ALGEBRAIC PROPERTIES OF QUANTUM QUASIGROUPS

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Abstract. Quantum quasigroups provide a self-dual framework for the unification of quasigroups and Hopf algebras. This paper furthers the transfer program, investigating extensions to quantum quasigroups of various algebraic features of quasigroups and Hopf algebras. Part of the difficulty of the transfer program is the fact that there is no standard model-theoretic procedure for accommodating the coalgebraic aspects of quantum quasigroups. The linear quantum quasigroups, which live in categories of modules under the direct sum, are a notable exception. They form one of the central themes of the paper.

From the theory of Hopf algebras, we transfer the study of grouplike and setlike elements, which form separate concepts in quantum quasigroups. From quasigroups, we transfer the study of conjugate quasigroups, which reflect the triality symmetry of the language of quasigroups. In particular, we construct conjugates of cocommutative Hopf algebras. Semisymmetry, Mendelsohn, and distributivity properties are formulated for quantum quasigroups. We classify distributive linear quantum quasigroups that furnish solutions to the quantum Yang-Baxter equation. The transfer of semisymmetry is designed to prepare for a quantization of web geometry.

1. Introduction and background

1.1. **Introduction.** In any symmetric, monoidal category, quantum quasigroups provide natural self-dual generalizations of Hopf algebras, without requirements of unitality, counitality, associativity, or coassociativity. They embrace diverse phenomena such as octonion multiplication, function spaces on the 7-sphere, new solutions to the quantum Yang-Baxter equation, and the combinatorial structures of quasigroups and Latin squares.

One of the main programs in the study of quantum quasigroups involves the transfer to them of various model-theoretic aspects such as quasigroup

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identities. The transfer would be routine if it were the case that quantum quasigroups only involved algebraic structure, but the parallel coalgebraic structure makes the transfer non-trivial in general. Nevertheless, here, we identify *linear quantum quasigroups*, and particularly quantum quasigroups in symmetric monoidal categories of finitely generated modules under the direct sum, as targets for a more amenable transfer of identities and model-theoretic properties.

Quasigroups and Latin squares have a rich equational theory, endowed with a triality symmetry or S_3-action that corresponds to interchanges of the respective roles of row labels, column labels, and body labels in a Latin square. The current paper initiates an extension of the concept of triality symmetry to quantum quasigroups (§4). But just as the group-theoretic concept of an inverse, which provides an exact left/right duality for groups, in a Hopf algebra becomes the antipode which need not be involutory, or even invertible, so we find that the triality of quasigroups may break down in quantum quasigroups. Nevertheless, Theorems 4.8 and 6.4 respectively show that it works to a certain degree for cocommutative Hopf algebras, and very well for linear quantum quasigroups.

The equationally defined class of semisymmetric quasigroups is especially important from many points of view. Within the triality symmetry, its defining equations are invariant under the alternating group A_3. In fact, there are equivalent left- and right-handed defining equations. Following the general approach used on Moufang identities in [1], we propose left- and right-handed versions of semisymmetry for quantum quasigroups in this paper (§5.2). Theorem 5.9 establishes the independence of these two versions of semisymmetry in categories of vector spaces under the direct sum, even though the two versions are equivalent in the category of sets under the direct product.

Idempotent semisymmetric quasigroups are equivalent to combinatorial designs known as Mendelsohn triple systems. Combining an earlier notion of quantum idempotence [8, 28] with our new concepts of semisymmetry, we obtain left- and right-handed versions of a Mendelsohn property applying to quantum quasigroups. In contrast with the situation of Theorem 5.9, Theorems 6.4 and 7.1 show that these two versions coincide respectively in categories of modules under the direct sum, and for linear quantum quasigroups. Theorem 6.8 classifies Mendelsohn linear quantum quasigroups in terms of an automorphism μ of the underlying module.

Earlier work [28, 29] identified quantum quasigroup properties, that are known under the name of *quantum distributivity* (in both left- and right-handed versions), which produce solutions of the quantum Yang-Baxter equation. Theorem 8.8 classifies the linear Mendelsohn quantum quasigroups that are quantum distributive, in terms of an equation (8.7) that has to be satisfied by the automorphism μ of Theorem 6.8.

1.2. **Quantum quasigroups unify quasigroups and Hopf algebras.** Cancellativity properties of groups have been extended in two directions: using linearization or "quantization" to yield Hopf algebras [13, 23], or by relaxation of the associativity requirement to obtain quasigroups [24]. A *quasigroup* $(Q, \cdot, /, \backslash)$ is a set Q equipped with respective binary operations of *multiplication*, *right division* and *left division*, such that the identities

$$\text{(1.1)} \qquad \begin{array}{llll} \text{(SL)} & x \cdot (x\backslash z) = z\,, & \text{(SR)} & z = (z/x) \cdot x\,, \\ \text{(IL)} & x\backslash(x \cdot z) = z\,, & \text{(IR)} & z = (z \cdot x)/x \end{array}$$

are satisfied. Nonempty quasigroups with associative multiplication are groups, with $x/y = xy^{-1}$ and $x\backslash y = x^{-1}y$. On the other hand, within a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, say a category of vector spaces with the tensor product, or the category of sets with the direct product, a *Hopf algebra* $(A, \nabla, \eta, \Delta, \varepsilon, S)$ embraces (linearized) monoid (A, ∇, η) and comonoid (A, Δ, ε) structures that are mutually homomorphic. Then the cancellativity is captured by the *antipode* $S \colon A \to A$, which is an inverse to the identity $1_A \colon A \to A$ in the *convolution monoid* $\big(\mathbf{V}(A, A), *, \varepsilon\eta\big)$ with product $f * g = \Delta(f \otimes g)\nabla$, as in (2.4) below.[1] A group $(Q, \cdot, e, {}^{-1})$ is a Hopf algebra in the category of sets under direct product, with multiplication $\nabla \colon x \otimes y \mapsto x \cdot y$ (using the tensor product symbol for ordered pairs), trivial subgroup $\mathbf{1}\eta = \{e\}$, *diagonal* comuliplication $\Delta \colon x \mapsto x \otimes x$, and inversion $S \colon x \mapsto x^{-1}$.

Quantum quasigroups were introduced to provide a self-dual unification of quasigroups and Hopf algebras [27]. In a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, a *quantum quasigroup* (A, ∇, Δ) is an object A, equipped with *multiplication* $\nabla \colon A \otimes A \to A$ and *comultiplication* $\Delta \colon A \to A \otimes A$ that are mutually homomorphic, where the *left composite*

$$\text{(1.2)} \qquad \mathsf{G} \colon A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A$$

("G" for "Gauche") and the *right composite*

$$\text{(1.3)} \qquad \mathsf{Ə} \colon A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes 1_A} A \otimes A$$

("Ə" for "Droite") morphisms are invertible. Quasigroups (equipped with diagonal comultiplication) are quantum quasigroups in the category of sets [11], while any Hopf algebra $(A, \nabla, \eta, \Delta, \varepsilon, S)$ reduces to a quantum quasigroup (A, ∇, Δ). Previously studied nonassociative generalizations of Hopf

[1] Throughout the paper, we default to algebraic notation, with functions to the right of their arguments, and composing morphisms in natural reading order from left to right.

algebras, including the *Hopf quasigroups* of Majid *et al.*, also reduce to quantum quasigroups [1, 3, 14, 15, 22]. However, these earlier concepts are not self-dual. Moreover, since they are all based on the use of an antipode, they really ought to be considered as linearizations of inverse property loops (as in Definition 5.4 below), rather than quasigroups as such. Nevertheless, since the term "Hopf quasigroup" is already in use, the term "quantum quasigroup" has been adopted for the general concept.

1.3. **Algebraic and coalgebraic properties of quantum quasigroups.** The current paper forms part of a central program within the study of quantum quasigroups: the transfer to them of various algebraic properties of quasigroups. For example, the quasigroup-theoretic property of (self)-distributivity transfers to *quantum distributivity*, where the composites of the quantum quasigroup are required to satisfy the quantum Yang-Baxter equation (QYBE) [28, 29] — compare §8.1 below. Various quantum quasigroups thus provide new solutions to the QYBE. As noted above, the Hopf quasigroups studied by Majid *et al.* transfer inverse property loops to the quantum setting, while the *Hopf algebras with triality* of Benkart *et al.* transfer Moufang loops to the quantum setting [1]. The diversity of these examples already hints at the inherent difficulty of the transfer program: the interplay of algebraic and coalgebraic structure in quantum quasigroups mean that, in general, there is no standard model-theoretic procedure for transferring quasigroup properties to quantum quasigroup properties.

Nevertheless, there is one important case where the coalgebraic aspects of quantum quasigroups may be suppressed. This is the so-called *linear* case, where the underlying symmetric monoidal category is taken to be a category of modules over a commutative, unital ring, but under direct sums (with the zero module as the monoidal unit), rather that under the tensor product (with the module reduct of the ring as monoidal unit). Since it is the tensor product that provides entanglement, the linear case is not of direct interest for applications to the study of quantum processes. From the theoretical point of view, however, the bug becomes a feature. Thus the linear case provides an amenable proving ground for the exploration of various aspects of quantum quasigroups. Here *bimagmas*, namely objects carrying mutually homomorphic multiplications and comultiplications, just correspond to bimodules (as discussed in §3.4 below). In the linear case, the coalgebraic structure is actually algebraic.

Since quantum quasigroups provide a unification of quasigroups and Hopf algebras, a parallel program seeks algebraic properties of quantum quasigroups that correspond to diverse aspects of the theory of Hopf algebras. In this paper, attention is focused for the first time on grouplike and setlike elements of quantum quasigroups, as in §3.1 below. While the two concepts are usually conflated in the study of Hopf algebras, they are separated in the study of quantum quasigroups.

In the opposite direction, we reverse the information flow from quantum quasigroups to Hopf algebras. First, quasigroup conjugacy, which reflects the triality or S_3-symmetry of the theory of quasigroups, is transferred from quasigroups to quantum quasigroups in §4.2. Then, §4.4 examines how quantum quasigroup conjugacy appears in the context of cocommutative Hopf algebras.

1.4. **Plan of the paper.** The paper begins with a brief reprise in Section 2, particularly for the benefit of readers more familiar with quasigroups from the combinatorial side, of the key aspects of symmetric monoidal categories that underlie Hopf algebras and quantum quasigroups. In particular, we draw attention to our special version $\Delta\colon x \mapsto x^L \otimes y^R$ of the Sweedler comultiplication notation, adapted to the situation where coassociativity is not necessarily required, and to our use of Jay-type set-theoretical notation [12] within symmetric monoidal categories.

Section 3 initiates the study of grouplike and setlike elements in quantum quasigroups, within categories of modules under the tensor product or the direct sum. Theorem 3.4 establishes that the set of grouplike elements of a quantum quasigroup in a category of vector spaces under the tensor product actually forms a quasigroup. Attention then turns to categories of modules under the direct sum. As previously noted, §3.4 shows that bimagmas are equivalent to certain bimodules in this setting, namely over algebras that are generated respectively by endomorphisms L, R as appearing in our Sweedler notation for the comultiplication, and endomorphisms ρ, λ that represent multiplication by 0 on the right or on the left.

In §3.5 one of the central topics of the paper, a *linear quantum quasigroup* $A(\rho, \lambda, L, R)$, is defined as a bimodule where the endomorphisms ρ, λ, L, R are all invertible. Linear quantum quasigroups form quantum quasigroups in categories of modules under the direct sum. Conversely, in categories of finitely generated modules (under the direct sum), quantum quasigroups are linear quantum quasigroups (as noted in Theorem 3.14). Setlike elements of linear quantum quasigroups are then examined in §3.6. Section 3 concludes with a brief study of the adjunction connecting linear quasigroups and linear quantum quasigroups.

If $(Q, \cdot, /, \backslash)$ is a quasigroup with a given multiplication $(x, y) \mapsto x \cdot y$, then further quasigroup structures on the set Q are furnished by taking right division $(x, y) \mapsto x/y$ or left division $(x, y) \mapsto x\backslash y$ as multiplications. Additional quasigroup structures on the set Q take the opposite $(x, y) \mapsto y{\cdot}x$ of the original multiplication, or the opposites $(x, y) \mapsto y/x$ or $(x, y) \mapsto y\backslash x$ of the original divisions, as the multiplication. These six quasigroups are known as *conjugates* or "parastrophes" [24, §1.3]. Section 4 of the current paper presents the analogous notion — *quantum conjugates* — for quantum quasigroups (Definition 4.2). Quantum conjugates of cocommutative Hopf algebras are described in §4.4, while §4.5 examines quantum conjugates of linear quantum quasigroups.

A magma is described as (*left*) *semisymmetric* if it satisfies $(xy)x = y$, and (*right*) *semisymmetric* if it satisfies $x(yx) = y$. The two properties are equivalent, each implying that the magma is a quasigroup [24, §1.4]. Semisymmetry plays an important role in the theory of quasigroups, most particularly in the application to web geometry [25]. Thus transfer of semisymmetry to quantum quasigroups becomes fundamental for the task of quantizing web geometry.

Section 5 presents one approach to the transfer of semisymmetry to bimagmas $(A, \nabla, \Delta, \varepsilon)$ that are equipped with an *augmentation* morphism $\varepsilon \colon A \to \mathbf{1}$ in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. The semisymmetry concepts (see Definition 5.3) are quite direct, requiring just one level of comultiplication. Typical examples of semisymmetric quantum quasigroups in our sense are provided by commutative inverse property loops (for example, commutative Moufang loops), as shown by Theorem 5.7. In general, the left and right-handed semisymmetry properties are not equivalent, and do not entail that a bimagma becomes a quantum quasigroup (Theorem 5.9). Exhibited in a category of vector spaces under the direct sum, this behavior is contrasted with the case of counital bimagmas in the category of sets (§5.5), where the two semisymmetry conditions are equivalent, and force the bimagma to be a quantum quasigroup.

Idempotent semisymmetric quasigroups are equivalent to combinatorial designs known as *Mendelsohn triple systems*, covered by oriented cycles of three elements [5, 6, 20]. Section 6 transfers the Mendelsohn property to quantum quasigroups, combining the quantum idempotence property [28, Defn. 5.1] (originally introduced in [8] under a different name) with the semisymmetry properties from the previous section. In the presence of quantum idempotence, the left and right semisymmetry properties turn out to be equivalent in the linear setting, and to induce the structure of a linear quantum quasigroup (Theorem 6.4). Linear Mendelsohn quantum quasigroups $A(\rho, \lambda, L, R)$ are constructed and classified (§6.3). They are fully parametrized by a single automorphism (Theorem 6.8). Section 6 concludes with a brief discussion of setlike elements in linear Mendelsohn quantum quasigroups.

