Algebra univers. **48** (2002) 427–438
0002-5240/02/040427 – 12

Algebra Universalis

A coalgebraic approach to quasigroup permutation representations

Jonathan D. H. Smith

Abstract. The paper identifies the class of all permutation representations of a given finite quasigroup as a covariety of coalgebras. Each permutation representation decomposes as a sum of homomorphic images of homogeneous spaces. For a group, permutation representations in the present sense specialise to the classical concept. Burnside's Lemma, with a new proof, is extended from groups to quasigroups.

1. Introduction

Quasigroups are defined informally as "non-associative groups," as sets equipped with a binary operation whose multiplication table is a Latin square. One of the major programs in the study of quasigroups has been the extension to them of various aspects of the representation theory of groups. For summaries of character theory, see [7, 14]. For a summary of module theory, see [13]. An active theme of current work is to develop a theory of permutation representations for quasigroups. The initial stage of this research [15, 16] introduced a concept of homogeneous space for finite quasigroups. The key ideas are summarised in Section 2. Given a subquasigroup P of a finite quasigroup Q, the elements of the corresponding homogeneous space $P\backslash Q$ are the orbits on Q of the relative left multiplication group of P in Q, the group of permutations of Q generated by the left multiplications by elements of P. Each element of Q yields a Markov chain action on the homogeneous space $P\backslash Q$ as a set of states. The full structure is an instance of an iterated function system (IFS) in the sense of fractal geometry [1], formalised in the concept of a Q-IFS described in Section 3. If P is a subgroup of a group Q, then the quasigroup homogeneous space $P \backslash Q$ specialises to the usual notion of a homogeneous space or transitive permutation representation for groups, the transition matrices of the Markov chain actions becoming permutation matrices in this case.

Presented by H. P. Gumm.
Received March 13, 2002; accepted in final form September 18, 2002.
2000 *Mathematics Subject Classification*: 08C05, 20N05, 68Q85.
Key words and phrases: Loop, quasigroup, permutation representation, coalgebra, covariety.
This paper was written while the author was a guest of the Institute of Mathematics and Information Sciences at Warsaw University of Technology, on Faculty Professional Development Assignment from Iowa State University.

Following the determination of the homogeneous spaces over a finite quasigroup Q, current research focuses on an appropriate specification for general Q-sets or permutation representations of Q. For a group Q, arbitrary Q-sets are built up by taking disjoint unions of homogeneous spaces. Moreover, the class of Q-sets is closed under direct products. The class of all Q-sets admits a syntactical characterisation as a variety of universal algebras, the axioms essentially characterising a Q-set (X, Q) as a set X with a group homomorphism from Q to the group $X!$ of permutations of the set X. In [17], an attempt was made to extend this algebraic approach to more general quasigroups. However, because of the demand for closure of the class of Q-sets under products, it turned out to be necessary to require Q to be a loop (a quasigroup with an identity element)[1]. An additional feature of the approach taken in [17] was the infinite number of non-isomorphic irreducible Q-sets that kept appearing in repeated direct products. Burnside's Lemma, which holds for quasigroup homogeneous spaces (Theorem 5.1 of [15]), no longer holds for these new irreducible Q-sets. Thus the algebraic approach to the specification of quasigroup permutation representations has three disadvantages:

(1) The approach only works for loops, not for general quasigroups.
(2) Infinitely many non-isomorphic irreducible Q-sets appear.
(3) Burnside's Lemma fails.

The current paper is motivated by the desire to avoid these problems of the algebraic approach. The key clues are the facts that the complications arise on taking products, and that the variety of Q-sets over a group Q is closed under sums. Now covarieties of coalgebras are closed under sums, but not necessarily under products. It thus becomes natural to consider Q-sets over a quasigroup as the members of a covariety of coalgebras.

