 are vacuous, so that $C(\beta, b)$ consists of all 2×2 real matrices with zero in the top left hand corner. This set of matrices is not closed under the ternary commutator.

Theorem 6.4. *Let (E, β) be a formed module with symmetric bilinear form β, over a commutative ring R in which 2 is not a zero divisor. Then the isometry algebra of the formed module $(E \oplus R, \beta \oplus 1)$ decomposes as a Thomas sum*

$$\mathrm{O}(\beta \oplus 1) = CT(E, \beta) \circledT \mathrm{O}(\beta) \tag{6.3}$$

of subalgebras isomorphic to $CT(E, \beta)$ and $\mathrm{O}(\beta)$.

Proof. Use Corollary 6.2 to identify the algebra $\mathrm{O}(\beta)$ with its image inside $\mathrm{O}(\beta \oplus 1)$. Under the stated conditions, the complement $C(\beta, 1)$ of this image consists of endomorphisms of the form

$$(e, r) \mapsto (rx, -\beta(e, x))$$

for elements x of E. It is straightforward to verify that the injective module homomorphism

$$E \to \mathrm{O}(\beta \oplus 1);\ x \mapsto ((e, r) \mapsto (rx, -\beta(e, x))) \tag{6.4}$$

is actually a comtrans algebra homomorphism from the monic formed algebra $CT(E, \beta)$. Identifying elements of E with their images under this comtrans homomorphism (6.4), the module $\mathrm{O}(\beta \oplus 1)$ then decomposes as an internal direct sum $CT(E, \beta) \oplus \mathrm{O}(\beta)$. It remains to establish the mutual interactions of the subalgebras inside the isometry algebra $\mathrm{O}(\beta \oplus 1)$. Since this isometry algebra is monic, it suffices to consider the commutators alone. Now for x in E and θ, φ in $\mathrm{O}(\beta)$, one has

$$[x, \theta, \varphi] : (e, r) \mapsto (rx\theta\varphi, -\beta(e\varphi\theta, x))$$

by (6.4) and Corollary 6.2. On the other hand, $\beta(e\varphi\theta, x) = \beta(e\varphi, x\theta) = \beta(e, x\theta\varphi)$ for all e in E by (5.1). Thus the action of $\mathrm{O}(\beta)$ on $CT(E, \beta)$ is given by

$$[x, \theta, \varphi] = x\theta\varphi.$$

Similarly, for x, y in E and φ in $\mathrm{O}(\beta)$, (6.4) and Corollary 6.2 yield

$$[\varphi, x, y] : (e, r) \mapsto (-y\beta(e\varphi, x) - x\varphi\beta(e, y), r\,[\beta(x\varphi, y) + \beta(x, y\varphi)]).$$

The second component of the image vanishes since $\varphi \in \mathrm{O}(\beta)$. Consider the endomorphism

$$\psi : e \mapsto -y\beta(e\varphi, x) - x\varphi\beta(e, y)$$

of the module E. For elements e, f of E, one has

$$\begin{aligned}
&\beta(e\psi, f) + \beta(e, f\psi)\\
&\quad = -\beta(y, f)\beta(e\varphi, x) - \beta(e, y)\beta(x\varphi, f)\\
&\qquad - \beta(f\varphi, x)\beta(e, y) - \beta(f, y)\beta(e, x\varphi)\\
&\quad = -\beta(y, f)\,[\beta(e\varphi, x) + \beta(e, x\varphi)]\\
&\qquad - \beta(e, y)\,[\beta(f\varphi, x) + \beta(f, x\varphi)] = 0,
\end{aligned}$$

again since $\varphi \in \mathrm{O}(\beta)$. Thus $[\varphi, x, y] = \psi \in \mathrm{O}(\beta)$. □

7. Thomas duality. Given a comtrans algebra E, its *negation* $\overline{E}$ is the negation $-E$ of E in the R-module (3.5) of all comtrans algebra structures on the underlying R-module of E. Note that the negation of $\overline{E}$ is E itself. Although the negation of a comtrans algebra E is not generally isomorphic to E (unless R contains a square root of -1), the universal enveloping algebras of E and $\overline{E}$ coincide. For a formed space (E, β), equations (1.5) and (1.6) show that the negation of $CT(E, \beta)$ is $CT(E, -\beta)$. (Indeed, the map $\beta \mapsto CT(E, \beta)$ yields an R-module homomorphism from the module of bilinear forms on E to the module (3.5) of comtrans algebra structures on E.)

Definition 7.1. Let E and F be comtrans algebras. Let $G = E \circledT F$ be a Thomas sum of E and F, given by actions $\eta : U(E) \to \mathrm{End}_R(F)$ and $\varphi : U(F) \to \mathrm{End}_R(E)$. Then the dual $G^* = \overline{E} \circledT F$ is the Thomas sum of $\overline{E}$ and F given by the action $-\eta$ of $U(E) = U(\overline{E})$ on F and the action φ of $U(F)$ on the equal underlying R-modules of $\overline{E}$ and E.