Section 7 studies the semisymmetry of linear quantum quasigroups. In this context, left and right semisymmetry are equivalent (Theorem 7.1). The semisymmetrization process, constructing a semisymmetric quasigroup from an arbitrary quasigroup [25], is transferred to the setting of linear quantum quasigroups in §7.2. A similar process, constructing a linear Mendelsohn quasigroup or triple system from an arbitrary linear quantum quasigroup, is exhibited in §7.3.

A bimagma is said to be *quantum left* (or *right*) *distributive* whenever its left (or right) composite solves the quantum Yang-Baxter equation (§8.1). Section 8 investigates those linear quantum quasigroups that are quantum distributive. The left and right quantum distributivity conditions turn out to be equivalent in this setting (Theorem 8.5), yielding a *linear quantum distributive quasigroup*. These objects are classified by Theorem 8.7. Those with the Mendelsohn property are analyzed in §8.4. The parametrizing endomorphism from Theorem 6.8 satisfies the quartic equation (8.7) in this case.

2. Structures in symmetric monoidal categories

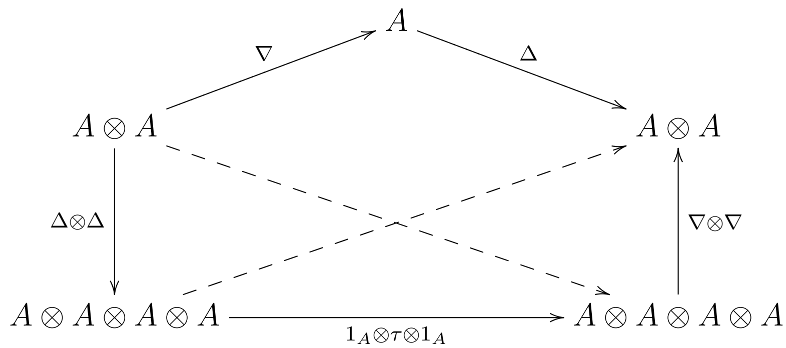
The general setting for the work of this paper is a (strict) symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, with a symmetry $\tau \colon A \otimes B \to B \otimes A$ for objects A and B. Primary examples are as follows:

- The category $(\underline{\underline{K}}, \otimes, K)$ of vector spaces over a field K, under the tensor product;
- The category $(\underline{\underline{S}}, \otimes, S)$ of modules over a commutative, unital ring S, under the tensor product, with the module reduct of the ring as the monoidal unit;
- The category $(\underline{\underline{S}}, \oplus, S)$ of modules over a commutative, unital ring S, under the direct sum, with the zero module as the monoidal unit;
- The category $(\mathbf{Set}, \times, \top)$ of sets under the cartesian product, with the terminal object $\top$ (a singleton set) as the monoidal unit. In this case it is often convenient to write the cartesian product as a tensor product, and to write $x \otimes y$ for an ordered pair (x, y).

This section records some basic definitions applying to objects A in $\mathbf{V}$. While these definitions are bread-and-butter for Hopf algebra experts, they may be less familiar to other algebraists. Following the lead of Jay [12], we often use a concrete, set-theoretical notation to record computations in a symmetric monoidal category, much as the Yoneda lemma is used to justify a set-theoretical notation in general categories.

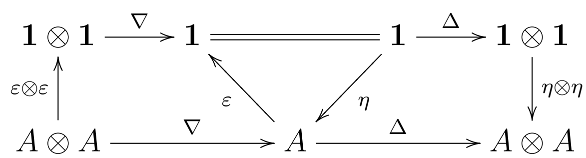
Extending the concept of a magma $(A, \nabla \colon A \otimes A \to A)$ in the category of sets, we have *magmas* $(A, \nabla \colon A \otimes A \to A)$ in $(\mathbf{V}, \otimes, \mathbf{1})$. Dually, we have *comagmas* $(A, \Delta\colon A \to A \otimes A)$ in $(\mathbf{V}, \otimes, \mathbf{1})$. Consider the *bimagma diagram*

(2.1)



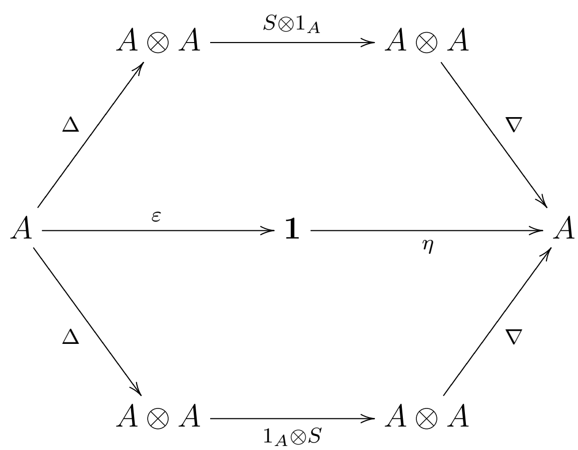
in the category $\mathbf{V}$, the *biunital diagram*

(2.2)



in the category $\mathbf{V}$, and the *antipode diagram*

(2.3)



in the category $\mathbf{V}$, all of which are commutative diagrams.

Definition 2.1. Consider a symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) A *bimagma* (A, ∇, Δ) is an object A carrying a magma structure (A, ∇) and a comagma structure (A, Δ), such that the bimagma diagram (2.1) commutes.
(b) A *unital magma* (A, ∇, η) is a magma (A, ∇) with a *unit* morphism $\eta\colon A \to \mathbf{1}$, such that the left hand trapezoid of the biunital diagram (2.2) commutes.

(b) Dually, a *counital comagma* (A, Δ, ε) is a comagma (A, Δ) with a *counit* morphism $\varepsilon\colon \mathbf{1} \to A$, such that the right hand trapezoid of the biunital diagram (2.2) commutes.

A *monoid* (A, ∇, η) in $(\mathbf{V}, \otimes, \mathbf{1})$ is an associative unital magma. Dually, a *comonoid* (A, Δ, ε) in $(\mathbf{V}, \otimes, \mathbf{1})$ is a coassociative counital magma. Suppose that $(A, \nabla, \eta, \Delta, \varepsilon)$ is a bimagma for which the biunital diagram commutes, with monoid and comonoid structures (A, ∇, η) and (A, Δ, ε). Then the antipode diagram (2.3) records that a morphism $S\colon A \to A$ is the inverse of $\varepsilon\eta\colon A \to A$ in the *convolution monoid* $\big(\mathbf{V}(A, A), *, \varepsilon\eta\big)$ with

$$f * g := \Delta(f \otimes g)\nabla \tag{2.4}$$

for $f, g \in \mathbf{V}(A, A)$ [13, Prop. III.3.1]. Note that we write M^* for the group of units of a monoid (in the category of sets). Thus for the endomorphism monoid $\mathbf{V}(A, A)$ of an object A in a (locally small) category $\mathbf{V}$, we write $\mathbf{V}(A, A)^*$ for the automorphism group of A in $\mathbf{V}$.

Remark 2.2. (a) Commuting of the bimagma diagram (2.1) in a bimagma (A, ∇, Δ) means that

$$\Delta\colon (A, \nabla) \to \big(A \otimes A, (1_A \otimes \tau \otimes 1_A)(\nabla \otimes \nabla)\big)$$

is a magma homomorphism (commuting of the upper-left solid and dotted quadrilateral), or equivalently, that

$$\nabla\colon \big(A \otimes A, (\Delta \otimes \Delta)(1_A \otimes \tau \otimes 1_A)\big) \to (A, \Delta)$$

is a comagma homomorphism (equivalent to commuting of the upper-right solid and dotted quadrilateral).

(b) A comagma comultiplication on A is often denoted by a version of Sweedler notation adapted to the general noncoassociative situation, namely $a\Delta = a^L \otimes a^R$. Note that, in concrete notation, coassociativity then takes the form

$$x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR} \tag{2.5}$$

for an element x of A. Given coassociativity, the classical Sweedler notation is recovered by replacing the superscripts, when taken in lexicographic order, by successive subscript numbers. For example, each side of (2.5) is then written as $x_1 \otimes x_2 \otimes x_3$.

(c) Magma multiplications on an object A of a monoidal category are often denoted concretely by juxtaposition, namely $(x \otimes y)\nabla = xy$, or with $x \cdot y$ as an infix notation, for elements x, y of A. The infix $\cdot$ binds less strongly than the juxtaposition.

(d) With the notations of (b) and (c), commuting of the bimagma diagram (2.1) in a bimagma (A,∇,Δ) amounts to

$$x^L y^L \otimes x^R y^R = (xy)^L \otimes (xy)^R \tag{2.6}$$

for x, y in A.

3. Grouplike and setlike elements

3.1. **Grouplike and setlike elements of comagmas.** Suppose that $\underline{\underline{S}}$ is the category of modules over a commutative, unital ring S. Let $(\underline{\underline{S}},\boxtimes,\mathbf{1})$ denote one of the two symmetric monoidal category structures $(\underline{\underline{S}},\oplus,\{0\})$ or $(\underline{\underline{S}},\otimes,S)$ on $\underline{\underline{S}}$.

Definition 3.1. Let (A,Δ) be a comagma within the symmetric monoidal category $(\underline{\underline{S}},\boxtimes,\mathbf{1})$.

(a) An element q of A is *grouplike* if $q \neq 0$ and $\Delta\colon q \to q \boxtimes q$.
(a) An element q of A is *setlike* if $\Delta\colon q \to q \boxtimes q$.
(b) Write A_1 for the set of grouplike elements of A.
(c) Write A_1^0 for the set of setlike elements of A.

Lemma 3.2. *Suppose that (A,∇,Δ) is a bimagma in $(\underline{\underline{S}},\boxtimes,\mathbf{1})$.*

(a) *The set A_1^0 forms a subbimagma (A_1^0,∇,Δ) of (A,∇,Δ).*
(b) *With $\nabla\colon q_1 \boxtimes q_2 \mapsto q_1 \cdot q_2$, the set A_1^0 forms a magma $(A_1^0,\cdot)$.*

Proof. Consider q_1, q_2 in A_1^0. The commuting of the bimagma diagram (2.1) yields

$$(q_1\cdot q_2)\Delta = (q_1\Delta \boxtimes q_2\Delta)\nabla = (q_1\boxtimes q_1)\boxtimes(q_2\boxtimes q_2)\nabla = (q_1\cdot q_2)\boxtimes(q_1\cdot q_2)\,,$$

so that $q_1\cdot q_2$ lies in A_1^0. Then since $0\Delta = 0 = 0\boxtimes 0$ and $q_1\Delta = q_1\boxtimes q_1$ for $q_1 \neq 0$, the comagma (A,Δ) contains (A_1^0,Δ) as a subcomagma. □

3.2. **Grouplike elements over fields.** Now let K be a field. The following result is adapted from [23, Lemma 2.1.12] by removing any reference to a counit.

Lemma 3.3. *If (A,Δ) is a comagma in $(\underline{\underline{K}},\otimes,K)$, the set A_1 of grouplike elements is linearly independent.*

Proof. Suppose that A_1 is not linearly independent, so there is a dependency relationship

$$h_0q_0 + h_1q_1 + \ldots + h_rq_r = 0 \tag{3.1}$$

between $r+1$ distinct elements $q_0, q_1, \ldots, q_r$ of A_1, with nonzero scalars $h_i \in K$. Suppose that r is minimal among all such dependency relationships.

Note that r is positive, since 0 does not lie in A_1, and that $\{q_1,\dots,q_r\}$ is linearly independent. The relationship (3.1) may be rewritten as

$$q_0 = k_1 q_1 + \dots + k_r q_r$$

with nonzero scalars $k_i \in K$.

Since q_0 is grouplike, the tensor rank of $q_0^\Delta = q_0 \otimes q_0$ is 1. On the other hand,

$$q_0^\Delta = (k_1 q_1 + \dots + k_r q_r)^\Delta = k_1 q_1 \otimes q_1 + \dots + k_r q_r \otimes q_r\,,$$

so the tensor rank of $q_0^\Delta = q_0 \otimes q_0$ is r by [23, Lemma 1.2.2], since $\{q_1,\dots,q_r\}$ is linearly independent. Thus $r = 1$ and $q_0 = k_1 q_1$. Then

$$k_1 q_1^\Delta = (k_1 q_1)^\Delta = q_0^\Delta = q_0 \otimes q_0 = (k_1 q_1) \otimes (k_1 q_1) = k_1^2 q_1^\Delta\,.$$

Since $q_1^\Delta \neq 0$, we have $k_1 = k_1^2$, so k_1 is one of the two roots $0, 1$ of the quadratic $X^2 - X = 0$. However, $q_0 = k_1 q_1$ is nonzero, so $k_1 = 1$ and $q_0 = q_1$. This contradicts the distinctness of the elements $q_0, q_1, \dots, q_r$. □

3.3. **Grouplike elements in quantum quasigroups over fields.**

Theorem 3.4. *Let K be a field. Let (A, ∇, Δ) be a quantum quasigroup in the symmetric monoidal category $(\underline{\underline{K}}, \otimes, K)$ of vector spaces over K. Then the set A_1 of grouplike elements forms a combinatorial quasigroup $(A_1, \cdot)$.*

Proof. By Lemma 3.3, it follows that A_1 is a linearly independent subset of A. The restriction of the left composite (1.2) to

$$A_1 \otimes A_1 := \{q_1 \otimes q_2 \mid q_1, q_2 \in A_1\}$$

acts as

$$\mathsf{G}\colon q_1 \otimes q_2 \xmapsto{\Delta\otimes 1_A} q_1 \otimes q_1 \otimes q_2 \xmapsto{1_A \otimes \nabla} q_1 \otimes q_1 \cdot q_2\,. \tag{3.2}$$

By Lemma 3.2, $q_1 \cdot q_2 \in A_1^0$ for $q_1, q_2 \in A_1$. If $q_1 \cdot q_2 = 0$ for some grouplike q_1, q_2, then $(q_1 \otimes q_2)^{\mathsf{G}} = 0$. Since G is invertible, this would imply $q_1 \otimes q_2 = 0$, a contradiction. Thus (3.2) corestricts to $\mathsf{G}_0\colon A_1 \otimes A_1 \to A_1 \otimes A_1$. It follows that the inverse G^{-1} restricts and corestricts to

$$\mathsf{G}_0^{-1}\colon A_1 \otimes A_1 \to A_1 \otimes A_1; p_1 \otimes p_2 \mapsto p_1 \otimes p_1 \backslash p_2\,.$$

The equation $\mathsf{G}_0 \mathsf{G}_0^{-1} = 1_{A_1 \otimes A_1}$ yields

$$\forall\ q_1, q_2 \in A_1\,,\ \ q_1 \backslash (q_1 \cdot q_2) = q_2\,, \tag{3.3}$$

while the equation $\mathsf{G}_0^{-1} \mathsf{G}_0 = 1_{A_1 \otimes A_1}$ yields

$$\forall\ p_1, p_2 \in A_1\,,\ \ p_1 \cdot (p_1 \backslash p_2) = p_2\,. \tag{3.4}$$

Dually, the equation $\mathsf{D}_0 \mathsf{D}_0^{-1} = 1_{A_1 \otimes A_1}$ yields

$$\forall\ q_1, q_2 \in A_1\,,\ \ (q_1 \cdot q_2)/q_2 = q_1\,, \tag{3.5}$$

while the equation $\partial_0^{-1}\partial_0 = 1_{A_1 \otimes A_1}$ yields

$$\forall\ p_1, p_2 \in A_1\ ,\ (p_1/p_2) \cdot p_2 = p_1\ . \tag{3.6}$$

Together, (3.3)–(3.6) imply that $(A_1, \cdot, /, \backslash)$ forms an equational quasigroup, so that $(A_1, \cdot)$ forms a combinatorial quasigroup. □

3.4. **Bimagmas in categories of modules under the direct sum.** In this paper, we will often have cause to consider bimagmas and quantum quasigroups in the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of S-modules under the direct sum, over a commutative unital ring S.

