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Bisemilattices of Subsemilattices

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Sets of subsemilattices of semilattices are given a natural meet-distributive bisemilattice structure. These bisemilattices are decomposed into constituent semilattices and distributive lattices. Their construction furnishes a left adjoint to the forgetful functor from meet-distributive bisemilattices to semilattices. Representation theorems for meet-distributive bisemilattices follow.

1. INTRODUCTION

A *semilattice* $(S, \cdot)$ is a commutative idempotent semigroup. Such a structure yields a partial order $\leqslant.$ on the set S on setting $x \leqslant. y$ iff $x \cdot y = x$. A semilattice with such a partial order is called a *meet-semilattice*. Similarly, given a semilattice $(S, +)$, one may consider the partial order $\leqslant_+$ given by $x \leqslant_+ y$ iff $x + y = y$. A semilattice with the partial order taken in this way is called a *join-semilattice*.

A set **S** with two semilattice operations $\cdot, +$ is called a *bisemilattice*. One regards $(\mathbf{S}, \cdot)$ as a meet-semilattice (and refers to $\cdot$ as "meet") and $(\mathbf{S}, +)$ as a join-semilattice (+ is the "join"). Examples are furnished by lattices $(\mathbf{L}, \vee, \wedge)$ with the usual meet and join operations (for which the two partial orders $\leqslant_\wedge$ and $\leqslant_\vee$ coincide with the usual order relation), and the bisemilattice $(S, \cdot, \cdot)$ obtained from a semilattice $(S, \cdot)$ by taking the same underlying set S with the semilattice operation considered twice, once as meet and once as join. In bisemilattice words written without parentheses, it is understood that the meet operations are to be carried out before the joins.

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One of the major classes of bisemilattices is that of the so-called *meet-distributive* bisemilattices, in which the meet operation $\cdot$ distributes over the join operation $+$:

$$x \cdot (y+z) = x \cdot y + x \cdot z. \tag{1.1}$$

As examples one has distributive lattices and the bisemilattices $(S, \cdot, \cdot)$ obtained from semilattices $(S, \cdot)$ (since semilattices are self-distributive). This paper is concerned with those meet-distributive bisemilattices that arise from the set $\mathbf{S}(V,\cdot)$ of all finite non-empty subsemilattices of a given semilattice $(V, \cdot)$ under the operations

$$S \cdot T = \{s \cdot t \mid s \in S, t \in T\} \tag{1.2}$$

and

$$S + T = S \cup T \cup S \cdot T \tag{1.3}$$

for elements S, T of $\mathbf{S}(V, \cdot)$. $S+T$ is the usual join of S and T obtained when regarding the set of subsemilattices in the classical way as a lattice, but $S \cdot T$ is not necessarily the usual intersection (for example, if the semilattice operation $\cdot$ on $V = \{0, 1, 2\}$ is defined by setting all products of distinct elements to be 0, then the singleton subsemilattices $\{1\}$ and $\{2\}$ satisfy $\{1\} \cdot \{2\} = \{0\}$, although they have empty intersection). It is immediate that $S \cdot T$ is indeed a subsemilattice of V, and that $(\mathbf{S}(V, \cdot), \cdot)$ is a semilattice. To check the meet-distributivity (U also a subsemilattice of V) one observes $S \cdot (T+U) = S \cdot (T \cup U \cup T \cdot U) = S \cdot T \cup S \cdot U \cup S \cdot T \cdot U = S \cdot T \cup S \cdot U \cup S \cdot T \cdot S \cdot U = S \cdot T + S \cdot U$. For certain purposes this meet-distributive bisemilattice structure $(\mathbf{S}(V, \cdot), +, \cdot)$ on the set of finite subsemilattices of a semilattice may well be more useful than the usual lattice structure on the set. Two algebraic operations are available, as with the lattice, and furthermore the semilattice $(V, \cdot)$ is *canonically embedded* in the bisemilattice:

$$\iota\colon (V, \cdot) \to (\mathbf{S}(V, \cdot), \cdot);\, v \mapsto \{v\}. \tag{1.4}$$

This paper sets out to discuss the abstract algebraic significance of these bisemilattices of subsemilattices within the variety of meet-distributive bisemilattices, and to give a detailed analysis of the structure of the bisemilattices that arise in this way. To begin with, however, the question of the two-sided distributivity of these bisemilattices is dealt with.

