

A GRAPH-THEORETIC APPROACH TO QUASIGROUP CYCLE NUMBERS

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ABSTRACT. Norton and Stein associated a number with each idempotent quasigroup or diagonalized Latin square of given finite order n , showing that it is congruent mod 2 to the triangular number $T(n)$. In this paper, we use a graph-theoretic approach to extend their invariant to an arbitrary finite quasigroup. We call it the *cycle number*, and identify it as the number of connected components in a certain graph, the *cycle graph*. The congruence obtained by Norton and Stein extends to the general case, giving a simplified proof (with topology replacing case analysis) of the well-known congruence restriction on the possible orders of general Schroeder quasigroups. Cycle numbers correlate nicely with algebraic properties of quasigroups. Certain well-known classes of quasigroups, such as Schroeder quasigroups and commutative Moufang loops, are shown to maximize the cycle number among all quasigroups belonging to a more general class.

1. INTRODUCTION

In a remarkable paper published in 1956, D.A. Norton and S.K. Stein defined a certain numerical invariant of each finite idempotent quasigroup or diagonalized Latin square [8]. Although they did not give a name to this invariant, it will be convenient to refer to it as the *cycle number*. By associating an oriented surface or 2-dimensional complex with each idempotent quasigroup of given finite order n , they showed that the cycle number of such a quasigroup is congruent modulo 2 to the triangular number $T(n) = n(n+1)/2$. Recently, the results of Norton and Stein were extended to arbitrary finite quasigroups, and used to show that certain permutation cycle types cannot be realized as quasigroup automorphisms [6].

The aim of the current paper is to present an alternative, graph-theoretic approach to the specification of the cycle number and 2-complex associated with a finite quasigroup. In Section 2, we begin

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with the free product G of three cyclic groups of order 2 that are generated by respective involutions t_1 , t_2 , and t_3 . The group G acts on the disjoint union (2.1) of three copies of the multiplication table of an arbitrary quasigroup Q . The undirected Cayley graph of this action with respect to the involutory generating set $\{t_1, t_2, t_3\}$ is denoted by Γ_Q . Certain so-called *stabilizing edges* (2.5) are then removed from Γ_Q to yield a graph C_Q known as the *cycle graph* of the quasigroup Q . The constructions of Γ_Q and C_Q yield functors from the category of quasigroup homomorphisms to the category of graph homomorphisms (Theorem 2.6).

For a finite quasigroup Q , the *cycle number* is defined as the number $\sigma(C_Q)$ of connected components in the cycle graph C_Q (Definition 2.1). Section 3 analyzes the structure of the cycle graph in the general case, and shows that its connected components are either cycles or doubly infinite paths. In Section 4, bounds for the cycle number of a finite quasigroup are given in terms of its algebraic properties. The cycle number of a quasigroup Q of finite order n is bounded above by n^2 , and this bound is attained precisely by the Schroeder quasigroups, i.e., quasigroups satisfying the identity $xy \cdot yx = x$ (Theorem 4.1). The cycle number of a commutative quasigroup Q of finite order n is bounded below by $n^2/2$ (Proposition 4.3). The cycle number of a quasigroup Q of order n is equal to the triangular number $T(n)$ if Q is totally symmetric (Proposition 4.6), or if Q is a commutative diassociative loop (Proposition 4.7). Indeed, commutative diassociative loops of a given finite order are characterized among all diassociative loops of that order by maximization of the cycle number (Theorem 4.9). Restrictions of the theorem similarly characterize abelian groups among all groups, or commutative Moufang loops among all Moufang loops.

The remainder of the paper constructs a 2-complex, associated with a finite quasigroup, that is dual to the complex previously studied by Norton, Stein, and the authors [6, 8]. This dual complex turns out to be somewhat easier to handle, allowing a more direct labeling of its elements. Using the new complex, we recover the result of Norton and Stein for finite idempotent quasigroups, and extended to general finite quasigroups by the authors, that the cycle number of a quasigroup of order n is congruent mod 2 to the triangular number $T(n)$ (Theorem 7.3). A sample application of this theorem gives a simple proof that the order of a general finite Schroeder quasigroup must be congruent to 0 or 1 modulo 4 (Corollary 7.5). Since this result was previously obtained by a detailed case analysis, we believe that our proof

may serve as a prototype for the use of topological techniques in combinatorics, either as a substitute for case analysis, or for the derivation of new results.

For concepts and conventions that are not otherwise explicitly given in this paper, see [10, 11].

2. MARKED TRIPLES

Consider the free product

$$G = \langle t_1, t_2, t_3 \mid t_1^2 = t_2^2 = t_3^2 = 1 \rangle$$

of three copies of the group of order two. The group G acts on the set $\underline{3} = \{1, 2, 3\}$ by the transpositions

$$t_1 = (2\ 3), \quad t_2 = (3\ 1), \quad t_3 = (1\ 2).$$

Let $(Q, \cdot, /, \backslash)$ be a quasigroup. Define the *marked multiplication table*

$$(2.1) \quad M_Q = \{(x, y, z, i) \in Q^3 \times \underline{3} \mid xy = z\}.$$

The elements of M_Q are called the *marked triples* of Q . It will often prove convenient to denote the respective marked triples $(x, y, z, 1)$, $(x, y, z, 2)$, and $(x, y, z, 3)$ by \underline{xyz} , xyz , and $xy\bar{z}$. In a marked triple (x_1, x_2, x_3, i) , the element x_i is called the *marked element*.