Definition 3.5. Consider the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules over a commutative, unital ring S. Then a magma and comagma structure (A, ∇, Δ) in $(\underline{\underline{S}}, \oplus, \{0\})$ with

$$\nabla\colon A \oplus A \to A; x \oplus y \mapsto x^\rho + y^\lambda \quad \text{and} \quad \Delta\colon A \to A \oplus A; x \mapsto x^L \oplus x^R \tag{3.7}$$

is written as $A(\rho, \lambda, L, R)$.

Remark 3.6. Note that the structure $A(\rho, \lambda, L, R)$ is a universal algebra, the S-module A together with its endomorphisms ρ, λ, L, R. This is why the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ furnishes such a useful proving ground for the study of magma and comagma properties: They reduce to classical universal algebra in this case.

The following result provides a first illustration of Remark 3.6.

Proposition 3.7. [26, Prop. 3.39] *The bimagma condition on (A, ∇, Δ) in $(\underline{\underline{S}}, \oplus, \{0\})$ reduces to the mutual commutativity of elements of the two (in general noncommutative) subalgebras $S(\rho, \lambda)$ and $S(L, R)$, respectively generated by $\{\rho, \lambda\}$ and $\{L, R\}$, in the endomorphism ring $\underline{\underline{S}}(A, A)$ of the S-module A.*

Corollary 3.8. *A bimagma (A, ∇, Δ) in $(\underline{\underline{S}}, \oplus, \{0\})$ may be identified as a bimodule ${}_{S(L,R)^{\mathsf{op}}}A_{S(\rho,\lambda)}$, where $S(L, R)^{\mathsf{op}}$ is the opposite of the subalgebra $S(L, R)$ of the endomorphism ring $\underline{\underline{S}}(A, A)$ of the S-module A.*

Lemma 3.9. *Let $A(\rho, \lambda, L, R)$ be a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$.*

(a) *The endomorphism*

$$\left(\begin{bmatrix} L & R \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} \rho \\ \lambda \end{bmatrix}\right) = \begin{bmatrix} L & R\rho \\ 0 & \lambda \end{bmatrix} \tag{3.8}$$

of A^2 is the left composite of $A(\rho, \lambda, L, R)$.

(b) *The endomorphism*

$$\left(\begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} L & R \end{bmatrix}\right)\left(\begin{bmatrix} \rho \\ \lambda \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}\right) = \begin{bmatrix} \rho & 0 \\ L\lambda & R \end{bmatrix} \tag{3.9}$$

of A^2 is the right composite of $A(\rho, \lambda, L, R)$.

Proposition 3.10. *Let $A(\rho, \lambda, L, R)$ be a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$.*

(a) *The bimagma is not uniquely specified by its left or right composites individually.*
(b) *The bimagma is uniquely specified by the conjunction of its left and right composites.*

Proof. (a) By (3.8), it is apparent that $A(\rho, \lambda, L, R)$ and $A(U^{-1}\rho, \lambda, L, RU)$, for any S-module automorphism U of A, have the same left composite.

(b) By (3.8), the left composite of $A(\rho, \lambda, L, R)$ determines L and λ. Dually, by (3.9), the right composite of $A(\rho, \lambda, L, R)$ determines ρ and R. $\square$

3.5. **Linear quantum quasigroups.** As before, consider the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of S-modules under the direct sum, over a commutative unital ring S.

Definition 3.11. Consider a bimagma $A(\rho, \lambda, L, R)$ in $(\underline{\underline{S}}, \oplus, \{0\})$. Suppose that:

(a) the endomorphisms ρ, λ of the S-module A are invertible, and
(b) the endomorphisms L, R of the S-module A are invertible.

Then $A(\rho, \lambda, L, R)$ is said to be a *linear quantum quasigroup*.

Proposition 3.12. *Suppose that (A, ∇, Δ) is a linear quantum quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$, interpreted as a bimodule ${}_{S(L,R)^{\mathsf{op}}}A_{S(\rho,\lambda)}$ using Corollary 3.8. It may be reconsidered as a bimodule ${}_{S\langle L,R\rangle}A_{S\langle\rho,\lambda\rangle}$ over the respective group rings $S\langle\rho, \lambda\rangle$ and $S\langle L, R\rangle$ of the subgroups $\langle\rho, \lambda\rangle$ and $\langle L, R\rangle$ of the S-module automorphism group $\underline{\underline{S}}(A, A)^*$ generated by $\{\rho, \lambda\}$ and $\{L, R\}$.*

Proof. Note the injective ring homomorphism $S(L, R)^{\mathsf{op}} \hookrightarrow S\langle L, R\rangle$ that is given by $L \mapsto L^{-1}$ and $R \mapsto R^{-1}$. $\square$

Proposition 3.13. *In $(\underline{\underline{S}}, \oplus, \{0\})$, a linear quantum quasigroup $A(\rho, \lambda, L, R)$ is indeed a quantum quasigroup.*

Proof. In (A, ∇, Δ), the left composite $\mathsf{G}\colon A \oplus A \to A \oplus A$ (1.2) acts as

$$x \oplus y \mapsto x^L \oplus x^R \oplus y \mapsto x^L \oplus (x^{R\rho} + y^\lambda)\,. \tag{3.10}$$

Then

$$a \oplus b \mapsto a^{L^{-1}} \oplus \left(b^{\lambda^{-1}} - a^{L^{-1}R\rho\lambda^{-1}}\right) \tag{3.11}$$

is its inverse. Dually, the right composite $\mathsf{Ə}\colon A \oplus A \to A \oplus A$ (1.3) acts as

$$x \oplus y \mapsto x \oplus y^L \oplus y^R \mapsto (x^\rho + y^{L\lambda}) \oplus y^R\,. \tag{3.12}$$

Then

$$a \oplus b \mapsto \left(a^{\rho^{-1}} - b^{R^{-1}L\lambda\rho^{-1}}\right) \oplus b^{R^{-1}} \tag{3.13}$$

is its inverse. □

Theorem 3.14. *Let $(\mathbf{V}, \oplus, \{0\})$ be the symmetric monoidal category of finitely generated modules over a commutative, unital ring S. Suppose that $A(\rho, \lambda, L, R)$ is a bimagma in $(\mathbf{V}, \oplus, \{0\})$. Consider (A, ∇, Δ) with the notation of* (3.7)*. Then the structure (A, ∇, Δ) forms a quantum quasigroup in $(\mathbf{V}, \oplus, \{0\})$ if and only if $A(\rho, \lambda, L, R)$ is a linear quantum quasigroup.*

Proof. The sufficiency of the linear quantum quasigroup property is given by Proposition 3.13. Conversely, consider a quantum quasigroup (A, ∇, Δ) in $(\mathbf{V}, \oplus, \{0\})$, with the notation of (3.7). Thus the matrices (3.8) and (3.9) are invertible, say

$$\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\begin{bmatrix}L & R\rho\\ 0_A & \lambda\end{bmatrix} = \begin{bmatrix}1_A & 0_A\\ 0_A & 1_A\end{bmatrix} = \begin{bmatrix}L & R\rho\\ 0_A & \lambda\end{bmatrix}\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix} \tag{3.14}$$

and

$$\begin{bmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{bmatrix}\begin{bmatrix}\rho & 0_A\\ L\lambda & R\end{bmatrix} = \begin{bmatrix}1_A & 0_A\\ 0_A & 1_A\end{bmatrix} = \begin{bmatrix}\rho & 0_A\\ L\lambda & R\end{bmatrix}\begin{bmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{bmatrix} \tag{3.15}$$

for endomorphisms α_{ij} and β_{ij} of the module A, with $1 \le i, j \le 2$.

Consider:

- the $(1,1), (2,1), (2,2)$-components of the left-hand equation of (3.14);
- the $(1,1), (1,2), (2,2)$-components of the left-hand equation of (3.15).

These equations yield

$$\begin{aligned} \alpha_{11}L &= 1_A\,, & \beta_{11}\rho + \beta_{12}L\lambda &= 1_A\,,\\ \alpha_{21}L &= 0_A\,, & \beta_{12}R &= 0_A\,,\\ \alpha_{21}R\rho + \alpha_{22}\lambda &= 1_A\,, & \beta_{22}R &= 1_A, \end{aligned} \tag{3.16}$$

respectively. Note that L retracts α_{11}, so it is a surjection. Then by Nakayama's lemma, it is an isomorphism [19, Th. 2.4]. Thus $\alpha_{21}L = 0_A$ implies $\alpha_{21} = 0_A$. The equation $\alpha_{21}R\rho + \alpha_{22}\lambda = \alpha_{22}\lambda = 1_A$ now exhibits λ as a surjection. Thus λ is also an isomorphism. Identical arguments applied to the three right-hand equations of (3.16) yield that R and ρ are invertible. □

3.6. Setlike elements in linear quantum quasigroups.

Definition 3.15. A linear quantum quasigroup (A, ∇, Δ) in $(\underline{\underline{S}}, \oplus, \{0\})$ is *classical* if $\Delta\colon x \mapsto x \oplus x$. In other words, in the structure $A(\rho, \lambda, L, R)$, we have $L = R = 1$.

We have the following counterpart of Theorem 3.4.

Theorem 3.16. *Suppose that (A, ∇, Δ) is a linear quantum quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$. Then the set A_1^0 of setlike elements of (A, ∇, Δ) forms a classical linear quasigroup (A_1^0, ∇, Δ).*

Proof. It is clear that comultiplication may be (co)restricted to a linear map $A_1^0 \to A_1^0 \oplus A_1^0$. The set A_1^0 is the set of fixed points of the action of the group $\langle L, R\rangle$ of automorphisms of the S-module A. As such, it is an S-submodule of A. In fact, by Proposition 3.7, we have

$$x^\sigma \Delta = x^{\sigma L} \oplus x^{\sigma R} = x^{L\sigma} \oplus x^{R\sigma} = x^\sigma \oplus x^\sigma$$

for $x \in A_1^0$ and $\sigma \in \langle \rho, \lambda\rangle$, making A_1^0 an $S\langle \rho, \lambda\rangle$-submodule of A. Thus $\nabla : x \oplus y \mapsto x^\rho + y^\lambda$ (co)restricts to a map $A_1^0 \oplus A_1^0 \to A_1^0$. □

3.7. **Classical reducts.** According to Definition 3.15, a linear quantum quasigroup (A, ∇, Δ) or $A(\rho, \lambda, L, R)$ in the category $(\underline{\underline{S}}, \oplus, \{0\})$ is classical if $L = R = 1_A$, or equivalently $\Delta\colon x \mapsto x \oplus x$.

Lemma 3.17. *If $A(\rho, \lambda, L, R)$ is a linear quantum quasigroup, then so is $A(\rho, \lambda, 1_A, 1_A)$.*

Definition 3.18. The linear quantum quasigroup $A(\rho, \lambda, 1_A, 1_A)$ is known as the *classical reduct* of the linear quantum quasigroup $A(\rho, \lambda, L, R)$.

Proposition 3.19. *Let $A(\rho, \lambda, L, R)$ be a linear quantum quasigroup, with set A_1^0 of setlike elements. Then the classical linear quasigroup (A_1^0, ∇, Δ) of Theorem 3.16 forms a subquasigroup of the classical reduct $A(\rho, \lambda, 1_A, 1_A)$ of $A(\rho, \lambda, L, R)$.*

Taking the classical reduct gives the object part of a functor CR, the *classical reduct functor*, from the category of linear quantum quasigroups over S to the category of (classical) linear quasigroups (piques) over S. The left adjoint LQ to CR is known as the *linear quantization* functor. It is constructed by the following theorem.

Theorem 3.20. *Consider the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules over a commutative, unital ring S. Let $S\langle r, l\rangle$ be the group algebra over S of the free group $\langle r, l\rangle$ on a two-element generating set $\{r, l\}$.*

(a) *The category of classical S-linear quasigroups is equivalent to the category of right $S\langle r, l\rangle$-modules $A_{S\langle r,l\rangle}$.*
(b) *There is an equivalence between the category of linear quantum quasigroups in $(\underline{\underline{S}}, \oplus, \{0\})$ and the category of left/right $S\langle r, l\rangle$-bimodules ${}_{S\langle r,l\rangle}A_{S\langle r,l\rangle}$.*
(c) *Via the equivalences, the (object part of the) classical reduct functor is given by the forgetful functor $\mathsf{CR}\colon {}_{S\langle r,l\rangle}A_{S\langle r,l\rangle} \mapsto A_{S\langle r,l\rangle}$.*

(d) *Using the equivalences, the tensor product functor*

$$\mathsf{Q}\colon A_{S\langle r,l\rangle} \mapsto S\langle r,l\rangle \otimes A_{S\langle r,l\rangle}$$

gives the (object part of the) linear quantization functor.