2. Two-Sided Distributivity

Given a meet-distributive bisemilattice $(\mathbf{S}, \cdot, +)$, it is natural to ask whether it is also *join-distributive*:

$$x + y \cdot z = (x + y) \cdot (x + z) \tag{2.1}$$

since such *(two-sidedly) distributive* bisemilattices have been characterised completely in [3, Theorem 3] (albeit under the no longer standard name "distributive quasi-lattice") as Płonka sums of distributive lattices. Theorem 2.2 below characterises those semilattices $(V, \cdot)$ for which $\mathbf{S}(V, \cdot)$ is a distributive bisemilattice. They are precisely the *antichain cones*, semilattices in which all products of distinct elements are a single element 0, so called because the corresponding partial order $\leqslant.$ on the underlying set V has as Hasse diagram the cone from 0 to the antichain $V - \{0\}$.

THEOREM 2.2. *The meet-distributive besemilattice* $(\mathbf{S}(V, \cdot), \cdot, +)$ *of finite non-empty subsemilattices of the semilattice* $(V, \cdot)$ *is join-distributive iff* $(V, \cdot)$ *is an antichain cone.*

Proof. First, consider an antichain cone $(V, \cdot)$. The set $\mathbf{A}_0(V)$ of finite subsemilattices S of V containing 0 forms a subbisemilattice of $\mathbf{S}(V)$ isomorphic to the distributive lattice $2^{V-\{0\}}$ of finite subsets of the antichain under the mapping $S \mapsto S - \{0\}$. For an element x of the antichain, let $\mathbf{A}_x(V)$ be the singleton bisemilattice $\{\{x\}\}$, and take $\phi_{x0}\colon \mathbf{A}_x(V) \to \mathbf{A}_0(V)$; $\{x\} \mapsto \{x, 0\}$. Defining a partial order $\leqslant^*$ on $\mathbf{T} = \{\mathbf{A}_v \mid v \in V\}$ by $\mathbf{A}_w \leqslant^* \mathbf{A}_v$ iff $v \leqslant. w$ for v, w in V then displays $\mathbf{S}(V, \cdot)$ as a Płonka sum of distributive lattices as in [3, Theorem 3]. Płonka's theorem then yields that $\mathbf{S}(V, \cdot)$ is a distributive bisemilattice.

Second, consider a 3-element semilattice $(\{0, 1, 2\} \cdot)$ with $0 \leqslant. 1 \leqslant. 2$. This has subsemilattices $X = \{0, 2\}$, $Y = \{0\}$, $Z = \{0, 1, 2\}$. Now $X + Y \cdot Z = \{0, 2\} + \{0\} = \{0, 2\}$, while $(X + Y) \cdot (X + Z) = \{0, 2\} \cdot \{0, 1, 2\} = \{0, 1, 2\}$, so that the join-distributivity (2.1) is not satisfied.

Finally, note that any semilattice which is not an antichain cone contains a subsemilattice isomorphic to $(\{0, 1, 2\}, \cdot)$, whence its bisemilattice of finite non-empty subsemilattices contains a subbisemilattice in which join-distributivity fails to hold. ∎

3. The Free Bisemilattice over a Semilattice

There is a forgetful functor from the category of meet-distributive bisemilattices to the category of (meet) semilattices obtained by forgetting the join structure. Theorem 3.1 below shows that the construction of the meet-distributive bisemilattice of finite non-empty subsemilattices of a semilattice provides a left adjoint to this functor. As natural corollaries of the adjointness one obtains a representation theorem for meet-distributive bisemilattices and the first stage of the structural analysis of the next section.

