Arch. Math., Vol. 59, 327–333 (1992)

0003-889X/92/5904-0327 $ 2.90/0

Comtrans algebras and bilinear forms

By

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1. Introduction. Comtrans algebras were introduced [9] in answer to a problem from differential geometry [2, Problem X.3.9] [6, p. 16]: finding the algebraic structure in the tangent bundle corresponding to the coordinate n-ary loop of an $(n+1)$-web [3, §3.7]. The algebraic structure consists of a system of comtrans algebras, interlaced with some of the "W-algebras" (now known as Akivis algebras) that had been introduced earlier by Akivis in correspondence with the coordinate binary loops of 3-webs [1] [2, §IX.6] [4]. The current paper is part of a programme (cf. [7], [8]) beginning a study of comtrans algebras from a purely algebraic point of view. It was noted in [9, Remark 3.1 (ii)] that a comtrans algebra arises from the repeated commutator [[,],] of a Lie algebra. For the Lie algebra of Euclidean space $\mathbb{R}^3$ under the "vector" or "cross" product $\times$, this repeated product is the "vector triple product"

$$(1.1)\qquad (\underset{\sim}{x} \times \underset{\sim}{y}) \times \underset{\sim}{z} = \underset{\sim}{y}(\underset{\sim}{x} \cdot \underset{\sim}{z}) - \underset{\sim}{x}(\underset{\sim}{y} \cdot \underset{\sim}{z}).$$

One could thus regard the vector triple product comtrans algebra as arising from the Euclidean inner product on $\mathbb{R}^3$ according to the right hand side of (1.1), rather than from the repeated Lie algebra commutator appearing on the left. The main topic of the present paper is a generalization (3.1–2) of this construction, producing a comtrans algebra $\mathrm{CT}(E, \beta)$ from a pair (E, β) consisting of a unital module E over a commutative ring R with 1 and a bilinear form $\beta: E^2 \to R$. A "transposed" comtrans algebra $\mathrm{CT}(E, \beta)^\tau$ is also given by the pair (E, β) (3.3–4). These constructions compare with the currently popular methods of making algebras out of spaces with forms, such as Jordan algebras or Clifford algebras. One major advantage of the comtrans algebras $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \beta)^\tau$ is that they do not require any extension of the underlying module E in order to achieve closure. By contrast, the underlying modules of Clifford algebras (for example) blow up exponentially in size.

Section 2 presents the definition of comtrans algebras (2.1–5), and covers some elementary topics that are needed, such as the notions of ideal, abelian algebras, and simple algebras. The transposition relationship between a pair of comtrans algebras, typified by $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \beta)^\tau$, is also described (2.6–7). The third section gives the basic construction of the comtrans algebras $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \beta)^\tau$ from a module (E, β) with a bilinear form. For tight connections between the form and the algebras, some restrictions on the underlying module E are required. Appropriate restrictions are encoded in the concept of "formed space" (Definition 3.3), making the underlying module free of rank more than 1. Theorem 3.4 shows how simplicity of the comtrans algebras is equivalent to non-degeneracy of the form and simplicity of the ring of scalars. In general, the radical of a bilinear form on a formed space may be described in comtrans algebra terms (Proposition 3.5). Theorem 3.6 and its corollary show that the automorphism groups of the formed space (E, β) and of the comtrans algebras $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \beta)^\tau$ coincide. The fourth section is concerned with the problem of recognizing when a comtrans algebra is a "form algebra", i.e. $\mathrm{CT}(E, \beta)$ or $\mathrm{CT}(E, \beta)^\tau$ for a formed space (E, β). The answer is given by Theorem 4.1. Consideration of the hyperbolic plane (Example 4.2) shows that the two-dimensional case is anomalous.

2. Comtrans algebras. Let R be a commutative ring with 1. A *comtrans algebra* over R is a unital R-module E equipped with a trilinear operation

(2.1) $$[\,,\,]: E^3 \to E;\ (x, y, z) \mapsto [x, y, z]$$

called the *commutator* and a trilinear operation

(2.2) $$\langle\,,\,\rangle: E^3 \to E;\ (x, y, z) \mapsto \langle x, y, z\rangle$$

called the *translator*. The commutator is *left alternative*, in the sense that

(2.3) $$\forall x, z \in E,\ [x, x, z] = 0.$$

The translator satisfies the *Jacobi identity*:

(2.4) $$\forall x, y, z \in E,\ \langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0.$$

Finally, the commutator and translator together satisfy the *comtrans identity*:

(2.5) $$\forall x, y \in E,\ [x, y, x] = \langle x, y, x\rangle.$$

A submodule J of a comtrans algebra E is said to be an *ideal* (notation $J \lhd E$) if, for all j in J and x, y in E, the elements $[y, x, j]$, $\langle y, x, j\rangle$ and $\langle j, x, y\rangle$ of E lie in J. A submodule J of E is an ideal iff it is the kernel of the underlying module homomorphism of a comtrans algebra homomorphism with domain E [7, Prop. 3.1].