An action of G on M_Q is defined by

$$(2.2) \quad (x, y, z, i)t_1 = (y/z, z, y, it_1);$$

$$(2.3) \quad (x, y, z, i)t_2 = (z, z \backslash x, x, it_2);$$

$$(2.4) \quad (x, y, z, i)t_3 = (y, x, y \cdot x, it_3).$$

Let Γ_Q denote the undirected Cayley graph of this action. (In actual figures, it suffices to label edges with 1, 2, 3 rather than t_1, t_2, t_3 .)

We now define a subgraph of Γ_Q on the vertex set M_Q , known as the *cycle graph* C_Q of the quasigroup Q . An edge of Γ_Q labeled t_i between marked triples of the form (x, y, z, i) is said to be a *stabilizing edge*:

$$(2.5) \quad (, , , i) \xrightarrow{i} (, , , i)$$

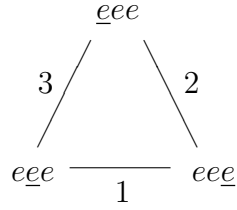
The subgraph C_Q of Γ_Q is obtained by removing all the stabilizing edges. The fundamental definition is as follows.

Definition 2.1. Let Q be a (possibly infinite) quasigroup.

- (a) For an element q of Q , a *cycle* of q is defined as a connected component of C_Q containing a marked triple in which q is the marked element. We let C_q denote the union of all cycles of q .
- (b) The *cycle number* of Q is the (possibly infinite) number $\sigma(C_Q)$ of connected components in the cycle graph C_Q .

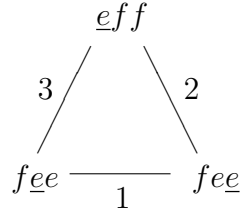
Remark 2.2. For finite Q , Proposition 3.1(c) below will show that the connected components of C_Q are actual cycles. If Q is infinite, the “cycles” of Definition 2.1 may be infinite paths (compare Corollary 4.5). Nevertheless, the term “cycle” is retained here to correlate with the usage of Norton and Stein [9].

Example 2.3. Suppose that e is an idempotent element of Q . Then the fragment



of Γ_Q is a cycle of e . Such a cycle is described as an *idempotent cycle*.

Example 2.4. Suppose that $ef = f$ and $fe = e$ in Q . In this case, the pair $\{e, f\}$ is said to form a *left couplet* in Q . (As a mnemonic, note that e is a left unit for f , and *vice versa*.) The fragment



of Γ_Q is a cycle of e , a so-called *left-couplet cycle*. Right couplets, and right-couplet cycles, are defined dually.

Proposition 2.5. *A quasigroup homomorphism $\theta : Q \rightarrow Q'$ induces graph homomorphisms $\Gamma_\theta : \Gamma_Q \rightarrow \Gamma_{Q'}$ and $C_\theta : C_Q \rightarrow C_{Q'}$.*

Proof. The quasigroup homomorphism $\theta : Q \rightarrow Q'$ induces a map

$$(2.6) \quad M_\theta : M_Q \rightarrow M_{Q'}; (x, y, z, i) \mapsto (x\theta, y\theta, z\theta, i).$$

In order to establish the proposition, we will show that the map M_θ is G -equivariant. Consider elements x, y, z of Q . By (2.2), one has

$$\begin{aligned}
 (z/y, y, z, i)t_1 M_\theta &= (y/z, z, y, it_1)M_\theta = ((y/z)\theta, z\theta, y\theta, it_1) \\
 &= (y\theta/z\theta, z\theta, y\theta, it_1) = (z\theta/y\theta, y\theta, z\theta, i)t_1 \\
 &= ((z/y)\theta, y\theta, z\theta, i)t_1 = (z/y, y, z, i)M_\theta t_1.
 \end{aligned}$$

By (2.3), one has

$$\begin{aligned} (x, x \setminus z, z, i) t_2 M_\theta &= (z, z \setminus x, x, i t_2) M_\theta = (z\theta, (z \setminus x)\theta, x\theta, i t_2) \\ &= (z\theta, z\theta \setminus x\theta, x\theta, i t_2) = (x\theta, x\theta \setminus z\theta, z\theta, i) t_2 \\ &= (x\theta, (x \setminus z)\theta, z\theta, i) t_2 = (x, x \setminus z, z, i) M_\theta t_2. \end{aligned}$$

Finally, by (2.4),

$$\begin{aligned} (x, y, xy, i) t_3 M_\theta &= (y, x, yx, i t_3) M_\theta = (y\theta, x\theta, (yx)\theta, i t_3) \\ &= (y\theta, x\theta, y\theta \cdot x\theta, i t_3) = (x\theta, y\theta, x\theta \cdot y\theta, i) t_3 \\ &= (x\theta, y\theta, (xy)\theta, i) t_3 = (x, y, xy, i) M_\theta t_3. \end{aligned}$$