Proof. (a) The structure of a right $S\langle r,l\rangle$-module A amounts to a selection of abelian group automorphisms $\rho, \lambda \in \mathrm{Aut}(A)$, yielding a classical linear quasigroup $A(\rho, \lambda, 1_A, 1_A)$. Conversely, any classical S-linear quasigroup furnishes a structure map $S\langle r,l\rangle \to \mathrm{End}(A); r \mapsto \rho, l \mapsto \lambda$. To see the functoriality of this correspondence, consider an S-module homomorphism $f : A \to B$. Now f is an $S\langle r,l\rangle$-homomorphism if and only if there are S-linear automorphisms ρ, λ of A and ρ', λ' of B such that for all $a \in A$, $(af)\rho' = (af)\cdot r = (a \cdot r)f = (a\rho)f$ and $(fa)\lambda' = (af)\cdot l = (a\cdot l)f = (a\lambda)f$. Note that these intertwining conditions $f\rho' = \rho f$ and $f\lambda' = \lambda f$ are precisely what is required of a bimagma homomorphism from $A(\rho, \lambda, 1_A, 1_A)$ into $B(\rho', \lambda', 1_A, 1_A)$.

(b) Given an S-linear quantum quasigroup $A(\rho, \lambda, L, R)$, we may use the fact that $S\langle \rho, \lambda\rangle$ and $S\langle L, R\rangle$ are quotients of the free group ring $S\langle r, l\rangle$ to establish bijections $\mathbf{Ring}(S\langle r,l\rangle, \mathrm{End}(A)) \cong \mathbf{Ring}(S\langle \rho,\lambda\rangle, \mathrm{End}(A))$ and $\mathbf{Ring}(S\langle r,l\rangle, \mathrm{End}(A)) \cong \mathbf{Ring}(S\langle L,R\rangle, \mathrm{End}(A))$. The equivalence follows from Proposition 3.12.

(c) The forgetfulness of CR corresponds to $A(\rho, \lambda, L, R) \mapsto A(\rho, \lambda, 1_A, 1_A)$ forgetting comultiplication.

(d) In terms of categories of modules, Q is the left adjoint of CR (compare [17, Sec. IV.2]). This fact, in conjunction with (a) and (b), completes the proof. □

4. Quantum conjugates

Quasigroup conjugacy reflects the triality or S_3-symmetry of the theory of quasigroups [24, §1.3]. This section investigates the transfer of quasigroup conjugacy to quantum quasigroups.

4.1. **Inverting composites of linear quantum quasigroups.** We begin the study of conjugates with an examination of how inverses of composites of linear quantum quasigroups themselves appear as composites of linear quasigroups.

Proposition 4.1. *Consider the symmetric monoidal category* $(\underline{\underline{S}}, \oplus, \{0\})$ *of modules over a commutative, unital ring* S*. Let* (A, ∇, Δ) *with*

$$\nabla\colon A\oplus A \to A; x\oplus y \mapsto x^\rho + y^\lambda \quad \text{and} \quad \Delta\colon A \to A\oplus A; x \mapsto x^L \oplus x^R$$

be a linear quantum quasigroup structure in $(\underline{\underline{S}}, \oplus, \{0\})$*.*

(a) *The bimagma* (A, ∇_l, Δ_l) *with*

$$\nabla_l \colon A \oplus A \to A; x \oplus y \mapsto y^{\lambda^{-1}} - x^{\rho\lambda^{-1}} \text{ and } \Delta_l \colon A \to A \oplus A; x \mapsto x^{L^{-1}} \oplus x^{L^{-1}R}$$

has the inverse (3.11) *of the left composite* G *of* (A, ∇, Δ) *as its own left composite* G_l*. It forms a linear quantum quasigroup.*

(b) *The bimagma* (A, ∇_r, Δ_r) *with*

$$\nabla_r \colon A \oplus A \to A; x \oplus y \mapsto x^{\rho^{-1}} - y^{\lambda\rho^{-1}} \text{ and } \Delta_r \colon A \to A \oplus A; x \mapsto x^{R^{-1}L} \oplus x^{R^{-1}}$$

has the inverse (3.13) *of the right composite* $\reflectbox{\textsf{G}}$ *of* (A, ∇, Δ) *as its own right composite* $\reflectbox{\textsf{G}}_r$*. It forms a linear quantum quasigroup.*

Proof. (a) The left composite of (A, ∇_l, Δ_l) is

$$\mathsf{G}_l \colon a \oplus b \xmapsto{\Delta_l \oplus 1} a^{L^{-1}} \oplus a^{L^{-1}R} \oplus b \xmapsto{1 \oplus \nabla_l} a^{L^{-1}} \oplus \left(b^{\lambda^{-1}} - a^{L^{-1}R\rho\lambda^{-1}}\right),$$

which is inverted by

$$\mathsf{G} \colon x \oplus y \mapsto x^L \oplus x^R \oplus y \mapsto x^L \oplus (x^{R\rho} + y^{\lambda}).$$

The right composite of (A, ∇_l, Δ_l) is

$$\reflectbox{\textsf{G}}_l \colon a \oplus b \xmapsto{1 \oplus \Delta_l} a \oplus b^{L^{-1}} \oplus b^{L^{-1}R} \xmapsto{\nabla_l \oplus 1} \left(b^{L^{-1}\lambda^{-1}} - a^{\rho\lambda^{-1}}\right) \oplus b^{L^{-1}R},$$

which is inverted by

$$\reflectbox{\textsf{G}}_{lr} \colon x \oplus y \mapsto x \oplus y^{R^{-1}} \oplus y^{R^{-1}L} \mapsto \left(y^{R^{-1}\rho^{-1}} - x^{\lambda\rho^{-1}}\right) \oplus y^{R^{-1}L}. \tag{4.1}$$

Finally, since the subgroups

$$\langle \rho, \lambda \rangle \quad \text{and} \quad \langle L, R \rangle$$

of the automorphism group $\underline{\underline{S}}(A, A)^*$ centralize each other, the subgroups

$$\langle -\rho\lambda^{-1}, \lambda^{-1} \rangle \quad \text{and} \quad \langle L^{-1}, L^{-1}R \rangle$$

centralize each other, so that (A, ∇_l, Δ_l) is a linear quantum quasigroup. The proof of (b) is dual. □

4.2. Quantum conjugates of quantum quasigroups.

Definition 4.2. Let $(\mathbf{V}, \otimes, \mathbf{1})$ be a symmetric monoidal category. Let (A, ∇, Δ) be a quantum quasigroup in $(\mathbf{V}, \otimes, \mathbf{1})$, with respective left and right composites G and $\reflectbox{\textsf{G}}$.

(a) The *opposite* or *transpose* (A, ∇_t, Δ_t) of (A, ∇, Δ) is $(A, \tau\nabla, \Delta\tau)$.

(b) A quantum quasigroup (A, ∇_l, Δ_l) is a *quantum left conjugate* of (A, ∇, Δ) if the left composite G_l of (A, ∇_l, Δ_l) is inverse to the left composite G of (A, ∇, Δ).

(c) A quantum quasigroup (A, ∇_r, Δ_r) is a *quantum right conjugate* of (A, ∇, Δ) if the right composite $\reflectbox{\textsf{G}}_r$ of (A, ∇_r, Δ_r) is inverse to the right composite $\reflectbox{\textsf{G}}$ of (A, ∇, Δ).

The following is an immediate consequence of Definition 4.2.

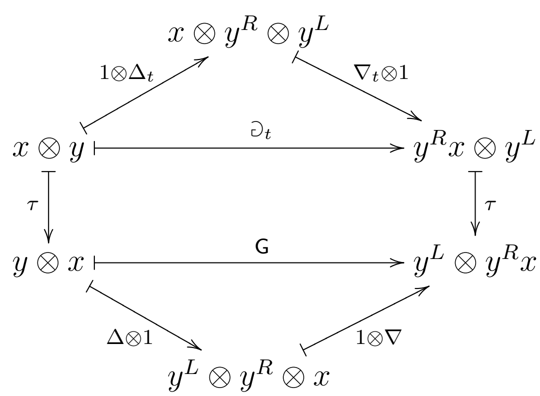
Lemma 4.3. *In a symmetric monoidal category* $(\mathbf{V}, \otimes, \mathbf{1})$*, let* (A, ∇, Δ) *be a quantum quasigroup.*

(a) *The transpose of the transpose* (A, ∇_t, Δ_t) *of* (A, ∇, Δ) *is* (A, ∇, Δ)*.*
(b) *Suppose that* (A, ∇, Δ) *has a quantum left conjugate* (A, ∇_l, Δ_l)*. Then* (A, ∇, Δ) *is a quantum left conjugate of* (A, ∇_l, Δ_l)*.*
(c) *Suppose that* (A, ∇, Δ) *has a quantum right conjugate* (A, ∇_r, Δ_r)*. Then* (A, ∇, Δ) *is a quantum right conjugate of* (A, ∇_r, Δ_r)*.*

4.3. **The opposite of a quantum quasigroup.** The composites of the opposite are related to the original composites as follows.

Lemma 4.4. *In a symmetric monoidal category* $(\mathbf{V}, \otimes, \mathbf{1})$*, let* (A, ∇, Δ) *be a quantum quasigroup, with composites* G *and* $\rotatebox[origin=c]{180}{\mathsf{G}}$*. Let* G_t *and* $\rotatebox[origin=c]{180}{\mathsf{G}}_t$ *be the respective left and right composites of the opposite. Then* $\tau\mathsf{G} = \rotatebox[origin=c]{180}{\mathsf{G}}_t\tau$ *and* $\tau\mathsf{G}_t = \rotatebox[origin=c]{180}{\mathsf{G}}\tau$*.*

Proof. Consider the commutative diagram



for the first equation, working with the Jay calculus [12].

The second equation follows from the first equation as applied to the opposite quantum quasigroup (A, ∇_t, Δ_t), using Lemma 4.3(a). □

Proposition 4.5. *Let* (A, ∇, Δ) *be a quantum quasigroup in a symmetric monoidal category* $(\mathbf{V}, \otimes, \mathbf{1})$*. The the opposite of* (A, ∇, Δ) *is a quantum quasigroup.*

Proof. The bimagma condition for the opposite of (A, ∇, Δ) is verified along the lines of [13, Prop. III.2.3] or [23, Ex. 5.1.2]. Since $\mathsf{G}_t = \tau\rotatebox[origin=c]{180}{\mathsf{G}}\tau$ by Lemma 4.4, one has $\mathsf{G}_t^{-1} = \tau\rotatebox[origin=c]{180}{\mathsf{G}}^{-1}\tau$. Similarly, $\rotatebox[origin=c]{180}{\mathsf{G}}_t^{-1} = \tau\mathsf{G}^{-1}\tau$. Thus (A, ∇_t, Δ_t) is a quantum quasigroup. □

4.4. **Quantum conjugates of Hopf algebras.** In general, the existence of quantum conjugates of a quantum quasigroup is nontrivial. We begin by offering some initial observations for the case of cocommutative Hopf algebras. If a quantum quasigroup (A, ∇, Δ) is the bimagma reduct of a Hopf algebra A or $(A, \nabla, \eta, \Delta, \varepsilon, S)$, then the opposite or transpose of (A, ∇, Δ) is the bimagma reduct of the Hopf algebra $A^{\mathrm{op\ cop}}$ (in the notation of [13, Cor. III.3.5]). On the other hand, Theorem 4.8 below exhibits left and right conjugates for cocommutative Hopf algebras.

Our proof of Theorem 4.8 relies on properties of the antipode of a Hopf algebra, as presented for Hopf algebras in a category of vector spaces under the tensor product in [13, Th. III.3.4], for example; these properties are listed in Lemma 4.6. However, because Kassel never invokes the structure of the ground field, we may consider his arguments to be expressions of the Jay calculus [12], meaning they apply to Hopf algebras in any symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

Lemma 4.6. *Let $(A, \nabla, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra in $(\mathbf{V}, \otimes, \mathbf{1})$.*

(a) *The antipode is an algebra antihomomorphism. Thus*

$$(xy)^S = y^S x^S \quad \text{and} \quad \eta S = \eta, \tag{4.2}$$

for $x, y \in A$.

(b) *If $x^{RS} x^L = x^{\varepsilon\eta}$ for all $x \in A$, then $S^2 = 1_A$.*

(c) *If $(A, \nabla, \eta, \Delta, \varepsilon, S)$ is cocommutative, then $S^2 = 1_A$.*

Remark 4.7. While Lemma 4.6(b) is not directly cited in the proof of Theorem 4.8, note (a) $\implies$ (b) $\implies$ (c). Hence, we list all three properties to make it easier for the reader, when cross-referencing [13, Th. III.3.4], to convince themselves of the validity of Lemma 4.6 in the general context.

Theorem 4.8. *Let $(A, \nabla, \eta, \Delta, \varepsilon, S)$ be a cocommutative Hopf algebra in a symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.*

(a) *The bimagma reduct (A, ∇, Δ) is a quantum quasigroup.*

(b) *With $\nabla_l = (S \otimes 1_A)\nabla \colon A \otimes A \to A; x \otimes y \mapsto x^S y$, there is a quantum left conjugate (A, ∇_l, Δ) of (A, ∇, Δ).*

(c) *With $\nabla_r = (1_A \otimes S)\nabla \colon A \otimes A \to A; x \otimes y \mapsto xy^S$, there is a quantum right conjugate (A, ∇_r, Δ) of (A, ∇, Δ).*

Proof. (a) See [27, Prop. 4.1].

(b) The bimagma condition for (A, ∇_l, Δ) is verified by the commuting of

$$
\begin{array}{ccccc}
x \otimes y & \xmapsto{\nabla_l} & x^S \cdot y & \xmapsto{\Delta} & (x^S y)^L \otimes (x^S y)^R \\
\downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \| \\
 & & & & x^{SL} y^L \otimes x^{SR} y^R \\
 & & & & \| \\
x^L \otimes x^R \otimes y^L \otimes y^R & \xmapsto{1\otimes\tau\otimes 1} & (x^L \otimes y^L) \otimes (x^R \otimes y^R) & \xmapsto{\nabla_l\otimes\nabla_l} & x^{LS} y^L \otimes x^{RS} y^R
\end{array}
$$

for $x, y \in A$, where the upper equality on the right hand side follows by the bimagma condition for (A, ∇, Δ), while the lower equality on the right hand side follows by the cocommutativity [23, Prop. 7.1.9(b)]. By [27, Prop. 4.1] or [10, Lemmas 4.2, 4.3], the inverse of the left composite G of (A, ∇, Δ) is

$$\mathsf{G}^{-1} \colon x \otimes y \xmapsto{\Delta\otimes 1} x^L \otimes x^R \otimes y \xmapsto{1\otimes S\otimes 1} x^L \otimes x^{RS} \otimes y \xmapsto{1\otimes\nabla} x^L \otimes x^{RS} y$$

which is realized by the left composite G_l of (A, ∇_l, Δ). In particular, G_l is invertible, so the bimagma (A, ∇_l, Δ) is a left quantum quasigroup.