The basic coalgebraic definitions are summarised in Section 4. The probabilistic aspect of quasigroup Q-sets is invoked via the free barycentric algebra functor B, as described briefly in Section 5. Theorem 6.3 then shows how to pass between Q-IFS and coalgebras over the functor B^Q, the $|Q|$-th direct power of the functor B. In particular (Corollary 6.4), quasigroup homogeneous spaces become B^Q-coalgebras. Section 7 examines the irreducibility of B^Q-coalgebras that are homomorphic images of homogeneous spaces. Section 8 notes that for quasigroups in general, in contrast with the group case, the homogeneous spaces do not necessarily arise as homomorphic images of the regular representation. Permutation representations or Q-sets over a finite quasigroup Q are then specified in Definition 9.1 as elements of the covariety of B^Q-coalgebras generated by the set of homogeneous spaces over Q.

[1]The technical reason was as follows: if P_i is a subloop of a loop Q_i for $i = 1, 2$, then the relative left multiplication group of $P_1 \times P_2$ in $Q_1 \times Q_2$ is a direct product of the relative left multiplication groups of P_1 and P_2. This property may break down in the absence of identity elements.

The main Structure Theorem 9.2 shows that general Q-sets are sums of homomorphic images of homogeneous spaces. Corollary 9.3 observes that there are only finitely many irreducible Q-sets (to within isomorphism). Corollary 9.4 notes that if Q is a finite group, then its quasigroup Q-sets in the present sense coincide with its Q-sets in the usual group sense. In the final section, Burnside's Lemma is extended from groups to quasigroups (Theorem 10.2). It may be of interest to contrast the proof of this generalisation with the usual combinatorial proofs for the group case.

For algebraic concepts and notations used in this paper, readers are referred to [18]. In particular, mappings are generally placed in the natural position on the right of their arguments, either in line or as an index. These conventions help to minimise the number of brackets, which otherwise proliferate in the study of non-associative systems such as quasigroups.

2. Quasigroup homogeneous spaces

The construction of a quasigroup homogeneous space for a finite quasigroup [15, 16] is analogous to the transitive permutation representation of a group Q (with stabiliser subgroup P) on the homogeneous space

$$P \setminus Q = \{Px \mid x \in Q\} \tag{2.1}$$

by the actions

$$R_{P\setminus Q}(q)\colon P \setminus Q \to P \setminus Q\,;\, Px \mapsto Pxq \tag{2.2}$$

for elements q of Q. Let P be a subquasigroup of a finite quasigroup Q. Let L be the relative left multiplication group of P in Q. Let $P \setminus Q$ be the set of orbits of the permutation group L on the set Q. If Q is a group, and P is nonempty, then this notation is consistent with (2.1). Let A be the incidence matrix of the membership relation between the set Q and the set $P\setminus Q$ of subsets of Q. Let A^+ be the pseudoinverse [9] of the matrix A. For each element q of Q, right multiplication in Q by q yields a permutation of Q. Let $R_Q(q)$ be the corresponding permutation matrix. Define a new matrix

$$R_{P\setminus Q}(q) = A^+ R_Q(q) A\,. \tag{2.3}$$

[In the group case, the matrix (2.3) is just the permutation matrix given by the permutation (2.2).] Then, in the homogeneous space of the quasigroup Q, each quasigroup element q yields a Markov chain on the state space $P\setminus Q$ with transition matrix $R_{P\setminus Q}(q)$ given by (2.3). The set of convex combinations of the states from $P \setminus Q$ forms a complete metric space, and the actions (2.3) of the quasigroup elements form an iterated function system or IFS in the sense of fractal geometry [1].

3. The IFS category

Let Q be a finite set. Define a Q-IFS (X, Q) as a finite set X together with an *action map* R or

$$R_X : Q \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}X); q \mapsto R_X(q) \tag{3.1}$$

from Q to the set of endomorphisms of the complex vector space with basis X (identified with their matrices with respect to the basis X), such that each *action matrix* $R_X(q)$ is stochastic. For Q non-empty, the *Markov matrix* of a Q-IFS (X, Q) is the arithmetic mean

$$M_{(X,Q)} = \frac{1}{|Q|} \sum_{q \in Q} R_X(q) \tag{3.2}$$

of the action matrices of the elements of Q. A Q-IFS (X, Q) is said to be *irreducible* if the Markov chain on X given by (3.2) is irreducible (cf. XV§4 of [3]).