The graph homomorphism Γ_θ acts by sending an edge

$$v \xrightarrow{i} vt_i$$

of Γ_Q to the edge

$$vM_\theta \xrightarrow{i} vt_iM_\theta = vM_\theta t_i$$

of $\Gamma_{Q'}$ (for $1 \leq i \leq 3$), while C_θ is just restricted from Γ_θ . \square

Given the form of the induced map (2.6), and the definitions of Γ_θ and C_θ in the proof of Proposition 2.5, it is straightforward to conclude:

Theorem 2.6. *The respective assignments of the graph homomorphisms $\Gamma_\theta : \Gamma_Q \rightarrow \Gamma_{Q'}$ and $C_\theta : C_Q \rightarrow C_{Q'}$ to a quasigroup homomorphism $\theta : Q \rightarrow Q'$ yield functors $\Gamma : \mathbf{Q} \rightarrow \mathbf{Graph}$ and $C : \mathbf{Q} \rightarrow \mathbf{Graph}$ from the category \mathbf{Q} of quasigroup homomorphisms to the category \mathbf{Graph} of graph homomorphisms .*

3. STRUCTURE OF THE CYCLE GRAPH

Proposition 3.1. *Let Q be a quasigroup.*

(a) *The cycle graph of Q is a disjoint union*

$$C_Q = \sum_{q \in Q} C_q$$

of the subgraphs C_q .

(b) *Each connected component of C_Q is composed of fragments of the form*

$$(3.1) \quad (, , , 2) \xrightarrow{1} (, , , 3) \xrightarrow{2} (, , , 1) \xrightarrow{3} (, , , 2)$$

Now suppose that Q has finite order n .

(c) *For each element q of Q , the graph C_q is a disjoint union of nontrivial cycles, the cycles of q in the sense of §2. Thus the graph C_Q is planar.*

- (d) *The length of each cycle of C_Q is a multiple of 3.*
- (e) *Each cycle of C_Q is oriented by defining the positive direction to be from left to right along each fragment (3.1)*
- (f) *For each element q of Q , the graph C_q has $3n$ vertices and $3n$ edges.*

Proof. (a): The subgraph C_q is induced on the set of marked triples in which the marked element is q . Note that in Γ_Q , the only edges that might connect marked triples with different marked elements are the stabilizing edges (2.5) — compare (2.2)–(2.4). These edges are excluded from C_Q .

(b): Since the stabilizing edges (2.5) are excluded from C_Q , just the fragments (3.1) remain.

(c): By (b), each vertex of the finite graph C_q has degree 2.

(d) and (e) follow from (b).

(f): For each of the three possible positions of q as the marked element of a marked triple, there is a unique such triple for each of the n elements of Q . Thus C_q has $3n$ vertices. Then by (c), C_q has as many edges as vertices. \square

Example 3.2. According to Proposition 3.1(e), the cycles displayed in Examples 2.3 and 2.4 are oriented in the counterclockwise direction.

Remark 3.3. For an element q of a quasigroup Q , let $\Gamma_{Q,q}$ denote the subgraph of Γ_Q induced on the vertex set of marked triples with q as the marked element. It may happen that a single connected component of $\Gamma_{Q,q}$ breaks up into distinct connected components of C_q . This is the case, for example, with the element 0 of the quasigroup Q displayed by the following multiplication table:

Q	0	1	2	3
0	0	3	1	2
1	1	2	3	0
2	2	1	0	3
3	3	0	2	1

The stabilizing edge

$$(3, 1, 0, 3) \text{ --- } \overset{3}{\text{---}} \text{ --- } (1, 3, 0, 3)$$

in $\Gamma_{Q,0}$ connects marked triples lying on distinct cycles in C_0 .

4. CYCLE NUMBERS

This section examines bounds on the cycle number for certain classes of finite quasigroups. We begin with the case of general quasigroups. Recall that a quasigroup satisfying the identity $xy \cdot yx = x$ is described as a *Schroeder quasigroup* [2, §7.2] [4, p.34]. (The identity $xy \cdot yx = x$ itself is known as *Schroeder's second law* [4, (2.26)].)

Theorem 4.1. *Let Q be a quasigroup of finite order n .*

(a) *The cycle number of Q satisfies the inequality*

$$(4.1) \quad \sigma(C_Q) \leq n^2.$$

(b) *Equality obtains in (4.1) iff Q is a Schroeder quasigroup.*

Proof. By Proposition 3.1(d), the length of each cycle is at least 3. Since there are $3n^2$ marked triples altogether, the bound (4.1) follows. Note that the bound is attained iff each cycle has length 3.