Theorem 7.2. *Let (E, β) be a formed module with symmetric bilinear form β, over a commutative ring R in which 2 is not a zero divisor. Then the isometry algebra of the formed module $(E \oplus R, \beta \oplus -1)$ decomposes as the Thomas sum*

(7.1) $$\mathrm{O}(\beta \oplus -1) = CT(E, -\beta) \circledT \mathrm{O}(\beta)$$

dual to (6.3).

Proof. As in the proof of Theorem 6.4, use Corollary 6.2 to identify the algebra $\mathrm{O}(\beta)$ with its image inside $\mathrm{O}(\beta \oplus 1)$. Under the hypotheses of Theorem 7.2, the complement $C(\beta, -1)$ of this image consists of endomorphisms of the form

$$(e, r) \mapsto (rx, \beta(e, x))$$

for elements x of E. It is again straightforward to verify that the injective module homomorphism

(7.2) $$E \to \mathrm{O}(\beta \oplus -1);\ x \mapsto ((e, r) \mapsto (rx, \beta(e, x)))$$

is actually a comtrans algebra homomorphism from the negation $\overline{E}$ of the monic formed algebra $CT(E, \beta)$. Identifying elements of $\overline{E}$ with their images under this comtrans homomorphism (7.2), the module $\mathrm{O}(\beta \oplus -1)$ then decomposes as an internal direct sum $\overline{E} \oplus \mathrm{O}(\beta)$. As before, the action of $\mathrm{O}(\beta)$ on $\overline{E} = CT(E, -\beta)$ is given by

$$[x, \theta, \varphi] = x\theta\varphi$$

for x in $\overline{E}$ and θ, φ in $\mathrm{O}(\beta)$. Now for x, y in E and φ in $\mathrm{O}(\beta)$, 7.2 and Corollary 6.2 yield

$$[\varphi, x, y] : (e, r) \mapsto (y\beta(e\varphi, x) + x\varphi\beta(e, y), -r\,[\beta(x\varphi, y) + \beta(x, y\varphi)]).$$

This is the negation of the action of $CT(E, \beta)$ on $\mathrm{O}(\beta)$ from (6.3), completing the proof that (7.1) is dual to (6.3). □

Example 7.3. The motivating instance of Theorem 7.2 from [5] is where (E, β) is 3-dimensional Euclidean space, so that $CT(E, \beta)$ is the vector triple product algebra. In this case, (7.1) is the algebra of infinitesimal isometries of Minkowski space, the Thomas dual of the algebra (6.3) of infinitesimal isometries of 4-dimensional Euclidean space. The Thomas summand $\mathrm{O}(\beta)$ in (7.1) represents the algebra of infinitesimal Thomas precessions (cf. [2, pp. 502–4]).

References

[1] J. Beck, Triples, Algebras, and Cohomology. Ph.D. thesis, Columbia University, New York 1967 (available online at http://www.tac.mta.ca/tac/reprints/articles/2/tr2abs.html as a Theory and Applications of Categories Reprint).
[2] R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications. New York 1974.
[3] V. V. Goldberg, Theory of Multicodimensional $(n+1)$-webs. Dordrecht 1988.
[4] K. W. Gruenberg and A. J. Weir, Linear Geometry. Princeton 1967.
[5] B. Im and J. D. H. Smith, Orthogonal ternary algebras and Thomas sums. Alg. Colloq. **11**, 287–296 (2004).
[6] B. Im and J. D. H. Smith, Simple ternary Grassman algebras. Alg. Colloq. **11**, 223–230 (2004).
[7] D. Mumford, Lectures on Curves on an Algebraic Surface. Princeton 1966.
[8] X. R. Shen, Classification of the comtrans algebras of low dimensionalities. Comm. Alg. **23**, 1751–1780 (1995).
[9] X. R. Shen and J. D. H. Smith, Representation theory of comtrans algebras. J. Pure Appl. Algebra **80**, 177–195 (1992).
[10] X. R. Shen and J. D. H. Smith, Comtrans algebras and bilinear forms. Arch. Math. **59**, 327–333 (1992).
[11] X. R. Shen and J. D. H. Smith, Simple multilinear algebras, rectangular matrices and Lie algebras. J. Algebra **160**, 424–433 (1993).
[12] X. R. Shen and J. D. H. Smith, Simple algebras of Hermitian operators. Arch. Math. **65**, 534–539 (1995).
[13] J. D. H. Smith, Multilinear algebras and Lie's Theorem for formal n-loops. Arch. Math. **51**, 169–177 (1988).
[14] J. D. H. Smith, Comtrans algebras and their physical applications. Banach Center Publ. **28**, 319–326 (1993).
[15] J. D. H. Smith and A. B. Romanowska, Post-Modern Algebra. New York 1999.
[16] E. Study, Über Systeme complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen, Wiener Monatshefte **1**, 8–10 (1899).
[17] E. Study, Geometrie der Dynamen. Leipzig 1903.

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