If P is a subquasigroup of a finite non-empty quasigroup Q, then the homogeneous space $P \setminus Q$ is a Q-IFS with the action map specified by (2.3). Each row of the Markov matrix of the Q-IFS $P \setminus Q$ takes the form

$$(|P_1|/|Q|, \dots, |P_r|/|Q|), \tag{3.3}$$

where $P_1, \dots, P_r$ are the orbits of the relative left multiplication group of P in Q. (Compare [17], Proposition 8.1, where this result was formulated for a loop Q. The proof given there applies to an arbitrary non-empty quasigroup Q.)

For a finite set Q, a *morphism*

$$\phi \colon (X, Q) \to (Y, Q) \tag{3.4}$$

from a Q-IFS (X, Q) to a Q-IFS (Y, Q) is a function $\phi \colon X \to Y$, whose graph has incidence matrix F, such that

$$R_X(q)F = FR_Y(q) \tag{3.5}$$

for each element q of Q. It is readily checked that the class of morphisms (3.4), for a fixed set Q, forms a concrete category $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$. For a group Q, it was shown in [17] that the category of finite Q-sets forms the full subcategory of $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$ consisting of those objects for which the action map (3.1) is a monoid homomorphism. Moreover, a Q-IFS (X, Q) is a Q-set if and only if it is isomorphic to a Q-set (Y, Q) in $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$.

For a fixed finite quasigroup Q, the category $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$ has finite products and coproducts. Consider objects (X, Q) and (Y, Q) of $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$. Their *disjoint union* $(X + Y, Q)$ consists of the disjoint union $X + Y$ of the sets X and Y together with the action map

$$q \mapsto R_X(q) \oplus R_Y(q) \tag{3.6}$$

sending each element q of Q to the direct sum of the matrices $R_X(q)$ and $R_Y(q)$. One obtains an object of $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$, since the direct sum of stochastic matrices is stochastic.

The *direct product* $(X \times Y, Q)$ of (X, Q) and (Y, Q) is the direct product $X \times Y$ of the sets X and Y together with the action map

$$q \mapsto R_X(q) \otimes R_Y(q) \tag{3.7}$$

sending each element q of Q to the tensor (or Kronecker) product of the matrices $R_X(q)$ and $R_Y(q)$. Again, one obtains an object of $\underline{\underline{\mathrm{IFS}}}_Q$, since the tensor product of stochastic matrices is stochastic. It is straightforward to check that the disjoint union, equipped with the appropriate insertions, yields a coproduct in $\underline{\underline{\mathrm{IFS}}}_Q$, and that the direct product, equipped with the appropriate projections, yields a product in $\underline{\underline{\mathrm{IFS}}}_Q$.

4. Coalgebras and covarieties

This section summarises the basic coalgebraic concepts required for the paper. For more details, readers may consult [4], [5] or [12]. Crudely speaking, coalgebras are just the duals of algebras: coalgebras in a category $\mathcal{C}$ are algebras in the dual category $\mathcal{C}^{\mathrm{op}}$.

Let $F\colon \underline{\underline{\mathrm{Set}}} \to \underline{\underline{\mathrm{Set}}}$ be an endofunctor on the category of sets and functions. Then an F-*coalgebra*, or simply a *coalgebra* if the endofunctor is implicit in the context, is a set X equipped with a function α_X or $\alpha\colon X \to XF$. This function is known as the *structure map* of the coalgebra X. (Of course, for complete precision, one may always denote a coalgebra by its structure map.) A function $f\colon X \to Y$ between coalgebras is a *homomorphism* if $f\alpha_Y = \alpha_X f^F$. A subset S of a coalgebra X is a *subcoalgebra* if it is itself a coalgebra such that the embedding of S in X is a homomorphism. A coalgebra Y is a *homomorphic image* of a coalgebra X if there is a surjective homomorphism $f\colon X \to Y$.

Let $(X_i \mid i \in I)$ be a family of coalgebras. Then the *sum* of this family is the disjoint union of the sets of the family, equipped with a coalgebra structure map α given as follows. Let $\iota_i\colon X_i \to X$ insert X_i as a summand in the disjoint union X of the family. For each i in I, let α_i be the structure map of X_i. Then the restriction of α to the subset X_i of X is given by $\alpha_i \iota_i^F$. (More generally, the forgetful functor from coalgebras to sets creates colimits — cf. Proposition 1.1 of [2].)