THEOREM 3.1. *Let* $(V, \cdot)$ *be a meet-semilattice, and* $(\mathbf{B}, +, \cdot)$ *a meet-distributive bisemilattice. Every semilattice morphism* $f\colon (V, \cdot) \to (\mathbf{B}, \cdot)$ *can be extended to a unique bisemilattice morphism* $\bar{f}\colon (\mathbf{S}(V, \cdot), +, \cdot) \to (\mathbf{B}, +, \cdot)$ *whose composite* $\bar{f} \circ \iota$ *with the canonical embedding* (1.4) *is* f.

Proof. Let $S = \{s_i \mid i \in I\}$, $T = \{t_i \mid j \in J\}$ denote typical elements of $\mathbf{S}(V, \cdot\ )$, I and J being finite non-empty sets. $\mathbf{S} = \sum_{i \in I} \iota(s_i)$, so in order that $\bar{f}$ be a bisemilattice morphism satisfying $\bar{f} \circ \iota = f$ one requires

$$\bar{f}(S) = \sum_{i \in I} f(s_i). \tag{3.2}$$

In particular, $\bar{f}$ is uniquely specified. It remains to prove that $\bar{f}$ is indeed a bisemilattice morphism. Now

$$\begin{aligned}
\bar{f}(S \cdot T) &= \sum_{(i,j) \in I \times J} f(s_i \cdot t_j) && \text{by (1.2) and (3.2)}\\
&= \sum_{(i,j) \in I \times J} f(s_i) \cdot f(t_j) && \text{since } f \text{ is a semilattice morphism}\\
&= \sum_{i \in I} f(s_i) \cdot \sum_{j \in J} f(t_j) && \text{by (1.1)}\\
&= \bar{f}(S) \cdot \bar{f}(T) && \text{by (3.2),}
\end{aligned}$$

so $\bar{f}$ is a meet-morphism, and then

$$\begin{aligned}
\bar{f}(S + T) &= \sum_{i \in I} f(s_i) + \sum_{j \in J} f(t_j) + \sum_{(i,j) \in I \times J} f(s_i \cdot t_j) && \text{by (1.3) and (3.2)}\\
&= \bar{f}(S) + \bar{f}(T) + \bar{f}(S \cdot T) && \text{by (1.2) and (3.2)}\\
&= \bar{f}(S) + \bar{f}(T) + \bar{f}(S) \cdot \bar{f}(T) && \text{since } \bar{f} \text{ is a meet-morphism}\\
&= (\bar{f}(S) + \bar{f}(T)) \cdot (\bar{f}(S) + \bar{f}(T)) && \text{by (1.1)}\\
&= \bar{f}(S) + \bar{f}(T),
\end{aligned}$$

so that $\bar{f}$ is also a join-morphism. ∎

COROLLARY 3.3 (Representation Theorem for meet-distributive bisemilattices). *Every meet-distributive bisemilattice is a quotient of the meet-distributive bisemilattice of non-empty finite subsemilattices of its meet-semilattice. Each element is represented by the congruence class of the singleton subsemilattice containing it.*

Proof. Taking the identity mapping $1_{\mathbf{B}}$ on a meet-distributive bisemilattice $(\mathbf{B}, +, \cdot)$ as the meet semilattice morphism $1_{\mathbf{B}}: (\mathbf{B}, \cdot) \to (\mathbf{B}, \cdot)$ for f in Theorem 3.1, the surjective bisemilattice morphism

$$\bar{1}_{\mathbf{B}}: (\mathbf{S}(\mathbf{B}, \cdot), +, \cdot) \to (\mathbf{B}, +, \cdot)$$

produces $(\mathbf{B}, +, \cdot)$ as the required quotient. Since $\bar{1}_{\mathbf{B}} \circ \iota = 1_{\mathbf{B}}$, each element b of $\mathbf{B}$ is the image under $\bar{1}_{\mathbf{B}}$ of the singleton $\{b\}$. ∎

One may display the Representation Theorem diagrammatically:

$$\begin{array}{ccc} (\mathbf{S}(\mathbf{B}, \cdot), +, \cdot) & \xleftarrow{\ \iota\ } & (\mathbf{B}, \cdot) \\ & \searrow^{\bar{1}_{\mathbf{B}}} & \downarrow 1_{\mathbf{B}} \\ & & (\mathbf{B}, +, \cdot) \end{array} \tag{3.4}$$

Note that this is a retract of meet semilattices.

