BURNSIDE ORDERS, BURNSIDE ALGEBRAS AND PARTITION LATTICES

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Abstract. The permutation representation theory of groups has been extended, through quasigroups, to one-sided left (or right) quasigroups. The current paper establishes a link with the theory of ordered sets, introducing the concept of a *Burnside order* that generalizes the poset of conjugacy classes of subgroups of a finite group. Use of the Burnside order leads to a simplification in the proof of key properties of the Burnside algebra of a left quasigroup. The Burnside order for a projection left quasigroup structure on a finite set is defined by the lattice of set partitions of that set, and it is shown that the general direct and restricted tensor product operations for permutation representations of the projection left quasigroup structure both coincide with the operation of intersection on partitions. In particular, the mark matrix of the Burnside algebra of a projection left quasigroup, a permutation-theoretic concept, emerges as dual to the zeta function of a partition lattice, an order-theoretic concept.

1. Introduction

The permutation representation theory of groups has been extended to quasigroups [10]–[13], [15], and subsequently to one-sided left (or right) quasigroups [14]. Recall that the class of left quasigroups ranges from groups at one extreme, through quasigroups, to sets (with projections) at the other. Permutation representation theory gains in richness as it is extended. For example, in (left) quasigroup actions, the permutation matrices of group actions become more general stochastic matrices. Similarly, the direct product of group actions splits into two distinct products: a (*direct*) *product* [14, p.402] and a *restricted tensor product* [15, p.124]. In each case, the underlying set of the product need not be the product of the underlying sets of the factors. Burnside's Lemma has been generalized to the left quasigroup context (yet still specializing to its classic form in the group case), with a linear-algebraic

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proof [14, §10]. Similarly, Burnside algebras have been extended from groups to left quasigroups, with two algebra products corresponding respectively to the product and restricted tensor product of permutation representations [14, §11].

The goal of the current paper is to establish a new link between the permutation representation theory of (left) quasigroups on the one hand, and the theory of ordered sets on the other. This link is the *Burnside order* (Section 4). The Burnside order is an ordered set of isomorphism types $[P\backslash Q]$ of homogeneous spaces $P\backslash Q$ associated with each finite left quasigroup Q. (The homogeneous space $P\backslash Q$ associated with a sub-(left)-quasigroup P of a left quasigroup Q is described in Section 2.) For a finite group, the Burnside order is the poset of conjugacy classes of subgroups.

A new, elementary survey of the permutation representation theory of (left) quasigroups is presented in Sections 3–5. The presentation avoids mention of the coalgebras that are needed for deeper questions. Furthermore, use of the new Burnside order concept leads to a simplification in the proof of a key result in the theory (Proposition 5.3).

The Burnside algebra of a left quasigroup, recalled in Section 5, is spanned as a vector space by the Burnside order. The main Theorem 6.5 then identifies the Burnside order for a finite set (under the left quasigroup operation of projection) as the lattice of partitions of the set. In particular, each finite lattice embeds as a sublattice of the Burnside order of a left quasigroup (Corollary 6.6). The mark matrix of the Burnside algebra of a projection left quasigroup, a permutation-theoretic concept, turns out to be dual to the zeta function of a partition lattice, an order-theoretic concept. Theorems 6.8 and 7.1 show that the Burnside algebra of the projection left quasigroup under both the direct product and the restricted tensor product is the bilinear extension of the operation of intersection on partitions.

Readers are referred to [15] and [16] for quasigroup-theoretic and general algebraic concepts and conventions that are not otherwise explicitly clarified here.

2. Left quasigroups and homogeneous spaces

Just like quasigroups, left quasigroups may be defined combinatorially or equationally. Combinatorially, a *left quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary multiplication such that for all x and z, there is a unique element y such that

$$x \cdot y = z \,. \tag{2.1}$$

Left quasigroups cover a wide spectrum of structures. Groups ($y = x^{-1} \cdot z$) form one end of this spectrum. At the other end are (*right*) *projection* (*left*) *quasigroups*, sets with the right projection operation $x \cdot y = y$. Quasigroups are left quasigroups.

Remark 2.1. Although this paper explicitly deals almost entirely with left quasigroups, one may also consider the chirally dual notion of a *right quasigroup*. Groups, and more generally quasigroups, are right quasigroups, as are sets equipped with the *left projection* operation $x \cdot y = x$.