A *covariety* of coalgebras is a class of coalgebras closed under the operations H of taking homomorphic images, S of taking subalgebras, and Σ of taking sums. (Note that homomorphic images are dual to subalgebras, while sums are dual to products.) If $\mathcal{K}$ is a class of F-coalgebras, then the smallest covariety containing $\mathcal{K}$ is given by $\mathrm{SH}\Sigma(\mathcal{K})$ (cf. Theorem 7.5 of [4] or Theorem 3.3 of [5]). This result is dual to the well-known characterisation of the variety generated by a class of algebras (cf. e.g., Exercise 2.3A of Chapter IV of [18] or Proposition 1.5.12 of [11]).

5. Barycentric algebras

This section briefly outlines the basic facts about barycentric algebras that are used in the paper. For more details, readers may consult [10] or [11]. Let I° denote the open unit interval $]0,1[$. For p in I°, define $p' = 1 - p$.

Definition 5.1. A *barycentric algebra* A or (A, I°) is an algebra of type $I^\circ \times \{2\}$, equipped with a binary operation

$$\underline{p} : A \times A \to A; (x,y) \mapsto xy\,\underline{p}$$

for each p in I°, satisfying the identities

$$xx\,\underline{p} = x \tag{5.1}$$

of *idempotence* for each p in I°, the identities

$$xy\,\underline{p} = yx\,\underline{p'} \tag{5.2}$$

of *skew-commutativity* for each p in I°, and the identities

$$xy\,\underline{p}\;z\,\underline{q} = x\;yz\,\underline{q/(p'q')'}\;\underline{(p'q')'} \tag{5.3}$$

of *skew-associativity* for each p, q in I°. The variety of all barycentric algebras, construed as a category with the homomorphisms as morphisms, is denoted by $\underline{\underline{\mathrm{B}}}$. The corresponding free algebra functor is $B\colon \underline{\underline{\mathrm{Set}}} \to \underline{\underline{\mathrm{B}}}$.

A convex set C forms a barycentric algebra (C, I°), with $xy\,\underline{p} = (1-p)x + py$ for x, y in C and p in I°. A semilattice $(S, \cdot)$ becomes a barycentric algebra on setting $xy\,\underline{p} = x \cdot y$ for x, y in S and p in I°.

For the following result, see [8], §2.1 of [10], §5.8 of [11]. The equivalence of the final two structures in the theorem corresponds to the identification of the barycentric coordinates in a simplex with the weights in finite probability distributions.

Theorem 5.2. *Let X be a finite set. The following structures are equivalent:*

(a) *The free barycentric algebra XB on X;*
(b) *The simplex spanned by X;*
(c) *The set of all probability distributions on X.*

6. Actions as coalgebras

Definition 6.1. Let Q be a finite set. The functor $B^Q\colon \underline{\underline{\mathrm{Set}}} \to \underline{\underline{\mathrm{Set}}}$ sends a set X to the set XB^Q of functions from Q to the free barycentric algebra XB over X. For a function $f\colon X \to Y$, its image under the functor B^Q is the function $fB^Q\colon XB^Q \to YB^Q$ defined by

$$fB^Q\colon (Q \to XB; q \mapsto w) \mapsto (Q \to YB; q \mapsto wf^B).$$

Some standard “coalgebraic” properties of the functor B^Q are listed for reference in the following proposition (cf. [19], where the functor B is denoted by $\mathcal{D}$).