We now prove the invertibility of $\reflectbox{\ensuremath{\partial}}_l = (1_A \otimes \Delta)(\nabla_l \otimes 1_A)$, the right composite of (A, ∇_l, Δ). Setting

$$\nabla_{tl} = (S \otimes 1_A)\nabla_t = (S \otimes 1_A)\tau\nabla \colon x \otimes y \mapsto y x^S \,,$$

we show that $\reflectbox{\ensuremath{\partial}}_{tl} = (1_A \otimes \Delta)(\nabla_{tl} \otimes 1_A)$ is inverse to $\reflectbox{\ensuremath{\partial}}_l$. Let $y \in A$. By coassociativity,

$$y^L \otimes y^{RL} \otimes y^{RR} = y^{LL} \otimes y^{LR} \otimes y^R. \tag{4.3}$$

Tensoring (4.3) on the right by an arbitrary element x and applying the map $(S \otimes 1_A \otimes 1_A \otimes 1_A)(\nabla \otimes 1_A \otimes 1_A)$ yields $(x \otimes y)\reflectbox{\ensuremath{\partial}}_l\reflectbox{\ensuremath{\partial}}_{tl} = (x^S y^L \otimes y^R)\reflectbox{\ensuremath{\partial}}_{tl} = (x^S y^{LL} \otimes y^{LR} \otimes y^R)(\nabla_{tl} \otimes 1_A) = y^{LR}(x^S y^{LL})^S \otimes y^R$, but now, the fact that S is an involutary algebra antihomomorphism leads to

$$
\begin{aligned}
(x \otimes y)\reflectbox{\ensuremath{\partial}}_l\reflectbox{\ensuremath{\partial}}_{tl} &= (y^{LR} y^{LLS}) x \otimes y^R \\
&= (y^{LL} y^{LRS}) x \otimes y^R \\
&= y^{L\varepsilon} x \otimes y^R \\
&= x \otimes y^{L\varepsilon} y^R \\
&= x \otimes y.
\end{aligned}
$$

The second equality is due to cocommutativity.

Finally, tensoring both sides of (4.3) on the left by x and applying the map $(S \otimes 1_A \otimes 1_A \otimes 1_A)(\tau \otimes 1_A \otimes 1_A)(\nabla \otimes 1_A \otimes 1_A)$ yields

$$\begin{aligned}(x \otimes y)\partial_{tl}\partial_l &= (y^L x^S \otimes y^R)\partial_l \\ &= (y^{LL} x^S \otimes y^{LR} \otimes y^R)(\nabla_l \otimes 1_A) \\ &= (y^{LL} x^S)^S y^{LR} \otimes y^R \\ &= x(y^{LLS} y^{LR}) \otimes y^R \\ &= x \otimes y^{L\varepsilon} y^R \\ &= x \otimes y \,,\end{aligned}$$

verifying $\partial_{tl}\partial_l = 1_{A \otimes A}$.

(c) is dual to (b). □

4.5. **Quantum conjugates of linear quantum quasigroups.** In the previous section, Theorem 4.8 provided partial results toward the existence of quantum conjugates for Hopf algebras. The current section observes that linear quantum quasigroups offer much more fertile ground for quantum conjugacy. Here, the notation of Definition 4.2 is reconciled with that of Proposition 4.1 in the following result, where parts (b) and (c) reformulate the proposition.

Theorem 4.9. *Consider the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules over a commutative, unital ring S. Let*

$$A = (A, \nabla, \Delta) = A(\rho, \lambda, L, R)$$

be a linear quantum quasigroup structure in $(\underline{\underline{S}}, \oplus, \{0\})$.

(a) *The transpose*

$$A_t = (A, \nabla_t, \Delta_t) = A(\lambda, \rho, R, L)$$

of (A, ∇, Δ) is a linear quantum quasigroup.

(b) *The linear quantum quasigroup*

$$A_l = (A, \nabla_l, \Delta_l) = A(-\rho\lambda^{-1}, \lambda^{-1}, L^{-1}, L^{-1}R)$$

of Proposition 4.1(a) is a quantum left conjugate to (A, ∇, Δ).

(c) *The linear quantum quasigroup*

$$A_r = (A, \nabla_r, \Delta_r) = A(\rho^{-1}, -\lambda\rho^{-1}, R^{-1}L, R^{-1})$$

of Proposition 4.1(b) is a quantum right conjugate to (A, ∇, Δ).

(d) *The transpose*

$$(A, \nabla_{rl}, \Delta_{rl}) = A(\lambda^{-1}, -\rho\lambda^{-1}, L^{-1}R, L^{-1})$$

of the quantum left conjugate (A, ∇_l, Δ_l) *of* (A, ∇, Δ) *provides a quantum left conjugate to the quantum right conjugate* (A, ∇_r, Δ_r) *of* (A, ∇, Δ).

(e) *The transpose*

$$(A, \nabla_{lr}, \Delta_{lr}) = A(-\lambda\rho^{-1}, \rho^{-1}, R^{-1}, R^{-1}L)$$

of the quantum right conjugate (A, ∇_r, Δ_r) *of* (A, ∇, Δ) *forms a quantum right conjugate to the quantum left conjugate* (A, ∇_l, Δ_l) *of* (A, ∇, Δ).

Proof. It will suffice to prove (e), since (d) is dual. Note that the right composite ∂_{lr} of $(A, \nabla_{lr}, \Delta_{lr})$ is presented in (4.1) as the inverse of ∂_l, the right composite of (A, ∇_l, Δ_l). Thus $(A, \nabla_{lr}, \Delta_{lr})$ is a quantum right conjugate of (A, ∇_l, Δ_l). □

Theorem 6.4 shows that, in the context of linear quantum quasigroups, quantum conjugates always exist. By Lemma 4.3, we may thus consider involutive transformations

$$A \mapsto A_t\,, \quad A \mapsto A_l\,, \quad A \mapsto A_r$$

from a linear quantum quasigroup A to its respective transpose, quantum left conjugate, and quantum right conjugate. The appropriate suffices t, l, r have already been applied to many relevant structures of linear quantum quasigroups, such as their comultiplications and right composites.

By Theorem 6.4(e), we have $A_{rt} = A_{lr}$, so $t = rlr$. By Theorem 6.4(d), we have $A_{lt} = A_{rl}$, so $t = lrl$. We thus obtain a symmetric group

$$S_3 = \langle l, r \mid l^2 = r^2 = (lr)^2 \rangle$$

of conjugations of linear quantum quasigroups, with $t = rlr = lrl$. From this point of view, the conjugation structure of linear quantum quasigroups is similar to that of classical quasigroups [24, §1.3].

5. Semisymmetry

5.1. **Augmentations.** Let $(\mathbf{V}, \otimes, \mathbf{1})$ be a symmetric, monoidal category.

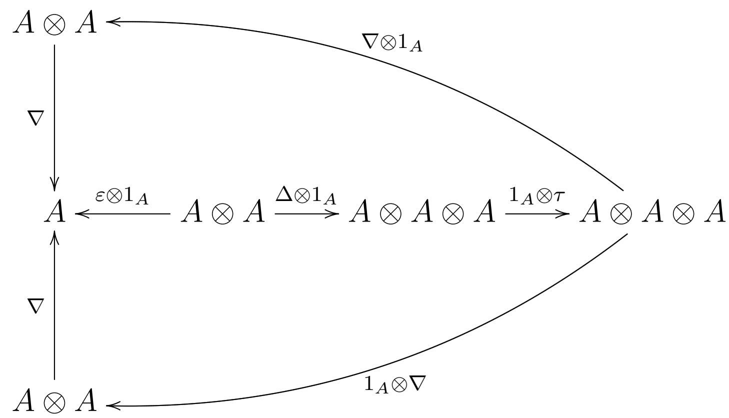
Definition 5.1. An *augmentation* on an object A of $\mathbf{V}$ is a $\mathbf{V}$-morphism $\varepsilon\colon A \to \mathbf{1}$.

Remark 5.2. (a) An augmentation on a comagma (A, Δ) is not necessarily required to serve as a counit for the comultiplication. On the other hand, if (A, Δ, ε) is a counital comagma, then the counit $\varepsilon\colon A \to \mathbf{1}$ is understood as the default augmentation on the object A.

(b) Suppose that the unit object $\mathbf{1}$ is terminal in $\mathbf{V}$, as in $(\mathbf{Set}, \times, \top)$, or the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of S-modules under the direct sum over a commutative unital ring S. Then each object A has a uniquely specified augmentation, which need not be mentioned explicitly.

5.2. **Left and right semisymmetry.** Let $(A, \nabla, \Delta, \varepsilon)$ be a bimagma with an augmentation in $\mathbf{V}$. Consider the diagram

(5.1)

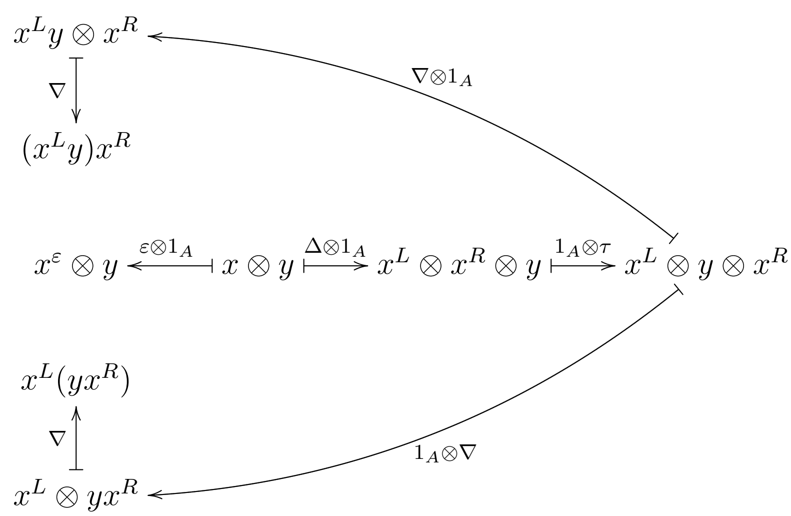


Definition 5.3. Let $(A, \nabla, \Delta, \varepsilon)$ be a bimagma with an augmentation.

(a) If the upper pentagon of the diagram (5.1) commutes in $\mathbf{V}$, then the bimagma satisfies the condition of *left semisymmetry*.
(b) If the lower pentagon of the diagram (5.1) commutes in $\mathbf{V}$, then the bimagma satisfies the condition of *right semisymmetry*.
(c) If the diagram (5.1) commutes in $\mathbf{V}$, then the bimagma satisfies the condition of *semisymmetry*.

For the record, it is convenient to have an elementary version of the diagram (5.1):

(5.2)



5.3. **Commutative inverse property loops.** We exhibit a key instance of semisymmetry for quantum quasigroups.

Definition 5.4. A loop $(A, \cdot, /, \backslash, 1)$ is an *inverse property loop* if

$$x^{-1}(xy) = y = (yx)x^{-1} \tag{5.3}$$

for all $x, y \in A$, with $x^{-1} = 1/x$ [2, §II.2].

Example 5.5. An important class of commutative inverse property loops is formed by *commutative Moufang loops*, loops satisfying $x^2(yz) = (xy)(xz)$ [18, I.1.4(4)] (cf. [2, Th. II.7B]).

The following appears in [2, §II.2], albeit with a misprint.

Lemma 5.6. *The relation $(x^{-1})^{-1} = x$ holds in inverse property loops.*

Working in the symmetric monoidal category $(\mathbf{Set}, \times, \top)$, we generally use tensor product notation for direct products and ordered pairs.

Theorem 5.7. *Let $(A, \cdot, /, \backslash, 1)$ be a commutative inverse property loop.*

(a) *Under the multiplication $\nabla \colon a \otimes b \mapsto a \cdot b$ and comultiplication $\Delta \colon A \to A \otimes A; a \mapsto a^{-1} \otimes a$, a quantum quasigroup within the category $(\mathbf{Set}, \times, \top)$ is formed by (A, ∇, Δ).*

(b) *The quantum quasigroup (A, ∇, Δ) is semisymmetric.*

Proof. (a) The statement follows by [27, Cor. 3.13], given that the inversion map $x \mapsto x^{-1}$ is an automorphism of a commutative inverse property loop [2, II(2.2)].

(b) The chase round the upper part of the diagram (5.2) takes the form

$$\begin{array}{ccccc} x \otimes y & \xmapsto{\Delta \otimes 1_A} & x^{-1} \otimes x \otimes y & \xmapsto{1_A \otimes \tau} & x^{-1} \otimes y \otimes x \\ \varepsilon \otimes 1_A \downarrow & & & & \downarrow \nabla \otimes 1_A \\ y & = & (x^{-1}y)x & \xleftarrow{\nabla} & x^{-1}y \otimes x \end{array}$$

where the commutativity of the diagram follows by $y = x(x^{-1}y) = (x^{-1}y)x$ using (5.3), Lemma 5.6, and the commutativity of the loop. Commuting of the lower part of the diagram is similar, but does not require any application of Lemma 5.6. □

Remark 5.8. Unless they happen to be Boolean groups (elementary abelian groups of exponent 2), commutative Moufang loops do not generally form semisymmetric quasigroups.

5.4. **Independence of left and right semisymmetry.**

Theorem 5.9. (a) *The left and right semisymmetry conditions are not equivalent in general.*

(b) *Neither left nor right semisymmetry guarantees that a bimagma with an augmentation is a quantum quasigroup.*

Proof. Let K be a field. Let A be a vector space over K, with a basis $\{e_n \mid n \in \mathbb{N}\}$. Consider the linear transformations

$$\lambda \colon A \to A; e_n \mapsto e_{n+1}$$

and

$$\rho \colon A \to A; e_n \mapsto \begin{cases} e_0 & \text{if } n = 0\,; \\ e_{n-1} & \text{otherwise.} \end{cases}$$

Thus

$$\lambda\rho = 1_A \tag{5.4}$$

and

$$e_0\rho\lambda = e_0\lambda = e_1\,. \tag{5.5}$$

In the symmetric monoidal category $(\underline{\underline{K}}, \oplus, \{0\})$ of vector spaces over K, take a multiplication $\nabla \colon A \oplus A \to A; x \oplus y \mapsto x^\rho + y^\lambda$ and comultiplication $\Delta = 0$ on A. The choice of comultiplication renders the commuting of the bimagma diagram (2.1) trivial.