Equationally, a left quasigroup $(Q, \cdot, \backslash)$ is a set Q equipped with binary operations of multiplication and *left division* $\backslash$, satisfying the identities

$$\text{(SL)} \quad x \cdot (x \backslash z) = z \qquad \text{and} \qquad \text{(IL)} \quad x \backslash (x \cdot z) = z\,.$$

A left quasigroup $(Q, \cdot, \backslash)$ is a set Q equipped with binary operations of multiplication and left division $\backslash$, satisfying the identities (SL) and (IL). These identities correspond respectively to the existence and uniqueness of the solution y to (2.1). Thus in groups, $x \backslash z = x^{-1}z$, while in right projection quasigroups, $x \backslash z = z$. When considering subsets of a left quasigroup that are closed under the multiplication and left division operations, the term *subquasigroup* will be used in place of the cumbersome "sub-left-quasigroup."

For each element x of a left quasigroup Q, the *right multiplication*

$$R(x) \colon Q \to Q; y \mapsto y \cdot x$$

and *left multiplication*

$$L(x) \colon Q \to Q; y \mapsto x \cdot y$$

are defined as for quasigroups. The left multiplications are elements of the group $Q!$ of bijections from the set Q to itself. The identity (SL) says that each $L(x)$ surjects, while (IL) gives the injectivity of $L(x)$. The *left multiplication group* of a left quasigroup Q is the subgroup $\mathrm{LMlt}\, Q = \langle L(q) \mid q \in Q \rangle_{Q!}$ of $Q!$ that is generated by the left multiplications. In right projection quasigroups, left multiplication groups are trivial. In a group Q, the map $Q \to \mathrm{LMlt}\, Q; x \mapsto L(x)^{-1}$ is a group isomorphism.

For a subquasigroup P of a left quasigroup Q, the subgroup $\mathrm{LMlt}_Q P$ of $\mathrm{LMlt}\, Q$ generated by $L_Q(P) = \{L(p) : Q \to Q \mid p \in P\}$ is called the *relative left multiplication group* of P in Q. If P is a subgroup of a group Q, then the homogeneous space

$$P \backslash Q = \{Px \mid x \in Q\} \tag{2.2}$$

of cosets of P is the set of orbits of $\mathrm{LMlt}_Q P$ on Q. In general, if P is a subquasigroup of a left quasigroup Q, then the *homogeneous space* $P\backslash Q$ will be defined as the set of orbits of $\mathrm{LMlt}_Q P$ on the set Q.

3. Iterated function systems

If P is a subgroup of a group Q, then the group Q has a permutation representation on the homogeneous space (2.2) by the actions

$$R_{P\backslash Q}(q) : P\backslash Q \to P\backslash Q\,;\; Px \mapsto Pxq \tag{3.1}$$

for elements q of Q. Now let P be a subquasigroup of a left quasigroup Q. For each element q of the left quasigroup Q, consider the Markov chain with transition matrix $R_{P\backslash Q}(q)$ on the state space $P\backslash Q$, where the probability of transition from an orbit A to an orbit B is given as

$$\big[R_{P\backslash Q}(q)\big]_{AB} = |A \cap R(q)^{-1}(B)|/|A|\ . \tag{3.2}$$

If Q is a group, the transition matrix $R_{P\backslash Q}(q)$ is the permutation matrix given by the permutation action (3.1). With the uniform distribution on the left quasigroup Q, the quotient (3.2) becomes the conditional probability of the event $xq \in B$ given $x \in A$. The set of convex combinations of the states from $P\backslash Q$ forms a complete metric space, and the actions $R_{P\backslash Q}(q)$ of the left quasigroup elements q form an iterated function system (IFS) in the sense of fractal geometry [1, 6].

Let Q be a finite set. Define a (*rational*) Q-IFS (X, Q) as a finite set X together with an *action map*

$$R\colon Q \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}X); q \mapsto R_X(q) \tag{3.3}$$

from Q to the set of endomorphisms of the rational vector space $\mathbb{Q}X$ with basis X (identified with their matrices with respect to the basis X), such that each *action matrix* $R_X(q)$ is stochastic. If P is a subquasigroup of a finite non-empty left quasigroup Q, then the homogeneous space $P\backslash Q$ is a Q-IFS with the action map specified by (3.2). A *morphism* or (Q-)*homomorphism*

$$\phi\colon (X, Q) \to (Y, Q) \tag{3.4}$$

from a Q-IFS (X, Q) to a Q-IFS (Y, Q) is a function $\phi : X \to Y$, whose graph has incidence matrix F, such that the intertwining relation

$$R_X(q)F = FR_Y(q) \tag{3.5}$$

is satisfied for each element q of Q. It is readily checked that the class of morphisms (3.4), for a fixed finite set Q, forms a concrete category $\mathbf{IFS}_Q$. The following proposition serves to define homomorphic images.