COROLLARY 3.5. *The meet-distributive bisemilattice* $(\mathbf{S}(V, \cdot), +, \cdot)$ *of non-empty finite subsemilattices of the semilattice* $(V, \cdot)$ *is the disjoint union of bisemilattice fibres over the meet-distributive bisemilattice* $(V, \cdot, \cdot)$.

Proof. In Theorem 3.1, take $(V, \cdot, \cdot)$ as the bisemilattice $(\mathbf{B}, +, \cdot)$, and the identity mapping on V as f. As $\bar{f}$ one obtains the bisemilattice morphism

$$\pi: (\mathbf{S}(V, \cdot), +, \cdot) \to (V, \cdot, \cdot); \{s_i \mid i \in I\} \mapsto \prod_{i \in I} s_i. \quad ∎ \tag{3.6}$$

4. THE STRUCTURE OF BISEMILATTICES OF SUBSEMILATTICES

This section analyses the structure of the meet-distributive bisemilattice $(\mathbf{S}(V, \cdot), +, \cdot)$ of non-empty finite subsemilattices of the semilattice $(V, \cdot)$. The starting point for this analysis is the decomposition of $(\mathbf{S}(V, \cdot), +, \cdot)$ obtained in Corollary 3.5 and (3.6) as the union of meet-distributive bisemilattice fibres $\pi^{-1}(v)$ indexed over the semilattice V. Such a fibre $\pi^{-1}(v)$ consists of all those finite subsemilattices T of V having v as the meet or product of all the elements of T. To describe the structure of these fibres it is helpful to consider a "pointed" version of the basic construction $\mathbf{S}(V, \cdot)$. A *pointed meet-semilattice* $(V, \cdot, 0)$ is a meet-semilattice $(V, \cdot)$ with element 0 having the property that for each element x of V, $0 \leqslant_{\cdot} x$. A morphism of pointed meet-semilatices is a meet-semilattice morphism preserving zeros. A *pointed bisemilattice* $(\mathbf{S}, +, \cdot, 0)$ is a bisemilattice $(\mathbf{S}, +, \cdot)$ and pointed meet-semilattice $(\mathbf{S}, \cdot, 0)$ with the additional property that $0 \leqslant_{+} x$ for all elements x of $\mathbf{S}$. For a pointed meet-semilattice $(V, \cdot, 0)$, let $\mathbf{S}_0(V, \cdot, 0)$ denote the set of all finite pointed subsemilattices of $(V, \cdot, 0)$, i.e., subsemilattices of $(V, \cdot)$ containing 0. Under the usual meet and join operations for bisemilattices of subsemilattices $\mathbf{S}_0(V, \cdot, 0)$ forms a meet-distributive bisemilattice. It may also be pointed by the element $\{0\}$. For an element v of the meet-semilattice $(V, \cdot)$, let V_v denote the set of elements x of V with $v \leqslant_{\cdot} x$. Then $(V_v, \cdot, v)$ is a pointed meet-semilattice, and $(\mathbf{S}_0(V_v, \cdot, v), +, \cdot)$ is just the fibre $(\pi^{-1}(v), +, \cdot)$.