The upper pentagon of (5.2) commutes, since $(x^L y)x^R = (y^\lambda)^\rho = y = 0 \oplus y$ for $x \oplus y \in A \oplus A$ by (5.4). Thus the bimagma (A, ∇, Δ) is left semisymmetric. On the other hand, starting from $e_0 \oplus e_0$ in $A \oplus A$, tracing round the bottom left of the diagram (5.2) leads to $e_0^L(e_0 e_0^R) = (e_0^\rho)^\lambda = e_1$ by (5.5). Then since $e_1 \neq e_0 = (e_0 \oplus e_0)(\varepsilon \oplus 1_A)$, the bimagma (A, ∇, Δ) is not right semisymmetric. Similarly, the opposite multiplication on A yields a right semisymmetric bimagma which is not left semisymmetric.

Finally, the left and right composites (1.2), (1.3) in (A, ∇, Δ) are the zero maps on the non-trivial space $A \oplus A$, so that (A, ∇, Δ) is not a quantum quasigroup. □

5.5. **Counital bimagmas.**

Theorem 5.10. *Let $(A, \nabla, \Delta, \varepsilon)$ be a counital bimagma in $(\mathbf{Set}, \times, \top)$.*

(*a*) *The left and right semisymmetry conditions are equivalent on the counital bimagma $(A, \nabla, \Delta, \varepsilon)$.*

(*b*) *If left and/or right semisymmetry conditions on $(A, \nabla, \Delta, \varepsilon)$ hold, then (A, ∇, Δ) is a quantum quasigroup.*

Proof. (a) Suppose $(A, \nabla, \Delta, \varepsilon)$ is left semisymmetric. Counitality requires $\Delta\colon a \mapsto (a, a)$ to be the diagonal embedding. The commuting of the upper pentagon (5.1) is thus equivalent to $(ab)a = b$ for all $a, b \in A$. This is just classical left semisymmetry for the magma (A, ∇). It follows [24, Prop. 1.2] that this magma is also classically right semisymmetric, which we may then translate into the commuting of the lower half of (5.1).

(b) By (a), left or right semisymmetry forces (A, ∇) to be a classically semisymmetric magma. Then $(A, \nabla, \nabla_t, \nabla_t)$ is an equational quasigroup, with $\nabla_t\colon (a, b) \mapsto ba$. Now counital quantum quasigroups in $(\mathbf{Set}, \times, \top)$ are just quasigroups [27, Prop. 3.11(a)], and thus (A, ∇, Δ) is a quantum quasigroup. □

Remark 5.11. (a) Definition 5.3 is designed to capture the classical concept of semisymmetry, as described by Theorem 5.10, with a single application of the comultiplication. Alternative versions of semisymmetry, invoking two comultiplications, are deferred for future consideration.

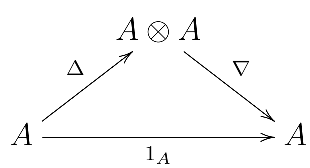
(b) Within the lower pentagon of the commutative diagram (5.1), a formal replacement of the morphism $1_A \otimes \tau$ by $S \otimes 1_A \otimes 1_A$ yields the first of the four equations used to specify a Hopf quasigroup [14, Defn. 4.1]. Note that the Hopf quasigroup definition requires the augmentation to be an algebra homomorphism.

6. The Mendelsohn property

6.1. Quantum idempotence and Mendelsohn properties.

Definition 6.1. Suppose that (A, ∇, Δ) is a bimagma in a symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) If the diagram



commutes in $\mathbf{V}$, then the bimagma is said to satisfy the condition of *quantum idempotence* [28, Defn. 5.1].

(b) With respect to an augmentation, the bimagma is said to have the *left Mendelsohn property* if it is quantum idempotent, and exhibits left semisymmetry.

(c) With respect to an augmentation, the bimagma is said to have the *right Mendelsohn property* if it is quantum idempotent, and exhibits right semisymmetry.

(d) With respect to an augmentation, the bimagma is said to have the *Mendelsohn property* when it has both the left and right Mendelsohn properties.

Remark 6.2. In Proposition 4.1, if (A, ∇, Δ) is a quantum Mendelsohn quasigroup, the quantum quasigroups (A, ∇_l, Δ_l) and (A, ∇_r, Δ_r) need not satisfy the quantum Mendelsohn properties.

Theorem 5.9 showed that in the symmetric monoidal category $(\underline{\underline{K}}, \oplus, \{0\})$ of vector spaces over a field K, left semisymmetry does not imply right semisymmetry. In contrast with Theorem 5.9, Theorem 6.4 below shows the power of quantum idempotence to relate the left and right Mendelsohn properties.

6.2. **Left and right Mendelsohn properties.**

Lemma 6.3. *Suppose that $(\underline{\underline{S}}, \oplus, \{0\})$ is the symmetric monoidal category of modules over a commutative, unital ring S. Suppose that (A, ∇, Δ) is a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$, written as $A(\rho, \lambda, R, L)$ according to Definition 3.5.*

(a) *The left semisymmetry property amounts to*

$$\lambda\rho = 1 \quad \textit{and} \quad L\rho^2 + R\lambda = 0 \tag{6.1}$$

in the endomorphism algebra $\underline{\underline{S}}(A, A)$.

(b) *The right semisymmetry property amounts to*

$$\rho\lambda = 1 \quad \textit{and} \quad L\rho + R\lambda^2 = 0 \tag{6.2}$$

in the endomorphism algebra $\underline{\underline{S}}(A, A)$.

(c) *The quantum idempotence property amounts to*

$$L\rho + R\lambda = 1 \tag{6.3}$$

in the endomorphism algebra $\underline{\underline{S}}(A, A)$.

Theorem 6.4. *Consider the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules over a commutative, unital ring S. Suppose that (A, ∇, Δ) is a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$. Suppose that (A, ∇, Δ) satisfies the left Mendelsohn property. Then:*

(a) *(A, ∇, Δ) satisfies the right Mendelsohn property;*

(b) *(A, ∇, Δ) is a quantum quasigroup.*

Proof. (a) Use the notation of (3.7). Under the commuting of $\{\rho, \lambda\}$ with $\{L, R\}$, together with the left semisymmetry property (6.1) and quantum idempotence (6.3), the right semisymmetry property (6.2) must be derived.

It is helpful to introduce

$$\mu = R\lambda = \lambda R \tag{6.4}$$

in $\underline{\underline{S}}(A,A)$, so that $R = R\lambda\rho = \mu\rho$ under our assumptions. Then $\mu = R\lambda = -L\rho^2$ by (6.1). Since both L and ρ^2 commute with ρ, the commutative subalgebra $S[\mu]$ generated by μ in the endomorphism S-algebra $\underline{\underline{S}}(A,A)$ commutes with the commutative subalgebra $S[\rho]$ generated by ρ.

By (6.3), one has $L\rho = 1 - R\lambda = 1 - \mu$. The second equation of (6.1) then becomes $0 = L\rho^2 + R\lambda = (1-\mu)\rho + \mu = \mu + \rho - \mu\rho$, so that

$$1 = 1 - \mu - \rho + \mu\rho = (1-\mu)(1-\rho) = (1-\rho)(1-\mu) \tag{6.5}$$

in $\underline{\underline{S}}(A,A)$, the latter equation following by the commuting of $S[\mu]$ with $S[\rho]$. Equation (6.5) shows that $1-\rho$ and $1-\mu$ are mutually inverse elements of the automorphism group of the S-module A, the group $\underline{\underline{S}}(A,A)^*$ of units of the monoid $\underline{\underline{S}}(A,A)$.

Since $\rho L = L\rho = 1 - \mu$, we have $\rho L(1-\rho) = 1$ by (6.5). Taken along with $\lambda\rho = 1$, this shows that ρ is invertible in $\underline{\underline{S}}(A,A)$. Thus $\lambda = L(1-\rho)$ and $\rho\lambda = 1$, the first equation of (6.2).

By (6.1), we have $(L\rho + R\lambda^2)\rho = L\rho^2 + R\lambda = 0$, which now implies $L\rho + R\lambda^2 = 0$ since ρ is invertible. This completes the verification of (6.2).

(b) Since $\lambda = \rho^{-1}$ and $\mu = R\lambda$, we have $R = \mu\lambda^{-1} = \mu\rho = -\rho^2(1-\rho)^{-1}$ invertible. Likewise, since $\mu = -L\rho^2$, we have $L = -\mu\rho^{-2}$ invertible. Thus $A(\rho, \lambda, L, R)$ is a quantum quasigroup by Proposition 3.13. □

6.3. **Construction and classification.** Theorem 6.4 and its proof lead to an identification of all Mendelsohn quantum quasigroups in the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules, over a commutative, unital ring S, under the direct sum.

Corollary 6.5. *An S-module A is endowed with a Mendelsohn quantum quasigroup structure $A(\rho, \lambda, L, R)$ in $(\underline{\underline{S}}, \oplus, \{0\})$ if and only if*

$$\lambda = \frac{1}{\rho}\,, \qquad L = \frac{1}{\rho(1-\rho)}\,, \qquad \textit{and} \qquad R = -\frac{\rho^2}{(1-\rho)} \tag{6.6}$$

for an endomorphism ρ of the S-module A such that both ρ and $1-\rho$ are invertible.

Proof. The proof of Theorem 6.4 serves to provide the necessity of the given conditions. Conversely, for any module automorphism ρ such that $1-\rho$ is also invertible, both the conditions (6.1), (6.2) and the commuting of $\{\rho, \lambda\}$ with $\{L, R\}$ are readily verified. □

An interesting "classical" consequence of Corollary 6.5 follows. Recall that a magma (A, ∇) in a symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is said to be *entropic* if $\nabla\colon A\otimes A\to A$ is a magma homomorphism, i.e., if the diagram

$$\begin{array}{ccccc} A\otimes A & \xrightarrow{\ \nabla\ } & A & \xleftarrow{\ \nabla\ } & A\otimes A \\ \uparrow{\scriptstyle \nabla\otimes\nabla} & & & & \uparrow{\scriptstyle \nabla\otimes\nabla} \\ A\otimes A\otimes A\otimes A & & \xrightarrow[1_A\otimes\tau\otimes 1_A]{} & & A\otimes A\otimes A\otimes A \end{array} \tag{6.7}$$

commutes. In the concrete notation of Remark 2.2(c), this amounts to the identity $x_1x_2\cdot x_3x_4 = x_1x_3\cdot x_2x_4$.

Proposition 6.6. *Suppose that* (A,∇,Δ) *is a Mendelsohn quantum quasigroup in* $(\underline{\underline{S}},\oplus,\{0\})$*. Then* (A,∇) *is an entropic quasigroup in* $(\mathbf{Set},\otimes,\top)$.

Proof. Chasing round the diagram (6.7) in $(\underline{\underline{S}},\oplus,\{0\})$ yields

$$\begin{array}{ccc} a_1\oplus a_2\oplus a_3\oplus a_4 & \xmapsto{\ \nabla\oplus\nabla\ } & a_1^\rho + a_2^\lambda \oplus a_3^\rho + a_4^\lambda \\ \Big\downarrow{\scriptstyle 1_A\oplus\tau\oplus 1_A} & & \downarrow{\scriptstyle\nabla} \\ & & \left(a_1^\rho + a_2^\lambda\right)^\rho + \left(a_3^\rho + a_4^\lambda\right)^\lambda \\ & & \| \\ & & \left(a_1^\rho + a_3^\lambda\right)^\rho + \left(a_2^\rho + a_4^\lambda\right)^\lambda \\ & & \uparrow{\scriptstyle\nabla} \\ a_1\oplus a_3\oplus a_2\oplus a_4 & \xmapsto{\ \nabla\oplus\nabla\ } & a_1^\rho + a_3^\lambda \oplus a_2^\rho + a_4^\lambda \end{array}$$

which commutes since ρ and $\lambda=\rho^{-1}$ commute. Thus (A,∇) is an entropic magma in $(\mathbf{Set},\otimes,\top)$. It is a quasigroup, since

$$\nabla_l\colon x\oplus z\mapsto z^\rho - x^{\rho^2} = y \quad\text{ and }\quad \nabla_r\colon z\oplus y\mapsto z^\lambda - y^{\lambda^2} = x$$

give respective unique solutions x and y to $z = xy = x^\rho + y^\lambda$. ☐

Remark 6.7. (a) The entropic quasigroup (A,∇) in Proposition 6.6, when augmented by a diagonal comultiplication $\Delta\colon x\to x\otimes x$, forms the classical reduct of the Mendelsohn quantum quasigroup (A,∇,Δ).

(b) Note that the classical quasigroup (A,∇) of Proposition 6.6 need not be semisymmetric, let alone a Mendelsohn quasigroup. Consider the case of $S=\mathbb{C}$ with $\rho=i$ and $\lambda=-i$, so $x\cdot y = i(x-y)$ in a complex vector space A. Then $(xy)x = y - x(1+i)$.

(c) Proposition 6.6 aligns with the discussion of [6], where it is noted that "affine" Mendelsohn quasigroups (in our case, $\mathbb{Z}$-linear Mendelsohn quasigroups) are entropic.

6.4. **An alternative parametrization.** In the context of the preceding section, it is actually more natural and instructive to take the automorphism μ, as in (6.4), for the fundamental parameter. Use the notation μ' for the *complement* $1-\mu$. Corollary 6.5 may then be reformulated as follows.

Theorem 6.8. *Consider the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules over a commutative, unital ring S. With the notations*

$$\nabla \colon A \oplus A \to A; x \oplus y \mapsto x^\rho + y^\lambda \quad \textit{and} \quad \Delta \colon A \to A \oplus A; x \mapsto x^L \oplus x^R$$

of (3.7)*, an S-module A is endowed with a Mendelsohn quantum quasigroup structure in $(\underline{\underline{S}}, \oplus, \{0\})$ if and only if*

$$\rho = -\mu/\mu' \,, \qquad \lambda = -\mu'/\mu \,, \qquad L = -(\mu')^2/\mu \,, \qquad \textit{and} \qquad R = -\mu^2/\mu'$$

for an endomorphism $\mu = R\lambda$ of the S-module A such that both μ and μ' are invertible.

Proof. Recall $(1-\mu)(1-\rho) = 1$ by (6.5), so $1-\rho = (1-\mu)^{-1}$ and

$$\rho = 1 - \frac{1}{1-\mu} = \frac{-\mu}{1-\mu} \,.$$

The rest follows from (6.6) on substituting for ρ in terms of μ. □

Note that $R = \mu\rho$ and $L = \mu'\lambda$. Complementation $\mu \mapsto \mu'$ takes both the quasigroup (A, ∇) and comagma (A, Δ) to their respective opposites $(A, \tau\nabla)$ and $(A, \Delta\tau)$. Expressed differently, the involutive complementation transformation $\mu \to \mu'$ induces $\rho \mapsto \lambda$ and $R \mapsto L$.