Proposition 3.1. *Let $\phi\colon (X,Q) \to (Y,Q)$ be a Q-IFS homomorphism. Let $Z = X\phi$. Then the subspace $\mathbb{Q}Z$ of $\mathbb{Q}Y$ is invariant under the set $\{R_Y(q) \mid q \in Q\}$ of actions in (Y,Q).*

Proof. Consider an element z of Z, say $z = x\phi$ for $x \in X$, and an element q of Q. Suppose $xR_X(q) = \sum_{t\in X} r_t t$ for rational numbers r_t. Then (3.5) implies $zR_Y(q) = x\phi R_Y(q) = xFR_Y(q) = xR_X(q)F = \left(\sum_{t\in X} r_t t\right) F = \sum_{t\in X} r_t(tF) = \sum_{t\in X} r_t(t\phi) \in \mathbb{Q}Z$. □

Definition 3.2. In the context of Proposition 3.1, the Q-IFS (Z,Q) with action map $R\colon Q \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}Z); q \mapsto R_Y(q)\big|_{\mathbb{Q}Z}$ is known as:

(a) the *homomorphic image* of (the Q-IFS (X,Q) under) the Q-IFS homomorphism $\phi\colon (X,Q) \to (Y,Q)$, and as
(b) a *sub-Q-IFS* of the Q-IFS (Y,Q).

Group permutation representations appear in the IFS context as follows [15, Prop. 5.1].

Proposition 3.3. *Let Q be a finite group.*

(a) *The category of finite Q-sets forms the full subcategory of* $\mathbf{IFS}_Q$ *consisting of those objects for which the action map* (3.3) *is a monoid homomorphism.*
(b) *A Q-IFS (X,Q) is a Q-set if and only if it is isomorphic to a Q-set (Y,Q) in* $\mathbf{IFS}_Q$.

For a fixed finite set Q, the category $\mathbf{IFS}_Q$ has finite sums or coproducts. Consider objects (X,Q) and (Y,Q) of $\mathbf{IFS}_Q$. Their *sum* or *disjoint union* $(X+Y,Q)$ consists of the disjoint union $X+Y$ of the sets X and Y together with the action map

$$q \mapsto R_X(q) \oplus R_Y(q) \tag{3.6}$$

sending each element q of Q to the direct sum of the matrices $R_X(q)$ and $R_Y(q)$. One obtains an object of $\mathbf{IFS}_Q$, since the direct sum of stochastic matrices is stochastic. The disjoint union, equipped with the appropriate insertions, yields a sum or coproduct in $\mathbf{IFS}_Q$ [15, Th. 5.1]. The *tensor product* $(X\otimes Y,Q)$ of (X,Q) and (Y,Q) is the direct product $X\times Y$ of the sets X and Y together with the action map

$$q \mapsto R_X(q) \otimes R_Y(q)$$

sending each element q of Q to the tensor (or Kronecker) product of the matrices $R_X(q)$ and $R_Y(q)$. In other words, one has

$$\big[R_{X\otimes Y}(q)\big]_{(A\times C)(B\times D)} = \big[R_X(q)\big]_{AB} \cdot \big[R_Y(q)\big]_{CD}$$

for $A, B \in X$ and $C, D \in Y$. Again, one obtains an object of $\mathbf{IFS}_Q$, since the tensor product of stochastic matrices is stochastic. In general, the tensor product does not yield a product in the category $\mathbf{IFS}_Q$ (compare [15, §5.2]).

Now let Q be a finite left quasigroup. Recall that for each subquasigroup P of Q, the homogeneous space $P\backslash Q$ is a Q-IFS with the action map specified by (3.2). In particular, the *regular space* is $\varnothing\backslash Q$. A Q-IFS is said to be a *basic Q-set* if it is a homomorphic image of a homogeneous space $P\backslash Q$ for a subquasigroup P of Q — compare Definition 3.2(a). Each basic Q-set is *irreducible* in the sense that it has no proper, non-empty subobjects [14, Cor. 8.2]. A Q-IFS is said to be a (*finite*) *Q-set* if it is a finite sum of basic Q-sets. A finite Q-set (Z, Q) is said to be a *Q-subset* or *sub-Q-set* of a finite Q-set (Y, Q) if (Z, Q) is a sub-Q-IFS of (Y, Q) — compare Definition 3.2(b). The *category* $\underline{\underline{Q}}_{\mathrm{fin}}$ *of finite Q-sets* is the full subcategory of $\mathbf{IFS}_Q$ induced on the class of finite Q-sets. (Note that the alternative definitions here agree with the earlier definitions of [14, 15] — compare [15, Th. 5.4]. Furthermore, if Q is associative, the present concept of a finite Q-set agrees with the concept as usually understood for groups [14, Cor. 9.5].)