The pointed version of the canonical embedding (1.4) is the pointed semilattice embedding

$$\eta: (V, \cdot, 0) \to (\mathbf{S}_0(V, \cdot, 0), +, \cdot, \{0\}); v \mapsto \{0, v\}. \tag{4.1}$$

The analogue of Theorem 3.1 is then

PROPOSITION 4.2. *Let* $(V, \cdot, 0)$ *be a pointed meet-semilattice, and* $(\mathbf{B}, +, \cdot, 0)$ *a pointed meet-distributive bisemilattice. Every pointed meet-semilattice morphism* $f: (V, \cdot, 0) \to (\mathbf{B}, \cdot, 0)$ can be extended to a unique pointed bisemilattice morphism $\bar{f}: (\mathbf{S}_0(V, \cdot, 0), +, \cdot, \{0\}) \to (\mathbf{B}, +, \cdot, 0)$ *whose composite* $\bar{f} \circ \eta$ *with the canonical embedding* (4.1) *is* f.

Proof. This is completely analogous to the proof of Theorem 3.1. Each element T of $\mathbf{S}_0(V, \cdot, 0)$ may be expressed as $\{0\} \cup \{t_i \mid i \in I\}$ for a finite set I. If I is non-empty, $T = \sum_{i \in I} \eta(t_i)$, so one may define $\bar{f}(T)$ as $\sum_{i \in I} f(t_i)$, and otherwise $\bar{f}(\{0\})$ is fixed as 0. ∎

COROLLARY 4.3 (Representation Theorem for pointed meet-distributive bisemilattices). *Every pointed meet-distributive bisemilattice is a quotient of the pointed meet-distributive bisemilattice of finite pointed subsemilattices of its pointed meet-semilattice. Each element is represented by the congruence class of the subsemilattice with underlying set containing just the element itself and the zero.*

Proof. As for the Representation Theorem for (unpointed) meet-distributive bisemilattices, Corollary 3.3. ∎

COROLLARY 4.4. *The fibre* $(\pi^{-1}(v), +, \cdot, \{v\})$ *is the free pointed meet-distributive bisemilattice over the pointed meet-semilattice* $(V_v, \cdot, v)$.

More detailed analysis of the fibres involves a concept of convexity. A pointed subsemilattice X of $(V, \cdot, 0)$ is said to be *convex* if the condition $0 \leqslant. v \leqslant. x \in X$ on an element v of V implies that v is in X. Let $\mathbf{C}_0(V, \cdot, 0)$ denote the set of all convex pointed subsemilattices of $(V, \cdot, 0)$. Let S and T be elements of $\mathbf{C}_0$. Then

$$S \cdot T = S \cap T \tag{4.5}$$

(where $S \cdot T$ is defined as in (1.2)), since $s \cdot t \in S \cdot T$ with $s \in S$, $t \in T$ implies $0 \leqslant. s \cdot t \leqslant. s$, whence convexity of S yields $s \cdot t \in S$, and similarly $s \cdot t \in T$. Conversely, $x \in S \cap T$ implies $x = x \cdot x \in S \cdot T$. $S \cap T$ is clearly an element of $\mathbf{C}_0$. Also,

$$S + T = S \cup T \tag{4.6}$$

(where $S + T$ is defined as in (1.3)), since $S + T = S \cup T \cup S \cdot T$, and by (4.5) $S \cdot T \in S \cup T$. Thus $S \cup T$ is also an element of $\mathbf{C}_0$ (convexity of $S \cup T$ is indeed immediate). This shows that $(\mathbf{C}_0(V, \cdot, 0), +, \cdot, \{0\}) = (\mathbf{C}_0(V, \cdot, 0), \cup, \cap, \{0\})$ is a sublattice of the power set distributive lattice $(2^V, \cup, \cap, \{0\})$.

Defining the *convex hull* $\kappa(T)$ of a finite pointed subsemilattice T of $(V, \cdot, 0)$ to be the least convex subsemilattice of $(V, \cdot, 0)$ containing T, i.e., $\kappa(T) = \{x \in V \mid \exists t \in T.\ 0 \leqslant. x \leqslant. t\}$, gives the mapping

$$\kappa\colon (\mathbf{S}_0(V, \cdot, 0), +, \cdot, \{0\}) \to (\mathbf{C}_0(V, \cdot, 0), \cup, \cap, \{0\}). \tag{4.7}$$