Proposition 6.9. *If (A, ∇, Δ) is a Mendelsohn quantum quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$, then so is its opposite $(A, \tau\nabla, \Delta\tau)$.*

Proof. If (A, ∇, Δ) is parametrized by μ via Theorem 6.8, then its opposite is parametrized by μ'. □

6.5. **Setlike elements in linear Mendelsohn quantum quasigroups.** Suppose that (A, ∇, Δ) is a classical Mendelsohn quantum quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$. According to Theorem 6.8, this corresponds to the equations

$$\mu^2 = -\mu' \quad \text{and} \quad (\mu')^2 = -\mu \,,$$

both of which are equivalent to the vanishing of

$$\mu^2 - \mu + 1 \tag{6.8}$$

in the endomorphism ring $\underline{\underline{S}}(A,A)$. In this case, the entropic quasigroup (A,∇) of §6.3 is indeed a Mendelsohn quasigroup, in sharp contrast to the situation described in Remark 6.7.

More generally, we have the following counterpart to Theorem 3.16.

Theorem 6.10. *Suppose that (A,∇,Δ) is a Mendelsohn quantum quasigroup in $(\underline{\underline{S}},\oplus,\{0\})$. Then the set A_1^0 of setlike elements of (A,∇,Δ) forms an entropic Mendelsohn quasigroup $(A_1^0,\cdot)$.*

Proof. Consider (A,∇,Δ) as being parametrized by an endomorphism μ of the S-module A, according to Theorem 6.8. Then A_1^0 is the kernel of the endomorphism (6.8) of the S-module A. As such, A_1^0 is invariant under the automorphisms μ and μ', and therefore carries $q_1\cdot q_2=q_1^\rho+q_2^\lambda$ as an entropic Mendelsohn quasigroup structure. □

7. Semisymmetry of linear quantum quasigroups

Throughout this section, we will be working in the symmetric monoidal category $(\underline{\underline{S}},\oplus,\{0\})$ of modules over a commutative, unital ring S. Recall that augmentations here are trivial.

7.1. **Left and right semisymmetry of linear quantum quasigroups.**

For a free S-module A of countable rank, the proof of Theorem 5.9(a) shows that the left and right semisymmetry conditions on a bimagma structure (A,∇,Δ) in $(\underline{\underline{S}},\oplus,\{0\})$ need not be equivalent if S is nontrivial. For linear quantum quasigroups, the situation is different.

Theorem 7.1. *Suppose that $A(\rho,\lambda,L,R)$ is a linear quantum quasigroup in $(\underline{\underline{S}},\oplus,\{0\})$. The left and right semisymmetry conditions on $A(\rho,\lambda,L,R)$ are equivalent.*

Proof. First suppose that $A(\rho,\lambda,L,R)$ is left semisymmetric. Then by Lemma 6.3(b), the left semisymmetry corresponds to

$$\lambda\rho=1 \quad\text{ and }\quad L\rho^2+R\lambda=0$$

in the endomorphism algebra $\underline{\underline{S}}(A,A)$. In particular, we have $\lambda=\rho^{-1}$ and $R\lambda=-L\rho^2$. Thus $\rho\lambda=1$ and $L\rho+R\lambda^2=L\rho-L\rho^2\lambda=L\rho(1-\rho\lambda)=0$, so $A(\rho,\lambda,L,R)$ is right semisymmetric by Lemma 6.3(c).

The converse implication follows by applying these considerations to the transpose $A(\lambda,\rho,R,L)$ of $A(\rho,\lambda,L,R)$. □

Corollary 7.2. *A linear quantum quasigroup in $(\underline{\underline{S}},\oplus,\{0\})$ is left and right semisymmetric if and only if it has the form $A(\rho,\rho^{-1},L,-L\rho^3)$ for mutually centralizing automorphisms ρ and L of the S-module A.*

7.2. **Semisymmetrizations of linear quantum quasigroups.** In this section, we exhibit semisymmetric linear quantum quasigroups constructed from an arbitrary linear quantum quasigroup.

Definition 7.3. Let $A(\rho,\lambda,L,R)$ be a linear quantum quasigroup in the category $(\underline{\underline{S}},\oplus,\{0\})$.

(a) Define an endomorphism $\Lambda_{LS}\colon A^3\to A^3$ by the matrix

$$\Lambda_{LS}=\begin{bmatrix}0 & -\rho\lambda^{-1} & 0\\ 0 & 0 & \lambda\\ \rho^{-1} & 0 & 0\end{bmatrix}. \tag{7.1}$$

(b) Define an endomorphism $\Lambda_{RS}\colon A^3\to A^3$ by the matrix

$$\Lambda_{RS}=\begin{bmatrix}0 & -\lambda\rho^{-1} & 0\\ 0 & 0 & \rho\\ \lambda^{-1} & 0 & 0\end{bmatrix}. \tag{7.2}$$

(c) The endomorphism $\Lambda_S\colon A^6\to A^6$ is defined as $\Lambda_S=\Lambda_{LS}\oplus\Lambda_{RS}$.

Lemma 7.4. *The endomorphisms Λ_{LS}, Λ_{RS} and Λ_S all satisfy the equation $X^3+1=0$. In particular, they are automorphisms of A^3, A^3, and A^6 respectively.*

Remark 7.5. The significance of the equation $X^3+1=0$ within the representation theory of classical semisymmetric quasigroups is discussed in [21, Ex. 4.6].

Lemma 7.6. *Suppose that $A(\rho,\lambda,L,R)$ is a linear quantum quasigroup in the category $(\underline{\underline{S}},\oplus,\{0\})$.*

(a) *Let C_{LS} commute with Λ_{LS} in the automorphism group $\underline{\underline{S}}(A^3,A^3)^*$ of the S-module A^3. Then there is a cocommutative linear quantum quasigroup structure $A^3(-\Lambda_{LS}^2,\Lambda_{LS},C_{LS},C_{LS})$ in $(\underline{\underline{S}},\oplus,\{0\})$.*

(b) *Let C_{RS} commute with Λ_{RS} in the automorphism group $\underline{\underline{S}}(A^3,A^3)^*$ of the S-module A^3. Then there is a cocommutative linear quantum quasigroup structure $A^3(-\Lambda_{RS}^2,\Lambda_{RS},C_{RS},C_{RS})$ in $(\underline{\underline{S}},\oplus,\{0\})$.*

(c) *Let C_S commute with Λ_S in the automorphism group $\underline{\underline{S}}(A^6,A^6)^*$ of the S-module A^6. Then there is a cocommutative linear quantum quasigroup structure $A^6(-\Lambda_S^2,\Lambda_S,C_S,C_S)$ in $(\underline{\underline{S}},\oplus,\{0\})$.*

Lemma 7.7. *The respective linear quantum quasigroups of Lemma 7.6(a), (b) and (c) are left and right semisymmetric.*

Proof. Generically, denote each of these structures as $A^d(-\Lambda^2,\Lambda,C,C)$. Note $\Lambda^{-1}=-\Lambda^2$ by Lemma 7.4. Then $C=-C(-\Lambda^2)^3$, so $A^d(-\Lambda^2,\Lambda,C,C)$ is left and right semisymmetric by Corollary 7.2. □

Recall the matrix

$$\mathrm{P}=\begin{bmatrix}0&0&1\\-1&0&0\\0&1&0\end{bmatrix}$$

of [9, (2.2)], pronounced "rho". For an S-module A, we thus define the automorphism

$$\mathrm{P}_A=\begin{bmatrix}0&0&1_A\\-1_A&0&0\\0&1_A&0\end{bmatrix}$$

of A^3. For S-module endomorphisms $\alpha\colon A\to A$ and $\theta\colon A^3\to A^3$, define the *left scalar multiplications*

$$\alpha\theta\colon A^3\to A^3; x\oplus y\oplus z\mapsto(x^\alpha\oplus y^\alpha\oplus z^\alpha)^\xi\,.$$

For automorphisms θ,φ,ψ of an S-module A, define the automorphism

$$D_{\theta,\varphi,\psi}\colon A^3\to A^3; x\oplus y\oplus z\mapsto x^\theta\oplus y^\varphi\oplus z^\psi$$

of A^3. Automorphisms of this type are described as (*twisted*) *diagonals*.

Lemma 7.8. *Suppose that $A(\rho,\lambda,L,R)$ is a linear quantum quasigroup in the category $(\underline{\underline{S}},\oplus,\{0\})$.*

(a) *An automorphism C_{LS} of A^3 centralizes Λ_{LS} in $\underline{\underline{S}}(A^3,A^3)^*$ if and only if it is of the form*

$$C_{LS}=D_{\rho,\lambda,1}\big(\alpha 1_{A^3}+\beta P_A+\gamma P_A^2\big)D_{1,\rho\lambda^{-1},\rho}$$

for endomorphisms α,β,γ of the S-module A such that

$$\alpha 1_{A^3}+\beta P_A+\gamma P_A^2$$

is invertible.

(b) *An automorphism C_{RS} of A^3 centralizes Λ_{RS} in $\underline{\underline{S}}(A^3,A^3)^*$ if and only if it is of the form*

$$C_{RS}=D_{\lambda,\rho,1}\big(\alpha' 1_{A^3}+\beta' P_A+\gamma' P_A^2\big)D_{1,\lambda\rho^{-1},\lambda}$$

for endomorphisms α',β',γ' of the S-module A such that

$$\alpha' 1_{A^3}+\beta' P_A+\gamma' P_A^2$$

is invertible.

Proof. Note that (b) follows by (a) applied to the transpose $A(\lambda,\rho,R,L)$ of $A(\rho,\lambda,L,R)$, so it suffices to prove (a).

Consider an endomorphism

$$X=\begin{bmatrix}\xi&\zeta'&\eta''\\\eta&\xi'&\zeta''\\\zeta&\eta'&\xi''\end{bmatrix}\tag{7.3}$$

of A^3 that commutes with Λ_{LS}, for given endomorphisms $\xi, \zeta', \dots, \xi''$ of the S-module A. The commuting is equivalent to $\Lambda_{LS}X =$

$$\begin{bmatrix} -\rho\lambda^{-1}\eta & -\rho\lambda^{-1}\xi' & -\rho\lambda^{-1}\zeta'' \\ \lambda\zeta & \lambda\eta' & \lambda\xi'' \\ \rho^{-1}\xi & \rho^{-1}\zeta' & \rho^{-1}\eta'' \end{bmatrix} = \begin{bmatrix} \eta''\rho^{-1} & -\xi\rho\lambda^{-1} & \zeta'\lambda \\ \zeta''\rho^{-1} & -\eta\rho\lambda^{-1} & \xi'\lambda \\ \xi''\rho^{-1} & -\zeta\rho\lambda^{-1} & \eta'\lambda \end{bmatrix} = X\Lambda_{LS}$$

or

$$-\rho\lambda^{-1}\xi' = -\xi\rho\lambda^{-1}\,, \qquad \rho^{-1}\xi = \xi''\rho^{-1}\,, \qquad \lambda\xi'' = \xi'\lambda\,, \tag{7.4}$$

$$\lambda\eta' = -\eta\rho\lambda^{-1}\,, \qquad -\rho\lambda^{-1}\eta = \eta''\rho^{-1}\,, \qquad \rho^{-1}\eta'' = \eta'\lambda\,, \tag{7.5}$$

$$\rho^{-1}\zeta' = -\zeta\rho\lambda^{-1}\,, \qquad \lambda\zeta = \zeta''\rho^{-1}\,, \qquad -\rho\lambda^{-1}\zeta'' = \zeta'\lambda\,. \tag{7.6}$$

If the commuting holds, the first two equations of (7.4) yield

$$\xi' = \lambda\rho^{-1}\xi\rho\lambda^{-1} \qquad \text{and} \qquad \xi'' = \rho^{-1}\xi\rho\,.$$

Conversely, if these two equations hold, then (7.4) is satisfied, including the third equation as $\lambda\xi'' = \lambda\rho^{-1}\xi\rho = \lambda\rho^{-1}\xi\rho\lambda^{-1}\lambda = \xi'\lambda$. In similar fashion, the three equations of (7.5) are seen to be equivalent to

$$\eta' = -\lambda^{-1}\eta\rho\lambda^{-1} \qquad \text{and} \qquad \eta'' = -\rho\lambda^{-1}\eta\rho\,,$$

while the three equations of (7.6) are seen to be equivalent to

$$\zeta' = -\rho\zeta\rho\lambda^{-1} \qquad \text{and} \qquad \zeta'' = \lambda\zeta\rho\,.$$

Summarizing, the matrix X of (7.3) commutes with Λ_{LS} if and only if

$$X = \begin{bmatrix} \xi & -\rho\zeta\rho\lambda^{-1} & -\rho\lambda^{-1}\eta\rho \\ \eta & \lambda\rho^{-1}\xi\rho\lambda^{-1} & \lambda\zeta\rho \\ \zeta & -\lambda^{-1}\eta\rho\lambda^{-1} & \rho^{-1}\xi\rho \end{bmatrix}$$

for endomorphisms ξ, η, ζ of the S-module A. Setting

$$\xi = \rho\alpha\,, \qquad \eta = -\lambda\beta\,, \qquad \zeta = -\gamma$$

then gives the desired result. □

Lemmas 7.6 – 7.8 combine to yield the following.

Theorem 7.9. *Suppose that $A(\rho, \lambda, L, R)$ is a linear quantum quasigroup in the category $(\underline{\underline{S}}, \oplus, \{0\})$.*

(a) *Consider Λ_{LS} as in (7.1) and C_{LS} as in Lemma 7.8(a). Then*

$$A^3(-\Lambda_{LS}^2, \Lambda_{LS}, C_{LS}, C_{LS})$$

forms a semisymmetric cocommutative linear quantum quasigroup structure in $(\underline{\underline{S}}, \oplus, \{0\})$.

(b) *Consider* Λ_{RS} *as in* (7.2) *and* C_{RS} *as in Lemma 7.8(b). Then*

$$A^3(-\Lambda_{RS}^2, \Lambda_{RS}, C_{RS}, C_{RS})$$

forms a semisymmetric cocommutative linear quantum quasigroup structure in $(\underline{\underline{S}}, \oplus, \{0\})$.

(c) *Consider* Λ_{LS} *and* C_{LS} *as in (a). Consider* Λ_{RS} *and* C_{RS} *as in (b). Then there is a semisymmetric cocommutative linear quantum quasigroup structure* $A^6(-\Lambda_S^2, \Lambda_S, C_S, C_S)$ *in* $(\underline{\underline{S}}, \oplus, \{0\})$.