4. Burnside orders

Let Q be a finite left quasigroup. For a finite Q-set X, let $[X]$ denote its isomorphism type within the category $\underline{\underline{Q}}_{\mathrm{fin}}$. Let B be the set of so-called *basic* types, the isomorphism types of basic Q-sets. The set B is finite [14, Cor. 9.4]. The *regular type* is $[\varnothing\backslash Q]$. Let $\underline{\underline{J}}$ be the full subcategory of $\underline{\underline{Q}}_{\mathrm{fin}}$ induced on the class of basic Q-sets. Define a new category $\widetilde{\underline{\underline{J}}}$ on the object class $\underline{\underline{J}}_0$ of $\underline{\underline{J}}$ by setting

$$\big|\widetilde{\underline{\underline{J}}}(X,Y)\big| = \begin{cases} 1 & \text{if } \underline{\underline{Q}}_{\mathrm{fin}}(X,Y) \neq \varnothing; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widetilde{\underline{\underline{J}}}$ is a pre-ordered class. It induces an order structure $(B, \sqsubseteq)$ on the set B (compare [16, I, Ex. 1.3H]) given explicitly by

$$[X] \sqsubseteq [Y] \quad \Leftrightarrow \quad \underline{\underline{Q}}_{\mathrm{fin}}(X,Y) \neq \varnothing\,. \tag{4.1}$$

(Antisymmetry follows from the fact that basic Q-sets are irreducible.)

Definition 4.1. The partially ordered set $(B, \sqsubseteq)$ of (4.1) is called the *Burnside order* of the finite left quasigroup Q.

Example 4.2. Let Q be a finite group. By Proposition 3.3, the left-quasigroup actions of Q coincide with the (right) group actions of Q.

The set B of basic types $[P\backslash Q]$ may be identified as the set of conjugacy classes P^Q of subgroups P of Q. Then the Burnside order of Q is given by

$$P_1^Q \sqsubseteq P_2^Q \quad \Leftrightarrow \quad \exists\, q \in Q\,.\; P_1^q \subseteq P_2\,,$$

i.e., by containment of subgroups within the conjugacy classes. The partial order $\sqsubseteq$ is written as $\subseteq_Q$ in the notation of [3].

Example 4.3. Let Q be the quandle of the trefoil knot [5, p.81], [7, §2]. This is the (left) quasigroup given by the field of order 3 under the multiplication operation $x \rhd y = -x - y$. The automorphism group of Q is the full symmetric group on the set Q, which of course acts doubly-transitively on Q. Let X_3 be the regular space, and let X_1 be the trivial space $Q\backslash Q$. There are three 2-element homogeneous spaces:

$$X_2 = \{0\}\backslash Q = \big\{\{0\},\{1,2\}\big\}\,;$$
$$X_2' = \{1\}\backslash Q = \big\{\{1\},\{2,0\}\big\}\,;$$
$$X_2'' = \{2\}\backslash Q = \big\{\{2\},\{0,1\}\big\}\,.$$

For example, X_2 consists of the two orbits $\{0\}$ and $\{\pm 1\}$ of the relative left multiplication group of $\{0\}$ in Q, the group consisting of the two scalar multiplications ± 1. The structure of the remaining two spaces X_2', X_2'' follows by applying automorphisms of Q to X_2. Note the right actions

$$(4.2)\quad R_{X_2}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R_{X_2}(2) = R_{X_2'}(0) = R_{X_2''}(1) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

— compare [15, §4.6, Ex. 3].

The set B of basic types is $\{[X_1],[X_2],[X_2'],[X_2''],[X_3]\}$. Certainly, homogeneous spaces with differing cardinalities are not isomorphic. To confirm the mutual non-isomorphism of the three homogenous spaces of cardinality 2, it suffices (by the double transitivity of the automorphism group of Q) to observe that there is no isomorphism $\phi\colon X_2 \to X_2'$. If there were such an isomorphism, its incidence matrix F would be a permutation matrix, with $R_{X_2}(0)F = F$. On the other hand, it is apparent from (4.2) that $FR_{X_2'}(0)$ is not a permutation matrix. Thus the intertwining relation (3.5) is broken, and there is no isomorphism.