Now consider elements x and y of Q . The cycle in C_x with the marked triple $(x, y, xy, 1)$ includes the following fragment:

$$\begin{array}{ccc} (x, y, xy, 1) & \xrightarrow{2} & (xy, xy \setminus x, x, 3) \\ & & \Big| \\ & & 3 \\ & & \Big| \\ (y, x, yx, 2) & \xrightarrow{1} & (x/yx, yx, x, 3) \end{array}$$

The cycle closes to a cycle of length 3 iff the equivalent conditions

(a) $xy = x/yx$, (b) $xy \cdot yx = x$, and (c) $yx = xy \setminus x$

obtain. In particular, each cycle has length 3 iff (b) holds identically in Q , i.e., iff Q is a Schroeder quasigroup. \square

A quaternion construction shows that the bound of Theorem 4.1 is attainable whenever n is a power of 81:

Proposition 4.2. *Suppose $n = 3^{4k}$ for some natural number k . Then there is a quasigroup Q of order n such that the bound (4.1) is attained.*

Proof. Let A be an elementary abelian group of order 3^k . Consider the automorphisms

$$i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad j = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

of A^4 — compare [11, p.157]. Define a quasigroup multiplication

$$x \cdot y = xi + yj$$

on A^4 — compare [10, (2.15)]. Since $ij = -ji$ and $i^2 + j^2 = -2 \equiv 1 \pmod{3}$, we have

$$xy \cdot yx = (xi + yj)i + (yi + xj)j = x(i^2 + j^2) + y(ji + ij) = x$$

for x and y in $Q = (A^4, \cdot)$, so $\sigma(C_Q) = n^2$ by Theorem 4.1(b). \square

For a given finite order n , R.D. Baker demonstrated the co-existence of idempotent Schroeder quasigroups and triple tournaments of that order [1]. As a contrast to the quaternion construction presented in Proposition 4.2, there are purely combinatorial constructions, primarily due to C.C. Lindner *et al.*, of Schroeder quasigroups for almost all finite orders n congruent to 0 or 1 modulo 4 [7], compare [2, §7.2] [4, II.3.5, 7.3]. Corollary 7.5 below shows the necessity of the congruence restriction.

Although this section is mainly concerned with maximization of the cycle number, as in Theorem 4.1, it is worth pointing out that a result of Norton and Stein yields a lower bound for the cycle number of a finite commutative quasigroup.

Proposition 4.3. *Let Q be a quasigroup of finite order n . Then*

$$\sigma(C_Q) \geq \frac{1}{2}n^2$$

if Q is commutative.

Proof. If Q is commutative, then each cycle has length 3 or 6 [9, Th.6.2]. (Note that we measure the length of a cycle in the graph-theory sense, as the number of edges, while the length according to Norton and Stein is one third as large.) Thus the number of cycles is no less than one sixth the total number $3n^2$ of vertices in C_Q . \square

A direct analysis of the cycles may be carried out for *semisymmetric quasigroups*, defined by the identity $(xy)x = y$ or $R(x)^{-1} = L(x)$ [10, 1.4]. For a given integer r , note that

$$x \cdot yR(x)^r = yR(x)^r L(x) = yR(x)^r R(x)^{-1} = yR(x)^{r-1}$$

and

$$x / (yR(x)^{r+1}) = xR(yR(x)^{r+1})^{-1} = xL(yR(x)^{r+1}) = yR(x)^{r+2}$$

within a semisymmetric quasigroup. Inductions on r and $-r$ then yield the following result.

Proposition 4.4. *Let Q be a (possibly infinite) semisymmetric quasigroup. Let x and y be elements of Q . Then the cycle of x that contains the marked triple*

$$(x, y, xy, 1) = (x, yR(x)^0, yR(x)^{-1}, 1)$$

is constructed from fragments of the form

$$\begin{aligned} & (x, yR(x)^r, yR(x)^{r-1}, 1) \xrightarrow{3} (yR(x)^r, x, yR(x)^{r+1}, 2) \xrightarrow{1} \\ & \xrightarrow{1} (yR(x)^{r+2}, yR(x)^{r+1}, x, 3) \xrightarrow{2} (x, yR(x)^{r+3}, yR(x)^{r+2}, 1) \end{aligned}$$

for an integer r .

Corollary 4.5. *Let x and y be distinct generators of a free semisymmetric quasigroup. Then the “cycle” that contains the marked triple $(x, y, xy, 1)$ is a doubly infinite path.*

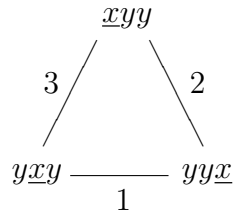
We now exhibit two classes of quasigroups, namely totally symmetric quasigroups and commutative diassociative loops, in which each member Q of finite order n has the triangular number $T(n)$ as its cycle number. The respective enumerations are similar in general pattern, but differ in the specific details. First recall that a quasigroup is *totally symmetric* if it satisfies the identities $xy = yx = x/y = y/x = x \setminus y = y \setminus x$.

Proposition 4.6. *Let Q be a totally symmetric quasigroup of finite order n . Then the cycle number $\sigma(C_Q)$ of Q is the triangular number $T(n)$.*

Proof. Since Q has n elements y , there are n pairs (x, y) in Q^2 that satisfy the equivalent conditions

- (a) $x = y \cdot y$, (b) $x = y/y$, (c) $xy = y$, and (d) $yx = y$.