Proposition 6.2. *Let Q be a finite set.*

(a) *The functor B^Q preserves weak pullbacks.*
(b) *The functor B^Q is bounded.*
(c) *Each covariety of B^Q-coalgebras is bicomplete.*

Proof. (a) By Appendix A of [19], the functor B preserves weak pullbacks. Thus the finite power B^Q of B also preserves weak pullbacks (cf. Lemma 8.11 of [4]).
(b) See the proof of Theorem 4.6 of [19].
(c) Since B^Q is bounded, the result follows according to §7.4 of [4]. □

Theorem 6.3. *Let Q be a finite set. Then the category $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$ is isomorphic with the category of finite B^Q-coalgebras.*

Proof. Given a Q-IFS (X,Q) with action map R as in (3.1), define a B^Q-coalgebra $L_X\colon X \to XB^Q$ with structure map

$$L_X\colon X \to XB^Q; x \mapsto (Q \to XB; q \mapsto xR_X(q)). \tag{6.1}$$

(Note the use of Theorem 5.2 interpreting the vector $xR_X(q)$, lying in the simplex spanned by X, as an element of XB.) Given a Q-IFS morphism $\phi\colon (X,Q) \to (Y,Q)$ as in (3.4), with incidence matrix F, one has

$$xL_X.\phi B^Q\colon Q \to YB; q \mapsto xR_X(q)F \tag{6.2}$$

for each x in X, by Definition 6.1. On the other hand, one also has

$$x\phi L_Y\colon Q \to YB; q \mapsto xFR_Y(q). \tag{6.3}$$

By (3.5), it follows that the maps (6.2) and (6.3) agree. Thus $\phi\colon X \to Y$ is a coalgebra homomorphism. These constructions yield a functor from $\underline{\underline{\underline{\mathrm{IFS}}}}_Q$ to the category of finite B^Q-coalgebras.

Conversely, given a finite B^Q-coalgebra with structure map $L_X\colon X \to XB^Q$, define a Q-IFS (X,Q) with action map

$$R_X\colon Q \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}X); q \mapsto (x \mapsto qL_X(x)), \tag{6.4}$$

well-defined by Theorem 5.2. Let $\phi\colon X \to Y$ be a coalgebra homomorphism with incidence matrix F. Then the maps (6.2) and (6.3) agree for all x in the basis X of $\mathbb{C}X$, whence (3.5) holds and $\phi\colon (X,Q) \to (Y,Q)$ becomes a Q-IFS morphism. In this way one obtains mutually inverse functors between the two categories. □

Corollary 6.4. *Each homogeneous space over a finite quasigroup Q yields a B^Q-coalgebra.*

Corollary 6.5. *Let Q be a finite group. Then the category of finite Q-sets embeds faithfully as a full subcategory of the category of B^Q-coalgebras.*

Proof. Apply Theorem 6.3, and Proposition 4.2 of [17]. □

7. Reachability

Definition 7.1. Let Q be a finite set. Let Y be a B^Q-coalgebra with structure map $L\colon Y \to YB^Q$. For elements y, y' of Y, the element y' is said to be *reachable* from y in Y if there is an element q of Q such that y' appears in the support of the distribution $qL(y)$ on Y. The *reachability graph* of Y is the directed graph of the reachability relation on Y. The coalgebra Y is said to be *irreducible* if its reachability graph is strongly connected.

Proposition 7.2. *If $P \setminus Q$ is a homogeneous space over a finite quasigroup Q, realised as a B^Q-coalgebra according to Corollary* 6.4*, then $P \setminus Q$ is irreducible.*

Proof. Let H be the relative left multiplication group of P in Q. For an arbitrary pair x, x' of elements of Q, consider the corresponding elements xH and $x'H$ of $P \setminus Q$. For $q = x \setminus x'$ in Q, the element $x'H$ then appears in the support of $qL(xH)$. □

Corollary 7.3. *Let Q be a finite quasigroup. Suppose that Y is a B^Q-coalgebra that is a homomorphic image of a homogeneous space S over Q. Then Y is irreducible.*

Proof. Since S and Y are finite, one may use the correspondence of Theorem 6.3. Let $\phi\colon S \to Y$ be the homomorphism, with incidence matrix F. Consider elements y and y' of Y. Suppose x and x' are elements of S with $x\phi = y$ and $x'\phi = y'$. By Proposition 7.2, there is an element q of Q with x' in the support of the distribution $xR_S(q)$. Then $yR_Y(q) = xFR_Y(q) = xR_S(q)F$, so the support of $yR_Y(q)$, as the image of the support of $xR_S(q)$ under ϕ, contains $x'\phi = y'$. □

8. Regular representations

For a quasigroup Q, the *regular* homogeneous space or permutation representation is the homogeneous space (Q, Q) or $(\varnothing \setminus Q, Q)$. (Note that the relative left multiplication group of the empty subquasigroup is trivial. If Q is a loop with identity element e, then the regular homogeneous space may also be described as $(\{e\} \setminus Q, Q)$. This definition was used in §7 of [17].) For a group Q, each homogeneous space $(P \setminus Q, Q)$ is obtained as a homomorphic image of the regular permutation representation. The following considerations show that the corresponding property does not hold for general quasigroups.