The singleton space X_1 is the codomain for a homomorphism from each homogeneous space. But there is no homomorphism from the regular space X_3 to any of the doubleton spaces X_2, X_2', or X_2''. Indeed, the regular space is *crisp* [15, Defn. 5.3], in the sense that its action matrices are 0-1-matrices. On the other hand, the doubleton spaces are not crisp, as exhibited by the latter set of action matrices in (4.2). Bearing in mind the irreducibility of the doubletons, and the fact that each homomorphic image of a crisp space is crisp [15, Prop. 5.4], it follows that $\underline{\underline{Q}}_{\mathrm{fin}}(X_3, X)$ is empty whenever $X \in \{X_2, X_2', X_2''\}$. Thus the Burnside order of Q contains an antichain $\{[X_2], [X_2'], [X_2''], [X_3]\}$ of elements, each dominated by $[X_1]$.

5. Burnside algebras

Let Q be a non-empty, finite left quasigroup. Using the theory of coalgebras [14, Prop. 7.3(c)], and a more special result [14, Cor. 11.4], it may be shown that the category $\underline{\underline{Q}}_{\mathrm{fin}}$ of finite Q-sets has finite sums and products. The sum and product of finite Q-sets X and Y will be denoted respectively by $X + Y$ and $X \times Y$. Note that $X + Y$ is just a disjoint union. If Q is a group, then $X \times Y$ is the usual direct or Cartesian product, but in general, the form of $X \times Y$ may be more subtle. (Compare Section 6 below for the case of projection left quasigroups.)

Let $A^+(Q)$ denote the set of isomorphism types of finite Q-sets within the category $\underline{\underline{Q}}_{\mathrm{fin}}$. It is often convenient to consider each basic type b of Q as represented by a specified irreducible Q-set H_b. Now

$$\forall\ [X] \in A^+(Q),\ \forall\ b \in B,\ \exists\ n_b \in \mathbb{N}.\ [X] = \sum_{b \in B} n_b b. \tag{5.1}$$

An inner product is defined on $A^+(Q)$ by

$$\Big\langle \sum_{b \in B} m_b b, \sum_{b \in B} n_b b \Big\rangle = \sum_{b \in B} m_b n_b. \tag{5.2}$$

With respect to this inner product, the set of basic types is orthonormal. The equation of (5.1) may then be rewritten as

$$[X] = \sum_{b \in B} \langle b, [X] \rangle b. \tag{5.3}$$

The proof of the following theorem relies on standard properties of sums and products in categories, and the definition of the category $\underline{\underline{Q}}_{\mathrm{fin}}$ (compare [15, Th. 5.5]).

Theorem 5.1. *Let Q be a finite left quasigroup.*

(a) *The set $A^+(Q)$ forms a commutative unital semiring, with zero $[\varnothing]$ and unit $[\{1\}]$, under the sum $[X] + [Y] = [X + Y]$ and the* product $[X] \cdot [Y] = [X \times Y]$.
(b) *The $\mathbb{N}$-semimodule $A^+(Q)$ is free over the basis B.*

The mark concept for left quasigroups in the following definition is a natural extension of Burnside's original [4, §180].

Definition 5.2. Let Q be a finite left quasigroup, and let X be a finite Q-set. For each basic Q-set type $b = [H_b]$, the *mark* of b in X or $x = [X]$ is defined to be the cardinality

$$Z_{xb} = \left|\underline{\underline{Q}}_{\text{fin}}(H_b, X)\right| \tag{5.4}$$

of the set of Q-homomorphisms from H_b to X. The *mark matrix* or *Z-matrix* Z or Z_Q of Q is the $|B| \times |B|$ matrix $[Z_{bc}]$ for b and c in B.

The following key proposition has an easy proof making use of the Burnside order. Earlier proofs ([14, Prop. 11.6], [15, Prop. 5.6]) were less natural.

Proposition 5.3. *With notation as in Definition 5.2:*

(a) *The set B may be ordered so that Z is triangular.*
(b) *The Z-matrix is invertible over $\mathbb{Q}$.*

Proof. (a): Extend the order $(B, \sqsubseteq)$ to a linear order (compare [2], [16, O, Prop. 3.5.4(a)]). With this linear ordering of B for its rows and columns, the Z-matrix becomes upper triangular.
(b): For $b = [H] \in B$, the identity map 1_H lies in $\underline{\underline{Q}}_{\text{fin}}(H, H)$, so the diagonal entries of the triangular matrix Z are all non-zero. □

Theorem 5.4. [14, Th. 11.7] *Let Q be a finite left quasigroup, with set B of basic types of Q-set. Then the* mark map

$$(A^+(Q), +, \cdot) \to \mathbb{Q}^B; x \mapsto (b \mapsto Z_{xb}), \tag{5.5}$$

with pointwise structure on its codomain, is an embedding of semirings.