For each such pair, there is a 3-cycle



in C_x . For the remaining $n^2 - n$ pairs (x, y) in Q^2 , there is a 6-cycle

$$\begin{array}{ccccc}
\underline{xyz} & \xrightarrow{2} & \underline{zyx} & \xrightarrow{1} & \underline{zxy} \\
3 \downarrow & & & & \downarrow 3 \\
\underline{yxz} & \xrightarrow{1} & \underline{yzx} & \xrightarrow{2} & \underline{xzy}
\end{array}$$

with $xy = z \neq y$. Since the same cycle also corresponds to the pair $(x, z) = (x, xy)$, there will be $(n^2 - n)/2$ such 6-cycles altogether. The cycle number $\sigma(C_Q)$ is $n + (n^2 - n)/2 = (n^2 + n)/2 = T(n)$. \square

A loop is said to be *diassociative* if the subloop generated by each pair of elements is associative (and thus forms a group). The study of cycles in a diassociative loop may be conducted as if the diassociative loop were a group. For example, in Proposition 4.8 below, we use the conjugation notations $x^y = y^{-1}xy$ and $x^{-y} = y^{-1}x^{-1}y$ for elements x and y of a diassociative loop.

Proposition 4.7. *Suppose that Q is a commutative diassociative loop of finite order n . Then the cycle number $\sigma(C_Q)$ of Q is the triangular number $T(n)$.*

Proof. Since Q has n elements y , there are n pairs (x, y) in Q^2 with $x = y^{-2}$. For each such pair, there is a 3-cycle

$$\begin{array}{ccc}
& \underline{xyy^{-1}} & \\
3 \swarrow & & \searrow 2 \\
\underline{xyy^{-1}} & \xrightarrow{1} & \underline{y^{-1}y^{-1}x}
\end{array}$$

in C_x . For the other $n^2 - n$ pairs (x, y) in Q^2 , there is a 6-cycle

$$\begin{array}{ccccc}
\underline{xyz} & \xrightarrow{2} & \underline{zy^{-1}x} & \xrightarrow{1} & \underline{z^{-1}xy^{-1}} \\
3 \downarrow & & & & \downarrow 3 \\
\underline{yxz} & \xrightarrow{1} & \underline{y^{-1}zx} & \xrightarrow{2} & \underline{xz^{-1}y^{-1}}
\end{array}$$

in C_x (note that $z = xy \neq y^{-2}y = y^{-1}$). Since the same cycle also corresponds to the pair $(x, z^{-1}) = (x, (xy)^{-1})$, there will be $(n^2 - n)/2$ such 6-cycles altogether. Thus the cycle number $\sigma(C_Q)$ is the total $n + (n^2 - n)/2 = (n^2 + n)/2 = T(n)$. \square

The following result describes the cycles of a given element of a general diassociative loop.

Proposition 4.8. *Suppose that x and y are elements of a (possibly infinite) diassociative loop Q . Then the cycle of x that contains the marked triple $(x, y, xy, 1)$ consists of fragments of the form*

$$\begin{aligned} & (x, y^{x^r}, xy^{x^r}, 1) \xrightarrow{3} (y^{x^r}, x, y^{x^r}x, 2) \xrightarrow{1} (y^{-x^r}, y^{x^r}x, x, 3) \xrightarrow{2} \\ & (x, x^{-1}y^{-x^r}, y^{-x^r}, 1) \xrightarrow{3} (x^{-1}y^{-x^r}, x, y^{-x^{r+1}}, 2) \xrightarrow{1} \\ & (xy^{x^{r+1}}, y^{-x^{r+1}}, x, 3) \xrightarrow{2} (x, y^{x^{r+1}}, xy^{x^{r+1}}, 1) \end{aligned}$$

for an integer r .

Proof. Use induction on r and $-r$. □

Theorem 4.9. *Let Q be a diassociative loop of finite order n .*

(a) *The cycle number of Q satisfies the inequality*

$$(4.2) \quad \sigma(C_Q) \leq T(n).$$

(b) *Equality obtains in (4.2) if and only if Q is commutative.*

Proof. First, note that if Q is commutative, then equality holds in (4.2) by Proposition 4.7.

Now consider the general case. Proposition 3.1(b) shows that each cycle of Q contains a triple of the form $(, , , 1)$. By Proposition 4.8 (with $r = 0$), there is a 3-cycle including $(x, y, xy, 1)$ for elements x and y of Q if and only if $y = x^{-1}y^{-1}$ or $x = y^{-2}$. Since Q^2 contains only n pairs (x, y) with $x = y^{-2}$, there are just n such cycles. These cycles comprise $3n$ of the total number $3n^2$ of marked triples. By Proposition 3.1(d), the length of each remaining cycle is at least 6. Thus

$$(4.3) \quad \sigma(C_Q) \leq n + \frac{3n^2 - 3n}{6} = T(n),$$

proving (a).