A comtrans algebra E is *abelian* if its commutator and translator are identically zero. If the underlying module of a comtrans algebra E is cyclic, then (2.3) and (2.5), along with the trilinearity of the commutator and translator, show that the algebra E is abelian. Note that, for an abelian comtrans algebra E, each submodule of E forms an ideal of E. At the opposite extreme, a comtrans algebra is said to be *simple* if it is non-abelian, and if it has no proper non-trivial ideals.

Given a comtrans algebra E with commutator $[\,,\,]$ and translator $\langle\,,\,\rangle$, a new comtrans algebra E^τ, called the *transpose of* E, may be formed by equipping the underlying module E with a new commutator

(2.6) $$[x, y, z]^\tau = [z, y, x] + \langle y, z, x\rangle$$

and a new translator

(2.7) $$\langle x, y, z\rangle^\tau = -\langle x, z, y\rangle$$

for x, y, z in E. Note that $E^{\tau\tau} = E$. The algebras E and E^{τ} are "term equivalent" in the sense of universal algebra, so they have the same (congruences and) ideals [10, p. 13]. Moreover, given comtrans algebras E and F, a module homomorphism $f: E \to F$ is a comtrans algebra homomorphism $f: E \to F$ iff $f: E^{\tau} \to F^{\tau}$ is a comtrans algebra homomorphism.

3. Forms and algebras. A *formed module* (E, β) is a unital module E over a commutative ring R with 1, equipped with a bilinear form $\beta: E \times E \to R$. Associated with (E, β) is the comtrans algebra $\mathrm{CT}(E, \beta)$ having commutator

(3.1) $$[x, y, z] = y\beta(x, z) - x\beta(y, z)$$

and translator

(3.2) $$\langle x, y, z \rangle = y\beta(z, x) - x\beta(y, z).$$

There is also the transposed comtrans algebra $\mathrm{CT}(E, \beta)^{\tau}$ with commutator

(3.3) $$[x, y, z]^{\tau} = z(\beta(x, y) - \beta(y, x))$$

and translator

(3.4) $$\langle x, y, z \rangle^{\tau} = x\beta(z, y) - z\beta(y, x).$$

The *radical* of (E, β) is the submodule

(3.5) $$\mathrm{Rad}\,\beta = \{x \in E \mid \forall y \in E,\ \beta(x, y) = \beta(y, x) = 0\}.$$

Proposition 3.1. *The radical of* (E, β) *is an ideal of* $\mathrm{CT}(E, \beta)$ *and* $\mathrm{CT}(E, \beta)^{\tau}$.

Proof. Since the algebras $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \beta)^{\tau}$ are term equivalent, they have the same ideals [10, p. 13]. It thus suffices to prove $\mathrm{Rad}\,\beta \lhd \mathrm{CT}(E, \beta)$. Consider an element j of $\mathrm{Rad}\,\beta$. Then for x, y in E, one has $[j, x, y] = \langle j, x, y \rangle = -j\beta(x, y) \in \mathrm{Rad}\,\beta$ and $\langle y, x, j \rangle = 0$, as required. □

Corollary 3.2. *If either* $\mathrm{CT}(E, \beta)$ *or* $\mathrm{CT}(E, \beta)^{\tau}$ *is simple, then* $\mathrm{Rad}\,\beta = \{0\}$.

Proof. By the term equivalence of $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \beta)^{\tau}$, it suffices to consider $\mathrm{CT}(E, \beta)$ alone. If $\mathrm{CT}(E, \beta)$ were simple with $\mathrm{Rad}\,\beta > \{0\}$, Proposition 3.1 would imply $\mathrm{Rad}\,\beta = E$, whence $\beta = 0$ and the contradiction that $\mathrm{CT}(E, \beta)$ would be abelian. □

If the underlying module E of a comtrans algebra is cyclic, the algebra is necessarily abelian. In this case, it is clear that all bilinear forms β on E yield the same comtrans algebra $\mathrm{CT}(E, \beta)$. For tighter connections between formed modules (E, β) and the comtrans algebras $\mathrm{CT}(E, \beta)$ or $\mathrm{CT}(E, \beta)^{\tau}$, some restrictions on the possible modules E are required. The following definition serves to impose such restrictions.