PROPOSITION 4.8. *κ is a pointed bisemilattice morphism.*

Proof. Let $S, T \in \mathbf{S}_0(V, \cdot, 0)$. Now $x \in \kappa(S \cdot T)$ implies the existence of s in S, t in T such that $0 \leqslant. x \leqslant. s \cdot t$. But $s \cdot t \leqslant. s$, $s \cdot t \leqslant. t$, so $x \leqslant. s$, $x \leqslant. t$, i.e., $x \in \kappa(S) \cap \kappa(T)$. Conversely, let $x \in \kappa(S) \cap \kappa(T)$, say $x \leqslant. s \in S$, $x \leqslant. t \in T$. Then $x = x \cdot x = x \cdot s \cdot x \cdot t = x \cdot s \cdot t \leqslant. s \cdot t$, so $x \in \kappa(S \cdot T)$. Thus κ is a meet-morphism. Further

$$\begin{aligned}\kappa(S + T) &= \kappa(S \cup T \cup S \cdot T)\\ &= \{x \in V \mid \exists s \in S, t \in T.\ x \leqslant. s \text{ or } x \leqslant. t \text{ or } x \leqslant. t \text{ or } x \leqslant. s \cdot t\}\\ &= \{x \in V \mid \exists s \in S, t \in T.\ x \leqslant. s \text{ or } x \leqslant. t\}\\ &= \kappa(S) \cup \kappa(T),\end{aligned}$$

so that κ is also a join-morphism. Finally $\kappa\{0\} = \{0\}$. ∎

The image of the mapping κ is thus a pointed distributive lattice $(\kappa\mathbf{S}_0(V, \cdot, 0), \cup, \cap, \{0\})$. The next proposition shows that this is the free pointed distributive lattice over the pointed meet-semilattice $(V, \cdot, 0)$, embedded via $\kappa \circ \eta\colon V \to \mathbf{S}_0(V)$; $v \mapsto \{x \in V \mid 0 \leqslant. x \leqslant. v\}$.

PROPOSITION 4.9. *Let* $(V, \cdot, 0)$ *be a pointed meet-semilattice, and* $(L, \vee, \wedge, 0)$ *a pointed distributive lattice. Every pointed meet-semilattice morphism* $f\colon (V, \cdot, 0) \to (L, \wedge, 0)$ *may be extended to a unique pointed lattice morphism* $\bar{\bar{f}}\colon (\kappa \mathbf{S}_0(V, \cdot, 0), \cup, \cap, \{0\}) \to (L, \vee, \wedge, 0)$ *satisfying* $\bar{\bar{f}} \circ \kappa \circ \eta = f$.

Proof. Using Proposition 4.2, f may be extended uniquely to a pointed bisemilattice morphism $\bar{f}\colon (\mathbf{S}_0(V, \cdot, 0), +, \cdot, \{0\}) \to (L, \vee, \wedge, 0)$ with $\bar{f} \circ \eta = f$. The gist of Proposition 4.9 is that $\bar{f}$ may be extended to a unique $\bar{\bar{f}}$ with $\bar{\bar{f}} \circ \kappa = \bar{f}$. Certainly $\bar{\bar{f}}$ is uniquely specified by this requirement, and since for S, T in $\mathbf{S}_0(V, \cdot, 0)$ one has $\bar{\bar{f}}(\kappa(S)) \vee \bar{\bar{f}}(\kappa(T)) = \bar{f}(S) \vee \bar{f}(T) = \bar{f}(S + T) = \bar{\bar{f}} \circ \kappa(S + T) = \bar{\bar{f}}(\kappa(S) \cup \kappa(T))$ and $\bar{\bar{f}}(\kappa(S)) \wedge \bar{\bar{f}}(\kappa(T)) = \bar{f}(S) \wedge \bar{f}(T) = \bar{f}(S \cdot T) = \bar{\bar{f}} \circ \kappa(S \cdot T) = \bar{\bar{f}}(\kappa(S) \cap \kappa(T))$, $\bar{\bar{f}}$ is a lattice morphism. ▮

The final proposition of this series shows that the fibres of κ are semilattices taken as bisemilattices.