Corollary 5.5. *Define $A(Q)$ as the $\mathbb{Q}$-vector space with basis B. Note that $A(Q)$ contains the free $\mathbb{N}$-semimodule $A^+(Q)$ of Theorem 5.1(b) as a subreduct. Then $A(Q)$ carries a $\mathbb{Q}$-algebra structure $(A(Q), +, \cdot)$ such that:*

(a) *The semiring $(A^+(Q), +, \cdot)$ is identified as a subreduct of the $\mathbb{Q}$-algebra $(A(Q), +, \cdot)$;*
(b) *The mark map (5.5) extends to a $\mathbb{Q}$-algebra isomorphism*

$$(A(Q), +, \cdot) \to \mathbb{Q}^B; \sum_{a \in B} r_a a \mapsto \Big(b \mapsto \sum_{a \in B} r_a Z_{ab}\Big). \tag{5.6}$$

Definition 5.6. For a finite left quasigroup Q, the (*rational*) *Burnside algebra* is defined to be the $\mathbb{Q}$-algebra $(A(Q), +, \cdot)$ of Corollary 5.5. The isomorphism (5.6) is known as the *mark isomorphism*.

Example 5.7. Let Q be a finite group. Then the Burnside algebra of Q in the left quasigroup sense of Definition 5.6 coincides with the Burnside algebra of Q in the classical group sense (see [3], for instance).

6. Projections

For a positive integer n, consider an n-element set $\mathbb{P}_n = \{x_1, \dots, x_n\}$ equipped with the right projection $\mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n; (x_0, x_1) \mapsto x_1$ as the multiplication and left division of a left quasigroup structure $\mathbb{P}_n$, a *projection* (*left*) *quasigroup* in the sense of Section 2. As noted in that section, each subset P of $\mathbb{P}_n$ forms a subquasigroup, and since $\mathrm{LMlt}_{\mathbb{P}_n}(P) = \{1\}$, the homogeneous space $P\backslash\mathbb{P}_n$ is the regular space $\mathbb{P}_n$ or $\varnothing\backslash\mathbb{P}_n$. Thus each $\mathbb{P}_n$-set is a sum of homomorphic images of the regular space.

Consider the lattice Π_n of equivalence relations on the set $\mathbb{P}_n$, with inclusion order induced from the power set of ${\mathbb{P}_n}^2$, the set of all binary relations on $\mathbb{P}_n$. For $\rho \in \Pi_n$, consider the set

$$Y_\rho = \{x^\rho \mid x \in \mathbb{P}_n\}$$

of ρ-classes $x^\rho = \{x' \mid (x, x') \in \rho\}$ for $x \in \mathbb{P}_n$. For $q \in \mathbb{P}_n$, define an action $R_{Y_\rho}(q)$ on Y_ρ by

$$yR_{Y_\rho}(q) = q^\rho \tag{6.1}$$

for each $y \in Y_\rho$. Note that if $\widehat{\mathbb{P}_n}$ is the diagonal $\{(x, x) \mid x \in \mathbb{P}_n\}$, the equality relation on $\mathbb{P}_n$, then $Y_{\widehat{\mathbb{P}_n}}$ is the regular space $\mathbb{P}_n$.

Proposition 6.1. *Up to isomorphism, the basic $\mathbb{P}_n$-sets are precisely the sets Y_ρ for $\rho \in \Pi_n$.*

Proof. For $\rho \in \Pi_n$, consider the natural projection $\phi \colon \mathbb{P}_n \to Y_\rho; x \mapsto x^\rho$. Then for $x, q \in \mathbb{P}_n$, the action (6.1) gives $x\phi R_{Y_\rho}(q) = q^\rho = xR_{\mathbb{P}_n}(q)\phi$, so ϕ is a surjective $\mathbb{P}_n$-homomorphism and Y_ρ is an irreducible $\mathbb{P}_n$-set.