Definition 3.3. A *formed space* (E, β) is a formed module whose underlying module E is free of rank more than 1.

The simplicity of the comtrans algebra $\mathrm{CT}(E, \beta)$ of a formed space may be characterized quite sharply.

Theorem 3.4. *Let (E, β) be a formed space over a commutative ring R with 1. Then the algebras $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \beta)^{\tau}$ are simple if and only if $\mathrm{Rad}\,\beta = \{0\}$ and R is a field.*

Proof. To begin, suppose that $\mathrm{CT}(E, \beta)$ is simple. By Corollary 3.2, $\mathrm{Rad}\,\beta = \{0\}$. Suppose that R is not a field, so that it has a proper non-zero ideal I. Since E is free of positive rank, the submodule IE is proper and non-trivial. One obtains the contradiction $IE \lhd \mathrm{CT}(E, \beta)$. Consider an element $j = i_1 x_1 + \cdots + i_n x_n$ of IE with $i_1, \ldots, i_n \in I$ and $x_1, \ldots, x_n \in E$. Then for y, z in E one has $[j, y, z] = y\beta(j, z) - j\beta(y, z) = i_1 y\beta(x_1, z) + \cdots + i_n y\beta(x_n, z) - j\beta(y, z) \in IE$. Similarly $\langle j, y, z\rangle \in IE$ and $\langle z, y, j\rangle \in IE$, so that $IE \lhd \mathrm{CT}(E, \beta)$.

Conversely, suppose that R is a field and $\mathrm{Rad}\,\beta = \{0\}$. Let j be a non-zero element of a non-trivial ideal J of $\mathrm{CT}(E, \beta)$. Since $j \notin \mathrm{Rad}\,\beta$, there is an element y of E such that $\beta(j, y) \neq 0$ or $\beta(y, j) \neq 0$. Consider an arbitrary element x of E. If $\beta(j, y) \neq 0$, one has $x = \beta(j, y)^{-1}([j, x, y] + j\beta(x, y)) \in J$. If $\beta(y, j) \neq 0$, one has $x = \beta(y, j)^{-1}(\langle j, x, y\rangle + j\beta(x, y)) \in J$. Thus J is improper: the only ideals of $\mathrm{CT}(E, \beta)$ are $\{0\}$ and E. Since $\dim_R E > 1$, there is a proper non-trivial subspace K of E. If $\mathrm{CT}(E, \beta)$ were abelian, then K would be an ideal. Thus $\mathrm{CT}(E, \beta)$ is non-abelian. □

The radical of a formed space (E, β) is determined by the comtrans algebra $\mathrm{CT}(E, \beta)$.

Proposition 3.5. *Let (E, β) be a formed space, with corresponding comtrans algebra $\mathrm{CT}(E, \beta)$. Then*

$$\mathrm{Rad}\,\beta = \{x \in E \mid \forall y, z \in E,\ \langle z, y, x\rangle = 0\}. \tag{3.6}$$

Proof. Let B be a basis for the free module E. Note $|B| > 1$. For any two distinct elements b, c of B, and for x in E, the equation $0 = \langle c, b, x\rangle = b\beta(x, c) - c\beta(b, x)$ forces $\beta(b, x) = 0 = \beta(x, c)$. Thus $\mathrm{Rad}\,\beta$ contains the right hand side of (3.6). Conversely, $x \in \mathrm{Rad}\,\beta \Rightarrow \langle z, y, x\rangle = y\beta(x, z) - z\beta(y, x) = 0$ for y, z in E. □

Finally, isomorphism of formed spaces is equivalent to isomorphism of the corresponding comtrans algebras.