Definition 7.10. The quantum quasigroups of Theorem 7.9(a)–(c) are known respectively as *left, right*, and *balanced semisymmetrizations* of the linear quantum quasigroup $A(\rho, \lambda, L, R)$ in $(\underline{\underline{S}}, \oplus, \{0\})$.

7.3. **Mendelsohnization of linear quantum quasigroups.** If we have a linear quantum quasigroup $A(\rho, \lambda, L, R)$ in the category $(\underline{\underline{S}}, \oplus, \{0\})$, define an endomorphism $M \colon A^2 \to A^2$ by the matrix

$$M = \begin{bmatrix} 0 & -\rho\lambda^{-1} \\ \lambda\rho^{-1} & 1 \end{bmatrix} . \tag{7.7}$$

Lemma 7.11. (a) *The endomorphisms* M *and* $1_{A^2} - M$ *of* A^2 *are mutually inverse.*

(b) *The endomorphism* M *satisfies* $M^3 = -1_{A^2}$.

Theorem 7.12. *Given a linear quantum quasigroup* $A(\rho, \lambda, L, R)$ *in the category* $(\underline{\underline{S}}, \oplus, \{0\})$*, the linear quantum quasigroup*

$$A^2\big(1_{A^2} - M, M, 1_{A^2}, 1_{A^2}\big)$$

forms a classical (and thus cocommutative and coassociative) Mendelsohn quantum quasigroup structure within the category $(\underline{\underline{S}}, \oplus, \{0\})$.

Proof. The result follows by Theorem 6.8, since Lemma 7.11 implies first that both M and $1-M$ are invertible, and then that $L = -(1-M)^2M^{-1} = 1_{A^2}$ and $R = -M^2(1-M)^{-1} = 1_{A^2}$. □

Definition 7.13. The linear quantum quasigroup $A^2\big(1_{A^2} - M, M, 1_{A^2}, 1_{A^2}\big)$ of Theorem 7.12 is called the (*classical*) *Mendelsohnization* of the linear quantum quasigroup $A(\rho, \lambda, L, R)$.

Remark 7.14. For a classical quasigroup $(Q, \cdot, /, \backslash)$, a classical quasigroup structure identified as semisymmetric on Q^2 has been defined by (a version of) the multiplication

$$(x_1, x_2) * (y_1, y_2) = \big((x_1 \cdot y_2)/x_2, y_1\backslash(x_1 \cdot y_2)\big)$$

in [16, §7]. This product is idempotent, and thus provides a Mendelsohn quasigroup structure on Q^2 that may be called the *Mendelsohnization* of the quasigroup $(Q, \cdot)$. Thus the Mendelsohnization of the linear quantum quasigroup (A, ∇, Δ) or $A(\rho, \lambda, L, R)$ appearing in Theorem 7.12 is just the Mendelsohnization of the quasigroup (A, ∇) in $(\mathbf{Set}, \times, \top)$.

8. Quantum distributivity of linear quantum quasigroups

8.1. **Quantum distributivity and the QYBE.** The quantum Yang-Baxter equation (QYBE) is

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} \tag{8.1}$$

[4, §2.2C]. It applies to an endomorphism

$$\mathcal{R} \colon A \otimes A \to A \otimes A$$

of the tensor square of an object A in a symmetric, monoidal category. For a given integer $n > 1$, the notation $\mathcal{R}^{ij}$, for $1 \leq i < j \leq n$, means applying $\mathcal{R}$ to the i-th and j-th factors in the n-th tensor power of A. Since the left and right composite morphisms of bimagmas are also endomorphisms of tensor squares, it is natural to seek conditions under which they satisfy the QYBE. Then, as anticipated by B.B. Venkov working in the category of sets with cartesian product [7, §9], the QYBE corresponds generally to various distributivity conditions on the products $\nabla \colon A \otimes A \to A$ appearing in the left and right composites, as in Example 8.2 below.

Definition 8.1. Suppose that (A, ∇, Δ) is a bimagma in a symmetric, monoidal category.

(a) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum left distributivity* if the left composite G of (A, ∇, Δ) satisfies the quantum Yang-Baxter equation (8.1).
(b) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum right distributivity* if the right composite $\circlearrowleft$ of (A, ∇, Δ) satisfies the quantum Yang-Baxter equation (8.1).
(c) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum distributivity* if it has both the left and right quantum distributivity properties.

Example 8.2. [28, Prop. 6.2] Let (A, ∇) be a magma in the category of sets with the cartesian product. Define $\Delta \colon A \to A \otimes A; a \mapsto a \otimes a$. Then the bimagma (A, ∇, Δ) is quantum left distributive if and only if the magma (A, ∇) is left distributive, in the sense that the identity

$$x(yz) = (xy)(xz) \tag{8.2}$$

is satisfied.

8.2. **Quantum distributivity of linear quasigroups.** For the remainder of this section, we will work in the symmetric monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of S-modules under the direct sum, over a commutative unital ring S.

Proposition 8.3. *Suppose that* $(A, \nabla, \Delta) = A(\rho, \lambda, L, R)$ *is a bimagma in* $(\underline{\underline{S}}, \oplus, \{0\})$.

(a) *The bimagma* (A, ∇, Δ) *is left quantum distributive if and only if the equations*

$$RL\rho = LR\rho\,, \quad R\rho = R^2\rho^2 + LR\rho\lambda\,, \quad R\lambda\rho = R\rho\lambda \tag{8.3}$$

are satisfied.

(b) *The bimagma* (A, ∇, Δ) *is right quantum distributive if and only if the equations*

$$L\rho\lambda = L\lambda\rho\,, \quad L\lambda = L^2\lambda^2 + RL\lambda\rho\,, \quad LR\lambda = RL\lambda \tag{8.4}$$

are satisfied.

Proof. (a) Note that

$$\mathsf{G}^{12} = \begin{bmatrix} L & R\rho & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{G}^{13} = \begin{bmatrix} L & 0 & R\rho \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \text{ and } \mathsf{G}^{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & L & R\rho \\ 0 & 0 & \lambda \end{bmatrix}$$

by (3.8). Then

$$\mathsf{G}^{12}\mathsf{G}^{13}\mathsf{G}^{23} = \begin{bmatrix} L^2 & RL\rho & R^2\rho^2 + LR\rho\lambda \\ 0 & L\lambda & R\lambda\rho \\ 0 & 0 & \lambda^2 \end{bmatrix}, \ \mathsf{G}^{23}\mathsf{G}^{13}\mathsf{G}^{12} = \begin{bmatrix} L^2 & LR\rho & R\rho \\ 0 & L\lambda & R\rho\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}.$$

The result follows by comparing the corresponding matrix entries.

(b) is dual to (a). Note that duality is implemented in the equations by the involution $(L\ \rho)(R\ \lambda)$ and a reversal of the order in products (corresponding to equating matrix entries following a transposition of the matrices in the QYBE). □

Corollary 8.4. *Suppose that* $(A, \nabla, \Delta) = A(\rho, \lambda, L, R)$ *is a linear quantum quasigroup in* $(\underline{\underline{S}}, \oplus, \{0\})$. *Then* (A, ∇, Δ) *is left quantum distributive if and only if the equations*

$$RL = LR\,, \quad 1 = R\rho + L\lambda\,, \quad \lambda\rho = \rho\lambda \tag{8.5}$$

are satisfied.

Proof. The outside equations of (8.5) are equivalent to the outside equations of (8.3) via multiplication by the automorphisms ρ and R or their inverses. Given the two commutation relationships expressed by the outside equations of (8.5), it is then seen that the middle equation of (8.5) is equivalent to the middle equation of (8.3) via multiplication by the automorphisms ρ and R or their inverses. □

Theorem 8.5. *Let (A, ∇, Δ) be a linear quantum quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$.*

(a) *The left and right quantum distributivity conditions on (A, ∇, Δ) are equivalent.*
(b) *If (A, ∇, Δ) is quantum distributive, then $(A, \tau\nabla, \Delta)$ and $(A, \nabla, \Delta\tau)$ are quantum idempotent.*

Proof. (a) Multiplication by the automorphisms L and λ or their inverses establishes the equivalence of (8.5) with (8.4), in the same way that the proof of Corollary 8.5 established the equivalence of (8.5) with (8.3).

(b) The quantum idempotence of each of the linear quantum quasigroups $(A, \tau\nabla, \Delta) = A(\lambda, \rho, L, R)$ and $(A, \nabla, \Delta\tau) = A(\rho, \lambda, R, L)$ is equivalent to $1 = L\lambda + R\rho$ by Lemma 6.3(c). This condition is the middle equation in the criterion (8.5) for the (left or right) quantum distributivity of (A, ∇, Δ). □

A quantum distributive linear quantum quasigroup will be described as a *linear quantum distributive quasigroup.*

8.3. Parametrization of linear quantum distributive quasigroups.

Lemma 8.6. *Let $(A, \nabla, \Delta) = A(\rho, \lambda, L, R)$ be a linear quantum distributive quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$. Define $\nu = R\rho$. Then $\nu' = 1 - \nu = L\lambda$.*

Proof. By the middle equation of (8.5), we have $\nu = R\rho = 1 - L\lambda$. □

Theorem 8.7. *Suppose that $(A, \nabla, \Delta) = A(\rho, \lambda, L, R)$ is a linear quantum quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$. Then (A, ∇, Δ) is quantum distributive if and only if there is an abelian group G, generated by a subset $\{l, n, r\}$, and a representation $\theta\colon G \to \underline{\underline{S}}(A, A)^*$, such that*

$$r^\theta = \rho\,, \quad l^\theta = L\,, \quad n^\theta = \nu\,, \quad R = \nu\rho^{-1}\,, \quad \lambda = L^{-1}\nu' \tag{8.6}$$

with $\nu' = 1 - \nu \in \underline{\underline{S}}(A, A)^*$.

Proof. First, suppose that $(A, \nabla, \Delta) = A(\rho, \lambda, L, R)$ is a linear quantum distributive quasigroup in $(\underline{\underline{S}}, \oplus, \{0\})$. Using the notation of Lemma 8.6, let G be the subgroup of the automorphism group $\underline{\underline{S}}(A, A)^*$ generated by $r = \rho$, $l = \lambda$, and $n = \nu$. Consider the embedding $\theta\colon G \hookrightarrow \underline{\underline{S}}(A, A)^*$. Then (8.6) holds by Lemma 8.6, and so $\nu' = L\lambda$ is invertible.

Conversely, suppose that A affords a representation $\theta\colon G \to \underline{\underline{S}}(A, A)^*$ as described in the theorem statement, with (8.6) holding. It may then be verified that the conditions (8.5) hold, so that $(A, \nabla, \Delta) = A(\rho, \lambda, L, R)$ is quantum distributive by Corollary 8.4. Indeed, $R\rho + L\lambda = \nu + \nu' = 1$, while $RL = LR$ and $\lambda\rho = \rho\lambda$ since ρ, λ, L, R all lie in the subalgebra of the endomorphism ring $\underline{\underline{S}}(A,A)$ generated by the image under θ of the abelian group G. $\square$

8.4. **Linear quantum distributive Mendelsohn quasigroups.** Recall the parametrization of linear Mendelsohn quantum quasigroups (A,∇,Δ) or $A(\rho,\lambda,L,R)$ in terms of an endomorphism μ of the S-module A with invertible μ and μ' (Theorem 6.8). We will describe a quantum distributive linear Mendelsohn quantum quasigroup as a *linear quantum distributive Mendelsohn quasigroup.*

Theorem 8.8. *Suppose that (A,∇,Δ) is a linear Mendelsohn quantum quasigroup in $(\underline{\underline{S}},\oplus,\{0\})$. Then (A,∇,Δ) is quantum distributive if and only if*

$$(\mu^2-\mu+1)(2\mu-1)^2=0 \tag{8.7}$$

in the endomorphism ring $\underline{\underline{S}}(A,A)$ of the S-module A.

Proof. Comparison of the specifications of ρ,λ,L,R from Theorem 6.8 and Theorem 8.7 yields the equations

$$-\frac{\mu^2}{\mu'}=R=\nu\rho^{-1}=-\frac{\nu\mu'}{\mu}\quad\text{and}\quad -\frac{\mu'}{\mu}=\lambda=L^{-1}\nu'=-\frac{\nu'\mu}{(\mu')^2}$$

so that

$$1=\nu+\nu'=\frac{\mu^3}{(1-\mu)^2}+\frac{(1-\mu)^3}{\mu^2}$$

or

$$0=4\mu^4-8\mu^3+9\mu^2-5\mu+1=(\mu^2-\mu+1)(2\mu-1)^2$$

as required. $\square$

Example 8.9. Let 2 be invertible in S. Consider the automorphism

$$\mu=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\frac14&\frac54&-\frac94&2\end{bmatrix}$$

of S^4, a companion matrix to the monic multiple $X^4-2X^3+\frac94X^2-\frac54X+\frac14$ of the polynomial on the left hand side of (8.7). Then the automorphisms

$$\rho=\begin{bmatrix}1&5&4&-4\\1&-4&4&-4\\1&-4&5&-4\\1&-4&5&-3\end{bmatrix},\quad \lambda=\begin{bmatrix}-4&9&-8&4\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$$

and

$$L = \begin{bmatrix} -3 & 8 & -8 & 4 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ \frac{1}{4} & -\frac{5}{4} & \frac{5}{4} & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & -4 & 4 & -4 \\ 1 & -4 & 5 & -4 \\ 1 & -4 & 5 & -3 \\ \frac{3}{4} & -\frac{11}{4} & \frac{11}{4} & -1 \end{bmatrix}$$

yield a linear quantum distributive Mendelsohn quasigroup $S^4(\rho, \lambda, L, R)$ in the category $(\underline{\underline{S}}, \oplus, \{0\})$.

Corollary 8.10. *Let (A, ∇, Δ) be a linear Mendelsohn quantum quasigroup in the category $(\underline{\underline{S}}, \oplus, \{0\})$. Then (A, ∇, Δ) is quantum distributive if either of the following holds:*

(a) *The quantum quasigroup (A, ∇, Δ) is setlike, with $\mu^2 - \mu + 1 = 0$;*
(b) *The quantum quasigroup (A, ∇, Δ) is not setlike, and $2\mu = 1_A$.*

The linear quantum distributive Mendelsohn quasigroup $S^4(\rho, \lambda, L, R)$ of Example 8.9 matches neither of the conditions (a), (b) of Corollary 8.10.

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