Finally, suppose that equality holds in (4.2). Then by (4.3), each cycle is of length 3 or 6. Let (x, y) be a pair of elements of Q . There are two cases to consider:

(a) If the marked triple $(x, y, xy, 1)$ lies in a cycle of length 3, then $x = y^{-2}$ and $xy = yx$ in this case;

(b) If the marked triple $(x, y, xy, 1)$ lies in a cycle of length 6, setting $r = 0$ in Proposition 4.8 shows that $y = x^{-1}yx$, so again $xy = yx$. □

Let n be a given finite order. Theorem 4.9 characterizes the abelian groups among all groups of order n by maximization of the cycle number, at $T(n)$. Similarly, recalling Moufang's Theorem that Moufang loops are diassociative [3, VII.4], Theorem 4.9 serves to characterize commutative Moufang loops among all Moufang loops of order n by maximization of the cycle number, again at $T(n)$.

5. THE UNMARKED MULTIPLICATION TABLE

The *unmarked multiplication table* of a quasigroup Q is defined as

$$K_Q^0 = \{(x, y, z) \in Q^3 \mid xy = z\}.$$

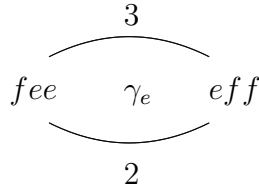
The *unmarking projection* is defined as

$$M_Q \rightarrow K_Q^0; (x, y, z, i) \mapsto (x, y, z).$$

As for the marked multiplication table, it is sometimes convenient to write an element (x, y, z) of the unmarked table simply as xyz .

From now on, suppose that Q is a quasigroup of finite order n . In this case, $|K_Q^0| = n^2$. Consider a cycle of C_Q which is not an idempotent cycle in the sense of Example 2.3. Its vertices are certain marked triples. Unmarking these vertices induces a quotient graph with loops. Deletion of the loops leaves a cycle, known as a *collapsed cycle*, which inherits the orientation provided by Proposition 3.1(e).

Example 5.1. Consider the couplet cycle of Example 2.4. It collapses to



with the inherited counterclockwise orientation. A loop labeled 1 has been deleted from the unmarked vertex fee .

Define K_Q^2 to be the union of the set of all collapsed cycles with the set of all idempotents of Q . By convention, the idempotents may also be considered as (degenerate) collapsed cycles. Note that $|K_Q^2| = \sigma(C_Q)$.

The unmarking projection induces a quotient graph of C_Q on the vertex set K_Q^0 . Delete all loops from this quotient graph, and let K_Q^1 denote the set of remaining edges. These remaining edges inherit labels t_1, t_2 or t_3 from C_Q (again just written as 1, 2, or 3 in actual figures).

Lemma 5.2. *Consider an edge*

$$(5.1) \quad (x_1, x_2, x_3, i) \xrightarrow{k} (y_1, y_2, y_3, j)$$

of C_Q . The following conditions are equivalent:

- (a) $x_i = x_j$;
- (b) $y_i = y_j$;
- (c) The edge (5.1) unmarks to a loop.

Proof. Since (5.1) is not a stabilizing edge, one has $t_k = (i j)$. Thus (a) and (b) are equivalent. If they hold, then $(x_1, x_2, x_3) = (y_1, y_2, y_3)$, so (5.1) unmarks to a loop. Conversely, if (c) holds, then $(x_1, x_2, x_3) = (y_1, y_2, y_3)$. Since $x_i = y_{it_k} = y_j$ and $x_j = y_{jt_k} = y_i$, (a) and (b) follow. \square

Corollary 5.3. *Let*

$$(5.2) \quad (x_1, x_2, x_3, i) \xrightarrow{k} (y_1, y_2, y_3, j)$$

be an edge of C_Q that does not unmark to a loop. Then

$$(5.3) \quad \{x_i, x_j\} = \{y_i, y_j\}$$

with $x_j = y_i \neq y_j = x_i$.

Proposition 5.4. *Let Q be a finite quasigroup of order n .*

- (a) $|K_Q^2| = \sigma(C_Q)$.
- (b) $|K_Q^1| = 3n(n - 1)/2$.
- (c) $|K_Q^0| = n^2$.

Proof. Since (a) and (c) have already been noted, it remains to verify (b). Consider an edge (5.1) of C_Q which survives the unmarking process to appear in K_Q^1 . There are three choices for k . For each such choice, there are $n(n - 1)/2$ choices for the doubleton (5.3). \square

6. THE DUAL COMPLEX

Let $\mathbb{Z}K_Q^r$ denote the free abelian group on K_Q^r , for $0 \leq r \leq 2$. We will define group homomorphisms $\partial_2 : \mathbb{Z}K_Q^2 \rightarrow \mathbb{Z}K_Q^1$ and $\partial_1 : \mathbb{Z}K_Q^1 \rightarrow \mathbb{Z}K_Q^0$ to yield a(n oriented) complex

$$(6.1) \quad K_Q^2 \xrightarrow{\partial_2} K_Q^1 \xrightarrow{\partial_1} K_Q^0$$

known as the (*dual*) *Norton-Stein complex* K_Q of the quasigroup Q .