Theorem 3.6. *Let (E, β) and (E, γ) be formed spaces on the same underlying module E. Then the following three conditions on a module automorphism $f: E \to E$ are equivalent:*
(a) *$f: (E, \beta) \to (E, \gamma)$ is an isomorphism of formed modules;*
(b) *$f: \mathrm{CT}(E, \beta) \to \mathrm{CT}(E, \gamma)$ is a comtrans algebra isomorphism;*
(c) *$f: \mathrm{CT}(E, \beta)^{\tau} \to \mathrm{CT}(E, \gamma)^{\tau}$ is a comtrans algebra isomorphism.*

Proof. The equivalence of the conditions (b) and (c) follows from the term equivalence of each comtrans algebra with its transpose. If (a) holds, the module automorphism f of E is such that $\forall x, y \in E$, $\gamma(xf, yf) = \beta(x, y)$. Let $[\,,\,,\,]_\beta$ and $[\,,\,,\,]_\gamma$ denote the commutators of $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \gamma)$ respectively. Then $\forall x, y, z \in E$, $[xf, yf, zf]_\gamma = yf\gamma(xf, zf) - xf\gamma(yf, zf) = yf\beta(x, z) - xf\beta(y, z) = (y\beta(x, z) - x\beta(y, z))f = [x, y, z]_\beta f$. Similarly, one obtains the equation $\langle xf, yf, zf\rangle_\gamma = \langle x, y, z\rangle_\beta f$ for the respective translators $\langle\,,\,,\,\rangle_\beta$ and $\langle\,,\,,\,\rangle_\gamma$ of $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \gamma)$. Thus (b) holds. Conversely, suppose that

(b) holds: there is a comtrans algebra isomorphism $f: \mathrm{CT}(E, \beta) \to \mathrm{CT}(E, \gamma)$. Let B be a basis for the free module E. Note that $|B| > 1$, and that Bf is also a basis for E. Given elements b, c of B, choose an element a of B distinct from b and an element θ of the dual module E^* with $af\theta = 1$ and $bf\theta = 0$. Then $[a, b, c]_\beta f = [af, bf, cf]_\gamma \Rightarrow (b\beta(a, c) - a\beta(b, c))f = bf\beta(a, c) - af\beta(b, c) = bf\gamma(af, cf) - af\gamma(bf, cf) \Rightarrow \beta(b, c) = (af\beta(b, c) - bf\beta(a, c))\theta = (af\gamma(bf, cf) - bf\gamma(af, cf))\theta = \gamma(bf, cf)$. Thus f is an isomorphism of formed modules, and (a) holds. □

Corollary 3.7. *For a formed space (E, β), the automorphism groups of (E, β), of* $\mathrm{CT}(E, \beta)$ *and of* $\mathrm{CT}(E, \beta)^\tau$ *coincide.*

Proof. Take $\gamma = \beta$ in Theorem 3.6. □

4. Recognizing form algebras. Given a comtrans algebra E, under what conditions is there a bilinear form β on E such that $E = \mathrm{CT}(E, \beta)$ or $E = \mathrm{CT}(E, \beta)^\tau$? It transpires that transposed form algebras $\mathrm{CT}(E, \beta)^\tau$ are slightly easier to recognize, but of course $E = \mathrm{CT}(E, \beta)$ if and only if $E^\tau = \mathrm{CT}(E, \beta)^\tau$. The main answer to the problem is given by the following theorem.

Theorem 4.1. *Let E be a free module over R of rank more than 2. Then there is a bilinear form β on E such that $E = \mathrm{CT}(E, \beta)^\tau$ if and only if the following two conditions obtain:*
(a) $\forall x, y, z \in E,\ [x, y, z] \in zR$;
(b) $\forall x, y, z \in E,\ \langle x, y, z\rangle \in xR + zR$.

Proof. By (3.3) and (3.4), conditions (a) and (b) are clearly necessary for transposed form algebras $\mathrm{CT}(E, \beta)^\tau$. Conversely, suppose that E carries a comtrans algebra structure satisfying conditions (a) and (b). Let B be a basis for E; in particular $|B| > 2$. By (a),

$$\exists\, \delta: B^3 \to R.\ \forall b, c, d \in B,\ [b, c, d] = d\delta(b, c, d). \tag{4.1}$$

For $d \neq d'$, $b, c \in B$, there is a scalar λ with $(d - d')\lambda = [b, c, d - d'] = [b, c, d] - [b, c, d'] = d\delta(b, c, d) - d'\delta(b, c, d')$, whence $\delta(b, c, d) = \lambda = \delta(b, c, d')$. Thus (4.1) may be rewritten as:

$$\exists\, \delta: B^2 \to R.\ \forall b, c, d \in B,\ [b, c, d] = d\delta(b, c). \tag{4.2}$$

By the trilinearity and left alternativity of the commutator, $\delta: B^2 \to R$ extends to a skew-symmetric form $\delta: E^2 \to R$ such that

$$\forall x, y, z \in E,\ [x, y, z] = z\delta(x, y). \tag{4.3}$$

By (b),

$$\exists\, \beta, \gamma: B^3 \to R.\ \forall b, c, d \in B,\ \langle b, c, d\rangle = b\beta(d, c, b) - d\gamma(c, b, d). \tag{4.4}$$