Conversely, suppose that Y is a basic $\mathbb{P}_n$-set, with surjective $\mathbb{P}_n$-homomorphism $\phi \colon \mathbb{P}_n \to Y$. Suppose $\rho = \ker \phi$. Then without loss of generality (using the First Isomorphism Theorem for sets [16, O, Th. 3.3.1]), one may take the underlying set Y to be Y_ρ, with ϕ as the natural projection by ρ. Now suppose that x^ρ, for some $x \in \mathbb{P}_n$, is an element of Y. The intertwining condition that $\phi \colon \mathbb{P}_n \to Y$ is a $\mathbb{P}_n$-homomorphism yields $x^\rho R_Y(q) = x\phi R_Y(q) = xR_{\mathbb{P}_n}(q)\phi = q\phi = q^\rho$, in agreement with (6.1). Thus the $\mathbb{P}_n$-sets Y and Y_ρ are isomorphic. □

Corollary 6.2. *The set of basic $\mathbb{P}_n$-types is Π_n, with $[Y_\rho] = \rho$ for $\rho \in \Pi_n$.*

Lemma 6.3. *For $\rho \subseteq \sigma$ in Π_n, the function*

$$\phi_{\rho\sigma} \colon Y_\rho \to Y_\sigma; x^\rho \mapsto x^\sigma \tag{6.2}$$

is well-defined.

It is convenient to refer to the function (6.2) as a *coarsening.*

Lemma 6.4. *Suppose $\rho, \sigma \in \Pi_n$.*

(a) *There is a $\mathbb{P}_n$-homomorphism $\phi\colon Y_\rho \to Y_\sigma$ if and only if $\rho \subseteq \sigma$.*
(b) *If $\rho \subseteq \sigma$, the $\mathbb{P}_n$-homomorphism $\phi\colon Y_\rho \to Y_\sigma$ is unique.*

Proof. If $\rho \subseteq \sigma$, the action (6.1) gives

$$x^\rho \phi_{\rho\sigma} R_{Y_\sigma}(q) = q^\sigma = x^\rho R_{Y_\rho}(q) \phi_{\rho\sigma}\,,$$

for $x, q \in \mathbb{P}_n$, so the coarsening $\phi_{\rho\sigma}$ is a $\mathbb{P}_n$-homomorphism.

Conversely, suppose there is a $\mathbb{P}_n$-homomorphism $\phi\colon Y_\rho \to Y_\sigma$. Let $(q_1, q_2) \in \rho$ and $y \in Y_\rho$. The action (6.1) gives

$$\begin{aligned} q_1^\sigma &= y\phi R_{Y_\sigma}(q_1) = y R_{Y_\rho}(q_1)\phi = q_1^\rho \phi \\ &= q_2^\rho \phi = y R_{Y_\rho}(q_2)\phi = y\phi R_{Y_\sigma}(q_2) = q_2^\sigma\,, \end{aligned} \tag{6.3}$$

so $(q_1, q_2) \in \sigma$. Thus $\rho \subseteq \sigma$, as required to complete the proof of (a). Line (6.3) shows that $\phi = \phi_{\rho\sigma}$, as required for (b). □

Lemma 6.4 shows that $\underline{\underline{\mathbb{P}_n}}_{\mathrm{fin}}(Y_\rho, Y_\sigma)$ is $\{\phi_{\rho\sigma}\}$ if $\rho \subseteq \sigma$, and empty otherwise. Together with Corollary 6.2, it may be summarized as follows.

Theorem 6.5. *Let n be a positive integer.*

(a) *The Burnside order of the projection left quasigroup $\mathbb{P}_n$ is the partition lattice Π_n.*
(b) *The mark matrix of $\mathbb{P}_n$ is the adjacency matrix of the containment relation on Π_n. In other words, for $\rho, \sigma \in \Pi_n$, one has*

$$Z_{\sigma\rho} = \begin{cases} 1 & \textit{if } \sigma \supseteq \rho; \\ 0 & \textit{otherwise,} \end{cases}$$

so $Z_{\sigma\rho}$ is the truth value $[[\rho \subseteq \sigma]]$.

Note that the statement of Theorem 6.5(b) is strictly stronger than the content of part (a). It identifies the mark matrix of $\mathbb{P}_n$ as the dual of the zeta function in the incidence algebra of the partition lattice Π_n [9], [16, Ex. III.1J]. Theorem 6.5(a) alone yields the following consequence of a well-known result of Pudlák-Tůma [8].