Definition 8.1. Let Q be a finite set. A Q-IFS (X, Q) is said to be *crisp* if, for each q in Q, the action matrix $R_X(q)$ is a 0-1-matrix. A B^Q-coalgebra $L\colon X \to XB^Q$ is said to be *crisp* if its structure map corestricts to $L\colon X \to X^Q$.

Note that crisp Q-IFS and finite crisp B^Q-coalgebras correspond under the isomorphism of Theorem 6.3.

Proposition 8.2. *Homomorphic images of finite crisp B^Q-coalgebras are crisp.*

Proof. Using Theorem 6.3, it is simpler to work in the category $\underline{\underline{\mathrm{IFS}}}_Q$. Let $\phi\colon X \to Y$ be a surjective $\underline{\underline{\mathrm{IFS}}}_Q$-morphism with incidence matrix F and crisp domain. For an element y of Y, suppose that x is an element of X with $x\phi = y$. Then for each element q of Q, one has $yR_Y(q) = x\phi R_Y(q) = xFR_Y(q) = xR_X(q)F$, using (3.5) for the last step. Since X is crisp, there is an element x' of X with $xR_X(q) = x'$. Then $yR_Y(q) = x'F = y'$ for the element $y' = x'\phi$ of Y. Thus Y is also crisp. □

For each finite quasigroup Q, the regular permutation representation is crisp. On the other hand, if Q is non-associative, then Q may have homogeneous spaces which are not crisp (cf. §3 of [15] or (6.2) of [17]). Proposition 8.2 shows that such spaces are not homomorphic images of the regular representation.

9. The covariety of Q-sets

Definition 9.1. Let Q be a finite quasigroup. Then the *category* $\underline{\underline{Q}}$ *of Q-sets* or of *permutation representations* of Q is defined to be the covariety of B^Q-coalgebras generated by the (finite) set of homogeneous spaces over Q.

For a finite quasigroup Q, the terms "Q-set" or "permutation representation of Q" are used for objects of the category of Q-sets, and also for those Q-IFS which correspond to finite Q-sets via Theorem 6.3. (For a finite loop Q, these terms were used in a different, essentially broader sense — at least for the finite case — in Definition 5.2 of [17]. If necessary, one may refer to "loop Q-sets" in that context, and to "proper Q-sets" or "quasigroup Q-sets" in the present context.)

Theorem 9.2. *For a finite quasigroup Q, the Q-sets are precisely the sums of homomorphic images of homogeneous spaces.*

Proof. Let $\mathcal{H}$ be the set of homogeneous spaces over Q. By Proposition 2.4 of [6], the covariety generated by $\mathcal{H}$ is $\mathrm{HS}\Sigma(\mathcal{H})$. By Proposition 2.5 of [6], the operators S and Σ commute. By Proposition 7.2, the homogeneous spaces do not contain any proper, non-empty subcoalgebras. Thus the covariety generated by $\mathcal{H}$ becomes $\mathrm{H}\Sigma(\mathcal{H})$. By Proposition 2.4(iii) of [6], one has $\Sigma\mathrm{H}(\mathcal{H}) \subseteq \mathrm{H}\Sigma(\mathcal{H})$. It thus remains to be shown that each homomorphic image of a sum of homogeneous spaces is a sum of homomorphic images of homogeneous spaces.

Let Y be a Q-set, with structure map L_Y, that is a homomorphic image of a sum X of homogeneous spaces under a homomorphism ϕ. It will first be shown that each element y of Y lies in a subcoalgebra Y_y of Y that is a homomorphic image of a homogeneous space. Since y lies in the image Y of X under ϕ, there is an element x of X such that $x\phi = y$. Since X is a sum of homogeneous spaces, the element x lies in such a space S. Consider the restriction of ϕ to S. Let Y_y be the image of this restriction. Then Y_y is a subcoalgebra of Y that is a homomorphic image of a homogeneous space (cf. Lemma 4.5 of [4]).