For a 3-element subset $\{b, b', d\}$ of B and c in B, there are scalars λ and μ with $(b - b')\lambda + d\mu = \langle b - b', c, d\rangle = \langle b, c, d\rangle - \langle b', c, d\rangle = b\beta(d, c, b) - b'\beta(d, c, b') - d(\gamma(c, b, d) - \gamma(c, b', d))$, whence $\beta(d, c, b) = \lambda = \beta(d, c, b')$. For a 3-element subset

$\{b, d, d'\}$ of B and c in B, there are scalars ξ and η with $b\xi - (d - d')\eta = \langle b, c, d - d'\rangle = \langle b, c, d\rangle - \langle b, c, d'\rangle = b(\beta(d, c, b) - \beta(d', c, b)) - d\gamma(c, b, d) + d'\gamma(c, b, d')$, whence $\gamma(c, b, d) = \eta = \gamma(c, b, d')$. Then (4.4) may be partly rewritten as:

$$(4.5) \qquad \exists \beta, \gamma: B^2 \to R.\ \forall b \neq d,\ c \in B,\ \langle b, c, d\rangle = b\beta(d, c) - d\gamma(c, b).$$

Suppose $b \neq c \neq d \in B$. By (4.4) and (4.5), the Jacobi identity gives $0 = \langle b, c, d\rangle + \langle c, d, b\rangle + \langle d, b, c\rangle = b\beta(d, c, b) - d\gamma(c, b, d) + c\beta(b, d) - b\gamma(d, c) + d\beta(c, b) - c\gamma(b, d)$. Equating coefficients of c yields

$$(4.6) \qquad \forall b, d \in B,\ \beta(b, d) = \gamma(b, d).$$

Suppose $b \neq d$, $c \in B$. By (4.3), (4.5) and (4.6), the comtrans identity gives $0 = [b, c, d] + [d, c, b] - \langle b, c, d\rangle - \langle d, c, b\rangle = d\delta(b, c) + b\delta(d, c) - b\beta(d, c) + d\beta(c, b) - d\beta(b, c) + b\beta(c, d)$. Equating coefficients of d yields

$$(4.7) \qquad \forall b, c \in B,\ \delta(b, c) = \beta(b, c) - \beta(c, b).$$

Now by (2.5), (4.2) and (4.7), $\langle b, c, b\rangle = [b, c, b] = b\delta(b, c) = b\beta(b, c) - b\beta(c, b)$. Together with (4.5) and (4.6), this yields

$$(4.8) \qquad \forall b, c, d \in B,\ \langle b, c, d\rangle = b\beta(d, c) - d\beta(c, b).$$

Extend $\beta: B^2 \to R$ to a bilinear form on E. By (4.3) and (4.7),

$$(4.9) \qquad \forall x, y, z \in E,\ [x, y, z] = z(\beta(x, y) - \beta(y, x)).$$

By (4.8) and the trilinearity of the translator,

$$(4.10) \qquad \forall x, y, z \in E,\ \langle x, y, z\rangle = x\beta(z, y) - z\beta(y, x).$$

Comparing (4.9) with (3.3) and (4.10) with (3.4), the algebra E is identified as $\mathrm{CT}(E, \beta)^{\tau}$. □

The following example shows the necessity of the rank condition in the statement of Theorem 4.1.

E x a m p l e 4.2. Let (H, α) be a hyperbolic plane [5, Def. II.9.7] over a field, generated by the basis $\{e, f\}$ of isotropic vectors with $\alpha(e, f) = -\alpha(f, e) = 1$. Define

$$(4.11) \qquad [x, y, z] = -z\alpha(y, x)$$

and

$$(4.12) \qquad \langle x, y, z\rangle = x\alpha(z, y).$$

Then H becomes a comtrans algebra, satisfying condition (a) and (b) of Theorem 4.1. If $H = \mathrm{CT}(H, \beta)^{\tau}$ for a bilinear form β on H, (3.4) and (4.12) would yield

$$(4.13) \qquad \forall x, y, z \in H,\ x\alpha(z, y) = x\beta(z, y) - z\beta(y, x).$$

For fixed x, y, choose z linearly independent of x. Then (4.13) would give $\beta(y, x) = 0$, leading to the contradiction that H would be abelian.

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Eingegangen am 4. 7. 1991

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