PROPOSITION 4.10. *The congruence classes of* κ *on* $\mathbf{S}_0(V, \cdot, 0)$ *form subbisemilattices of* $(\mathbf{S}_0(V, \cdot, 0), +, \cdot)$ *that are semilattices, i.e., the operations* $+$ *and* $\cdot$ *agree on* κ*-classes.*

Proof. Suppose $\kappa(S) = \kappa(T)$ for S and T in $\mathbf{S}_0(V, \cdot, 0)$. Let $s \in S$. Then $s \in \kappa(T)$, so there is an element t of T with $s \leqslant t$, i.e., $s = s \cdot t \in S \cdot T$. Thus $S \subseteq S \cdot T$, and similarly $T \subseteq S \cdot T$, whence $S + T = S \cup T \cup S \cdot T \subseteq S \cdot T \subseteq S \cup T \cup S \cdot T = S + T$. ▮

It is now possible to apply all these results to give a detailed description of the structure of the meet-distributive bisemilattice of finite non-empty subsemilattices of a meet-semilattice:

THEOREM 4.11. *Let* $(V, \cdot)$ *be a meet-semilattice. Let* $(\mathbf{S}(V, \cdot), +, \cdot)$ *be the meet-distributive bisemilattice of finite non-empty subsemilattices of* $(V, \cdot)$. $(\mathbf{S}(V, \cdot), +, \cdot)$ *is the free meet-distributive bisemilattice over the meet-semilattice* $(V, \cdot)$. *There is a bisemilattice projection*

$$\pi\colon (\mathbf{S}(V, \cdot), +, \cdot) \to (V, \cdot, \cdot) \tag{4.12}$$

onto the semilattice V *taken as a bisemilattice. For each* v *in* V *the fibre* $(\pi^{-1}(v), +, \cdot, \{v\})$ *of* π *is the free pointed meet-distributive bisemilattice over the pointed meet-semilattice* $(V_v, \cdot, v)$, *the set of elements of* V *greater than or equal to* v. *There is a pointed bisemilattice projection*

$$\kappa_v\colon (\pi^{-1}(v), +, \cdot, \{v\}) \to F_{PDL}(V_v, \cdot, v) \tag{4.13}$$

onto the free pointed distributive lattice over the pointed meet-semilattice $(V_v, \cdot, v)$, *the fibres of which are semilattices taken as bisemilattices.*

Figure 1 illustrates some of this description, for the meet-semilattice $(V, \cdot)$ shown, by means of Hasse diagrams of the partially ordered sets involved. Congruence classes of π on $\mathbf{S}(V, \cdot)$ and κ_z on $\pi^{-1}(z)$ are indicated by bold lines (or bold points for singleton congruence classes).