Corollary 6.6. *Each finite lattice embeds as a sublattice of the Burnside order of a left quasigroup.*

Corollary 6.7. *For $Q = \mathbb{P}_n$, the mark isomorphism takes the form*

$$(A(\mathbb{P}_n), +, \cdot) \to \mathbb{Q}^{\Pi_n};\ \sum_{\sigma \in \Pi_n} r_\sigma \sigma \mapsto \Big(\rho \mapsto \sum_{\rho \subseteq \sigma} r_\sigma\Big).$$

Proof. Consider an element $\sum_{\sigma\in\Pi_n} r_\sigma \sigma$ of $A(\mathbb{P}_n)$. Its image under the mark isomorphism sends an element ρ of Π_n to $\sum_{\sigma\in\Pi_n} r_\sigma Z_{\sigma\rho}$, which reduces to $\sum_{\rho\subseteq\sigma} r_\sigma$ by Theorem 6.5(b). □

Set-theoretic intersection provides a semilattice structure $(\Pi_n, \cap)$ on the set of basic types. Bilinearity extends this product to the entire Burnside algebra $A(\mathbb{P}_n)$.

Theorem 6.8. *Let n be a positive integer. Then the product in the Burnside algebra $A(\mathbb{P}_n)$ of the projection left quasigroup $\mathbb{P}_n$ is given by the bilinear extension of set-theoretic intersection.*

Proof. Consider partitions ρ, σ, and τ. By Corollary 6.7, the respective images of σ and τ under the mark isomorphism send ρ to the truth values $[[\rho \subseteq \sigma]]$ and $[[\rho \subseteq \tau]]$. Thus by Corollary 5.5, the image of the Burnside algebra product $\sigma \cdot \tau$ sends ρ to

$$[[\rho \subseteq \sigma]] \cdot [[\rho \subseteq \tau]] = [[\rho \subseteq (\sigma \cap \tau)]] \,.$$

Since the image of the intersection $\sigma \cap \tau$ has the same effect on ρ, it follows that $\sigma \cdot \tau = \sigma \cap \tau$. □

Corollary 6.9. *The set Π_n of basic types forms a subsemigroup of the Burnside algebra that coincides with the partition semilattice $(\Pi_n, \cap)$.*

7. Restricted tensor products

Let Q be a finite left quasigroup. Suppose that X or (X, Q) and Y or (Y, Q) are finite Q-sets. Then the *restricted tensor product* $X\widehat{\otimes}Y$ is the largest (finite) Q-set contained in the tensor product $(X \otimes Y, Q)$ [15, p.124]. The Burnside algebra $A(Q)$ is closed under the restricted tensor product [15, Theorem 5.5, 1(b)].

Theorem 7.1. *Let n be a positive integer. Then the restricted tensor product in the Burnside algebra $A(\mathbb{P}_n)$ of the projection left quasigroup $\mathbb{P}_n$ coincides with the product.*

Proof. Let ρ and σ be elements of Π_n. By [15, Cor. 5.4],

$$\rho\widehat{\otimes}\sigma \subseteq \rho \cdot \sigma = \rho \cap \sigma \,.$$

Since $Y_{\rho\cap\sigma}$ is irreducible, it suffices to show that the tensor product $\mathbb{P}_n$-IFS $Y_\rho \otimes Y_\sigma$ contains a $\mathbb{P}_n$-subset isomorphic to $Y_{\rho\cap\sigma}$. The First Isomorphism Theorem for sets applied to the function

$$\theta\colon Q \to Q^\rho \times Q^\sigma; x \mapsto (x^\rho, x^\sigma)$$

yields an isomorphism $b : Q^{\ker\theta} \to Q^\theta$. For elements s, t of Q, one has

$$(s,t) \in \ker\theta \quad \Leftrightarrow \quad (s^\rho, s^\sigma) = (t^\rho, t^\sigma) \quad \Leftrightarrow \quad (s,t) \in \rho \cap \sigma \,,$$

so $b : Q^{\rho\cap\sigma} \to Q^\theta; x^{\rho\cap\sigma} \mapsto (x^\rho, x^\sigma)$. Now if x and q are elements of Q, then $x^{\rho\cap\sigma}R_{Y_{\rho\cap\sigma}}(q)b = q^{\rho\cap\sigma}b = (q^\rho, q^\sigma) = \big(x^\rho R_{Y_\rho}(q), x^\sigma R_{Y_\sigma}(q)\big) = (x^\rho, x^\sigma)R_{Y_\rho\otimes Y_\sigma}(q) = x^{\rho\cap\sigma}bR_{Y_\rho\otimes Y_\sigma}(q)$, so b is the desired $\mathbb{P}_n$-isomorphism from $Y_{\rho\cap\sigma}$ to the $\mathbb{P}_n$-subset Q^θ of $Y_\rho \otimes Y_\sigma$. □

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