Now suppose that for elements y and z of Y, the corresponding images Y_y and Y_z of homogeneous spaces intersect non-trivially, say with a common element t. By Corollary 7.3, there is an element q of Q such that z lies in the support of $qL_Y(t)$. On the other hand, since t lies in the subcoalgebra Y_y, the support of the distribution $qL_Y(t)$ lies entirely in Y_y. Thus z is an element of Y_y, and for each q in Q, the support of the distribution $qL_Y(z)$ lies entirely in Y_y. It follows that Y_z is entirely contained in Y_y. Similarly, one finds that Y_y is contained in Y_z, and so the two images agree. Thus Y is a sum of such images. □

Corollary 9.3. *A finite quasigroup Q has only finitely many isomorphism classes of irreducible Q-sets.*

Proof. By Theorem 9.2, the irreducible Q-sets are precisely the homomorphic images of homogeneous spaces. Since Q is finite, it has only finitely many homogeneous spaces. The (First) Isomorphism Theorem for coalgebras (cf. Theorem 4.15 of [4]) then shows that each of these homogeneous spaces has only finitely many isomorphism classes of homomorphic images. □

Corollary 9.4. *For a finite group Q, the quasigroup Q-sets coincide with the group Q-sets.*

Proof. For a group Q, each homomorphic image of a homogeneous space is isomorphic to a homogeneous space, and each group Q-set is isomorphic to a sum of homogeneous spaces. □

In considering the final corollary of Theorem 9.2, recall that the intersection of a family of subcoalgebras of a coalgebra is not necessarily itself a subcoalgebra (cf. Corollary 4.9 of [4]).

Corollary 9.5. *Let y be an element of a Q-set Y over a finite quasigroup Q. Then the intersection of the subcoalgebras of Y containing y is itself a subcoalgebra of Y.*

Proof. In the notation of the proof of Theorem 9.2, this intersection is the subcoalgebra Y_y. □

10. Burnside's Lemma

Definition 10.1. For a Q-set Y over a finite quasigroup Q, the irreducible summands of Y given by Theorem 9.2 are called the *orbits* of Y. For an element y of Y, the smallest subcoalgebra of Y containing y (guaranteed to exist by Corollary 9.5) is called the *orbit* of the element y.

Burnside's Lemma concerns itself with finite permutation representations. In the quasigroup case, its formulation (and proof) rely on the identification given by Theorem 6.3. Recall that the classical Burnside Lemma for a finite group Q (cf. e.g., Theorem 3.1.2 in Chapter I of [18]) states that the number of orbits in a finite Q-set X is equal to the average number of points of X fixed by elements q of Q. The number of points fixed by such an element q is equal to the trace of the permutation matrix of q on X. In the IFS terminology of §3, this permutation matrix is the action matrix $R_X(q)$ of q on the corresponding Q-IFS (X, Q). Thus the following theorem does specialise to the classical Burnside Lemma in the associative case.

Theorem 10.2 (Burnside's Lemma for quasigroups)**.** *Let X be a finite Q-set over a finite, non-empty quasigroup Q. Then the trace of the Markov matrix of X is equal to the number of orbits of X.*

Proof. Consider the Q-IFS (X, Q). By Theorem 6.3, Theorem 9.2 and (3.6), its Markov matrix decomposes as a direct sum of the Markov matrices of its orbits. Thus it suffices to show that the trace of the Markov matrix of a homomorphic image of a homogeneous space is equal to 1.