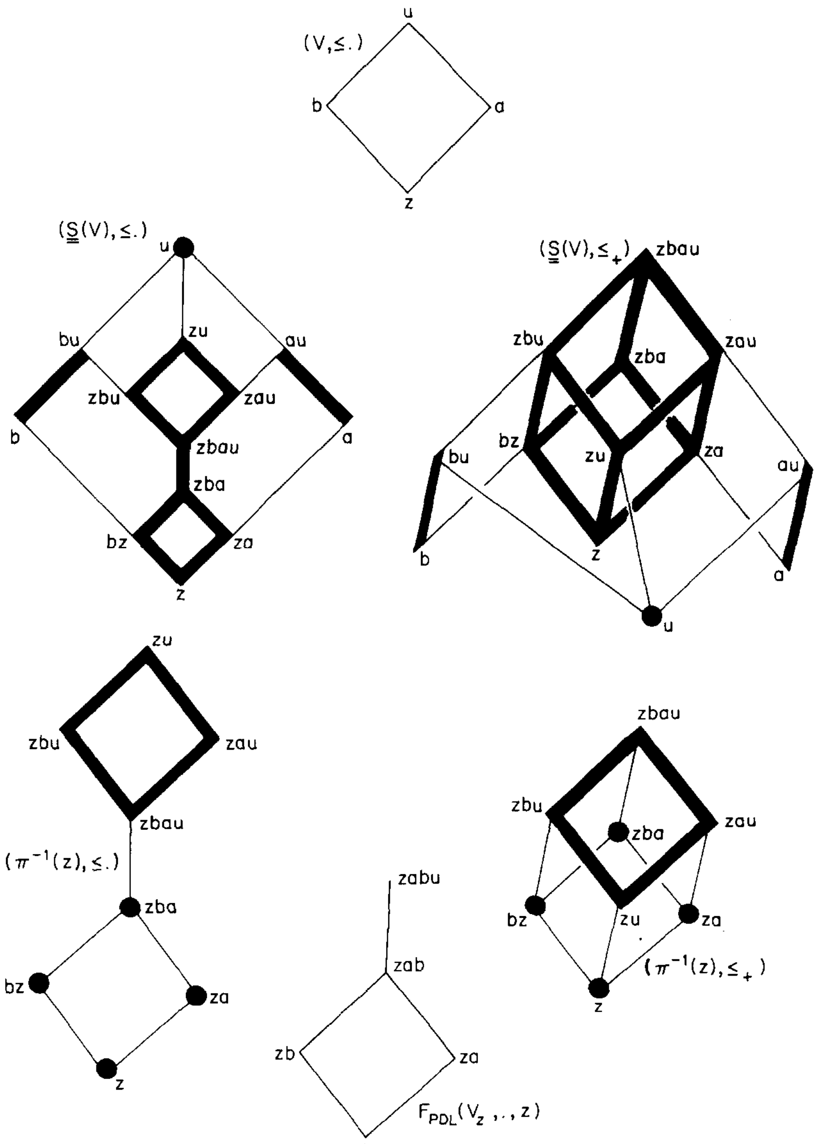

FIGURE 1

5. Conclusion

To summarize this material it is helpful to make a definition. Let $\mathscr{A}$, $\mathscr{B}$ denote classes of bisemilattices. A bisemilattice P is said to be in the *product class* $\mathscr{A} * \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ (cf. [1, p. 422]) if there is a bisemilattice projection $\theta: P \to B$ onto a bisemilattice B in the class $\mathscr{B}$ such that for each b in B the fibre $\theta^{-1}(b)$ is a bisemilattice in the class $\mathscr{A}$. Let $\mathsf{H}\mathscr{A}$ denote the class of homomorphic images of bisemilattices in the class $\mathscr{A}$. Denote the various special classes of bisemilattices as follows:

Notation	Class
$\mathscr{D}$	distributive lattices
$\mathscr{M}$	meet-distributive
$\mathscr{M}_0$	pointed meet-distributive
$\mathscr{S}$	semilattices taken as bisemilattices
$\mathscr{T}$	two-sidely distributive

The results of Section 4 display the free pointed meet-distributive bisemilattice as an element of the class $\mathscr{S} * \mathscr{D}$, and the free meet-distributive bisemilattice over a meet-semilattice as an element of the class $(\mathscr{S} * \mathscr{D}) * \mathscr{S}$. The Representation Theorem for meet-distributive bisemilattices supplies the inclusion $\mathscr{M} \subseteq \mathsf{H}((\mathscr{S} * \mathscr{D}) * \mathscr{S})$. It is interesting to compare this with [2, Theorem 2, p. 216] where it was shown that the join of $\mathscr{S}$ and $\mathscr{D}$ in the lattice of varieties of meet-distributive bisemilattices is the class $\mathscr{T}$, a proper subclass of $\mathscr{M}$. The Representation Theorem for pointed meet-distributive bisemilattices supplies the inclusion $\mathscr{M}_0 \subseteq \mathsf{H}(\mathscr{S} * \mathscr{D})$. For further studies of meet-distributive bisemilattices, see [4–6]. Note that the analysis of (absolutely) free meet-distributive bisemilattices in [6] is a special case of Theorem 4.11.

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