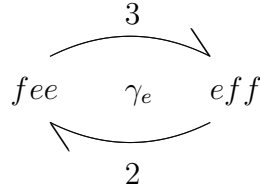
The orientation of the edges in K_Q^1 is based on a well-ordering (Q, \leq) of the underlying set of Q . Consider an element of K_Q^1 , obtained by unmarking an edge (5.2) of C_Q . Its orientation is defined as

$$(6.2) \quad x_1x_2x_3 \xrightarrow{k} y_1y_2y_3$$

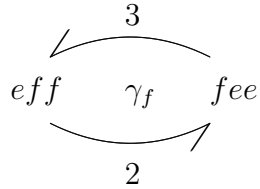
with $y_{k+1} < y_{k+2}$ (using addition modulo 3 for the suffixes). Of course, by Corollary 5.3, one then has $x_{k+2} < x_{k+1}$. It is convenient to write

$(k, y_{k+1} < y_{k+2})$ for the oriented edge. The group homomorphism ∂_1 takes $(k, y_{k+1} < y_{k+2})$ to the signed sum $y_1y_2y_3 - x_1x_2x_3$. The group homomorphism ∂_2 maps a non-degenerate collapsed cycle to a signed sum of the edges that constitute it. An edge takes a positive sign if its orientation (6.2) is consistent with the inherited orientation of the cycle, and a negative sign if the edge is oriented in the opposite direction to the cycle. Degenerate collapsed cycles corresponding to idempotents of Q are sent to 0 by ∂_2 .

Example 6.1. For a left couplet $\{e, f\}$ of Q , consider the collapsed cycle γ_e of Example 5.1. Suppose that $e < f$. The cycle becomes



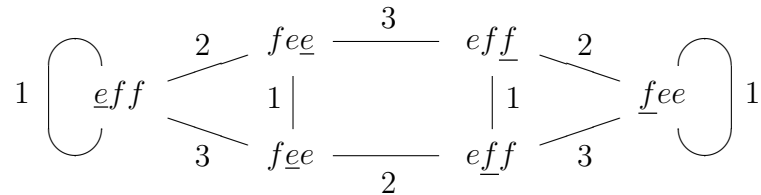
when its edges are oriented. It is sent to $-(2, e < f) + (3, e < f)$ by the group homomorphism ∂_2 . Now consider the corresponding couplet cycle γ_f . It becomes



when its edges are oriented, and maps to $(2, e < f) + (3, e < f)$ under ∂_2 . The fragment

$$(6.3) \quad \{\gamma_e, \gamma_f\} \xrightarrow{\partial_2} \{(2, e < f), (3, e < f)\} \xrightarrow{\partial_1} \{eff, fee\}$$

of the Norton-Stein complex corresponds to the connected component



of Γ_Q . Geometrically, the complex (6.3) is realized by a sphere. In geographical terms, one might say that γ_e is the southern hemisphere,

and γ_f is the northern hemisphere. Then $(2, e < f)$ is the eastern half of the equator, while $(3, e < f)$ is the western half. The hemispheres are oriented by normals emerging from the center of the earth. The orientation of the equator lines corresponds to the direction of rotation of the earth. The Gulf of Guinea contains eff , while fee lies somewhere in Kiribati.

Remark 6.2. In the sense of [5, p.20], the complex (6.1) is dual to the original Norton-Stein construction [8] as extended in [6]. In order to distinguish the two complexes, it is sometimes convenient to refer to the latter as the *primal* complex, while (6.1) is the *dual* complex.

7. THE DEPLETED COMPLEX

Let Q be a finite quasigroup, with set E_Q of idempotents. Let \tilde{K}_Q^2 denote the set of non-degenerate collapsed cycles of Q . Note that

$$(7.1) \quad |K_Q^2| = |\tilde{K}_Q^2| + |E_Q|.$$

Set $\tilde{K}_Q^1 = K_Q^1$. Let \tilde{K}_Q^0 be the complement in K_Q^0 of the set $\{eee \mid e \in E_Q\}$. Again, note that

$$(7.2) \quad |K_Q^0| = |\tilde{K}_Q^0| + |E_Q|.$$

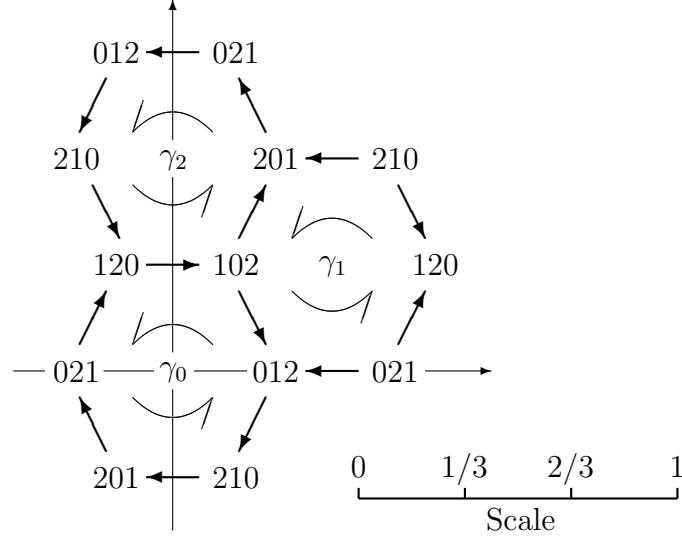
The *depleted Norton-Stein complex* \tilde{K}_Q or

$$(7.3) \quad \tilde{K}_Q^2 \xrightarrow{\partial_2} \tilde{K}_Q^1 \xrightarrow{\partial_1} \tilde{K}_Q^0$$

of Q is defined from (6.1) by restriction.