Consider a Q-set $Y = \{y_1, \dots, y_m\}$ which is the image of a homogeneous space $P \setminus Q$ under a surjective homomorphism $\phi\colon P \setminus Q \to Y$ with incidence matrix F. Let F^+ be the pseudoinverse of F. Note that each row sum of F^+ is 1. Suppose that the Markov matrix Π of $P \setminus Q$ is given by (3.3). By (3.5), one has

$$R_Y(q) = F^+ R_{P\setminus Q}(q) F$$

for each q in Q. Thus the trace of the Markov matrix of Y is given by

$$\begin{aligned} tr(F^+ \Pi F) &= \sum_{i=1}^{m} \sum_{j=1}^{r} \sum_{k=1}^{r} F^+_{ij} \Pi_{jk} F_{ki} \\ &= |Q|^{-1} \sum_{i=1}^{m} \Big(\sum_{j=1}^{r} F^+_{ij} \Big) \Big(\sum_{k=1}^{r} |P_k| F_{ki} \Big) \\ &= |Q|^{-1} \sum_{k=1}^{r} |P_k| = 1, \end{aligned}$$

the penultimate equality following since for each $1 \le k \le r$, there is exactly one index i (corresponding to $P_k\phi = y_i$) such that $F_{ki} = 1$, the other terms of this type vanishing. □

References

[1] M. F. Barnsley, *Fractals Everywhere,* Academic Press, San Diego, CA, 1988.

[2] M. Barr, *Terminal coalgebras in well-founded set theory*, Theoret. Comput. Sci. **114** (1993), 299–315.

[3] W. Feller, *An Introduction to Probability Theory and its Applications*, Wiley, New York, NY, 1961.

[4] H.-P. Gumm, *Elements of the general theory of coalgebras*, LUATCS'99, Rand Afrikaans University, Johannesburg, 1999.

[5] H.-P. Gumm, *Birkhoff's variety theorem for coalgebras*, Contributions to General Algebra **13** (2001), 159–173.

[6] H.-P. Gumm and T. Schröder, *Covarieties and complete covarieties*, in *Coalgebraic Methods in Computer Science*, (eds. B. Jacobs et al.), Electronic Notes in Theoretical Computer Science, vol. 11, Elsevier Science, 1998, pp. 43–56.

[7] K. W. Johnson, *Some historical aspects of the representation theory of groups and its extension to quasigroups*, in *Universal Algebra and Quasigroup Theory*, (eds. A. Romanowska and J. D. H. Smith), Heldermann, Berlin, 1992, pp. 101–117.

[8] W. D. Neumann, *On the quasivariety of convex subsets of affine spaces*, Arch. Math. **21** (1970), 11–16.

[9] R. Penrose, *A generalised inverse for matrices*, Proc. Camb. Phil. Soc. **51** (1955), 406–413.

[10] A. B. Romanowska and J. D. H. Smith, *Modal Theory*, Heldermann, Berlin, 1985.

[11] A. B. Romanowska and J. D. H. Smith, *Modes*, World Scientific, Singapore, 2002.

[12] J. J. M. M. Rutten, *Universal coalgebra: a theory of systems*, Theoret. Comput. Sci. **249** (2000), 3–80.

[13] J. D. H. Smith, *Representation Theory of Infinite Groups and Finite Quasigroups*, Université de Montréal, Montreal, 1986.

[14] J. D. H. Smith, *Combinatorial characters of quasigroups*, in *Coding Theory and Design Theory, Part I: Coding Theory*, (ed. D. Ray-Chaudhuri), Springer, New York, NY, 1990, pp. 163–187.

[15] J. D. H. Smith, *Quasigroup actions: Markov chains, pseudoinverses, and linear representations*, Southeast Asia Bulletin of Mathematics **23** (1999), 719–729.

[16] J. D. H. Smith, *Quasigroup homogeneous spaces and linear representations*, J. Alg. **241** (2001), 193–203.

[17] J. D. H. Smith, *Permutation representations of loops*, Preprint, 2001.

[18] J. D. H. Smith and A. B. Romanowska, *Post-Modern Algebra*, Wiley, New York, NY, 1999.

[19] E. P. de Vink and J. J. M. M. Rutten, *Bisimulation for probabilistic transition systems: a coalgebraic approach*, Theoret. Comput. Sci. **221** (1999), 271–293.

Jonathan D. H. Smith
Department of Mathematics, Iowa State University, Ames, Iowa 50011, U.S.A.
e-mail: jdhsmith@math.iastate.edu
URL: `http://www.math.iastate.edu/jdhsmith/`