Proposition 7.1. *Let Q be a finite quasigroup. Then the depleted Norton-Stein complex of Q is realized geometrically by an orientable surface.*

Proof. As a disjoint union of cycles, the set \tilde{K}_Q^2 of non-degenerate collapsed cycles forms a planar graph, which may be drawn on the plane so that each of the (disjoint) cycles is oriented in the counterclockwise direction. Each cycle bounds a disc. The geometric realization of \tilde{K}_Q is then obtained from the union of these discs by identifying the bounding edges from the set \tilde{K}_Q^1 . Each edge (6.2) appears twice, once on each of a pair of discs. As these discs are stitched together along the edge by the identification process, the orientations of the discs on each side of the edge are consistent. (Compare Figure 1 for an illustration.) \square

FIGURE 1. A toral component in the complex of $\mathbb{Z}/3\mathbb{Z}$.

Example 7.2. Let Q be the group $(\mathbb{Z}/3\mathbb{Z}, +)$ of integers modulo 3 under negated addition, with the natural order $0 < 1 < 2$. Take the complex cube root of unity $\omega = (-1 + i\sqrt{3})/2$. The depleted complex of Q has a geometric realization which includes a connected component given by the (real) torus (or complex elliptic curve) $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$, i.e., identifying points of the complex plane that differ by an integral multiple of 1 or an integral multiple of ω . This is illustrated in Figure 1, with 012 located at $1/3 + (\mathbb{Z} + \mathbb{Z}\omega)$ on the torus and 120 at $\omega/3 + (\mathbb{Z} + \mathbb{Z}\omega)$. For each element x of Q , the corresponding cycle in C_x is labeled γ_x , drawn with its orientation. The real and imaginary axes of the complex plane, and the scale, are also indicated.

The following result is the counterpart of [8, Th. II], as extended from the idempotent to the general finite case in [6].

Theorem 7.3. *Suppose that Q is a quasigroup of finite order n . Then the cycle number $\sigma(C_Q)$ of Q is congruent modulo 2 to the triangular number $T(n) = n(n+1)/2$.*

Proof. Proposition 7.1 shows that the Euler characteristic

$$|\tilde{K}_Q^2| - |\tilde{K}_Q^1| + |\tilde{K}_Q^0|$$

of the oriented geometric realization of \tilde{K}_Q is even. Thus by (7.1), (7.2), and Proposition 5.4, the integer

$$\begin{aligned} \sigma(C_Q) - 3n(n-1)/2 + n^2 &= |K_Q^2| - |K_Q^1| + |K_Q^0| \\ &= |\tilde{K}_Q^2| + |E_Q| - |\tilde{K}_Q^1| + |\tilde{K}_Q^0| + |E_Q| \end{aligned}$$

is even. The result follows, since $3n(n-1)/2 - n^2 \equiv T(n) \pmod{2}$. \square

The first application of Theorem 7.3 gives a simplified proof of a result of Stein [12], compare [6].

Corollary 7.4. *Suppose that Q is a quasigroup of finite order n , with a transitive group of automorphisms. Then n is congruent to 0, 1, or 3 modulo 4.*

Proof. Suppose that $n = 4k+2$ for some natural number k . Since Q has a transitive automorphism group, Proposition 2.5 shows that there is a constant c such that C_x has c connected components for each $x \in Q$. Hence $\sigma(C_Q) = \sum_{x \in Q} c = (4k+2)c \equiv 0 \pmod{2}$. However, Theorem 7.3 implies $\sigma(C_Q) \equiv (2k+1)(4k+3) \equiv 1 \pmod{2}$, a contradiction. \square

As a second application of Theorem 7.3, we give a direct proof of a congruence condition on the possible orders of general Schroeder quasigroups that was obtained by Lindner *et al.* using a combinatorial analysis [7, Th. 2]. Our method extends the ‘‘alternate proof’’ proposed by Norton and Stein for the idempotent case [9, Th.4.2].

Corollary 7.5. *If Q is a Schroeder quasigroup of finite order n , then n is congruent to 0 or 1 modulo 4.*

Proof. By Theorem 4.1, $\sigma(C_Q) = n^2$, while by Theorem 7.3, $\sigma(C_Q) \equiv n(n+1)/2 \pmod{2}$. However, $n^2 \equiv n(n+1)/2 \pmod{2}$ if and only if n is congruent to 0 or 1 modulo 4. \square

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