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The algebra of mode homomorphisms

Research Article

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Abstract: Modes are idempotent and entropic algebras. While the mode structure of sets of submodes has received considerable attention in the past, this paper is devoted to the study of mode structure on sets of mode homomorphisms. Connections between the two constructions are established. A detailed analysis is given for the algebra of homomorphisms from submodes of one mode to submodes of another. In particular, it is shown that such algebras can be decomposed as Płonka sums of more elementary homomorphism algebras. Some critical examples are examined.

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1. Introduction

There are two important and typical ways of constructing new modes (idempotent and entropic algebras) from given ones. One method is based on the fact that the set of subalgebras of a mode again forms a mode, providing so-called modes of submodes. The other method is based on the fact that the set of homomorphisms from one mode to another also has the structure of a mode, providing so-called (homo-)morphism algebras or homomorphism modes. While there is already a substantial body of results concerning modes of submodes and their generalizations (for basic results see [8] and [9], and for newer results [6]), there are only a few known results concerning algebras of mode homomorphisms.

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This paper is devoted to the study of algebras of mode homomorphisms. We investigate their structure, compare it with the structure of modes of submodes, and examine the fine structure of the sets of homomorphisms from subalgebras of one mode to subalgebras of another mode of the same type.

The main results are as follows. In Section 3, we define the algebra of homomorphisms between two modes A, B of the same type, and show that it can also be described as a certain algebra of submodes of $A \times B$. In general, this algebra does not coincide with the mode of submodes of $A \times B$. However, if A and B are diagonal modes, then the algebra of homomorphisms from A to B is a subalgebra of the algebra of submodes of $A \times B$ (Proposition 3.6).

The algebra of homomorphisms from submodes of a mode A to submodes of a mode B, both in the same prevariety $\mathbf{K}$, is defined in Section 4. When $\mathbf{K}$ is a variety $\mathbf{V}$, this algebra satisfies precisely the regular identities satisfied in $\mathbf{V}$ (Theorem 4.3). The fine structure of these homomorphism algebras is investigated in Section 5. We show that they are Płonka sums of $\mathbf{K}$-algebras (Theorem 5.4). In Section 6, the construction is specialized to the case of surjective homomorphism algebras (Theorem 6.6 and Corollary 6.8).

We refer readers to the monographs [8–10] for additional information about algebraic concepts used in the paper, especially those concerning convex sets, barycentric algebras, and affine spaces. Our notation generally follows the conventions established in these three monographs.

Throughout the paper, we assume that all the algebras have plural type $\tau\colon \Omega \to \mathbb{N}$. In other words, they always have at least one binary operation (there exists $\omega \in \Omega$ such that $\omega\tau > 1$) and no basic constants (for all $\omega \in \Omega$, $\omega\tau > 0$). If there is no danger of confusion, algebras and their underlying sets are denoted by the same symbol. The class of all algebras of type τ (or the category of τ-algebras) is denoted by $\underline{\tau}$. For algebras A and B in $\underline{\tau}$, the symbol $\underline{\tau}(A, B)$ denotes the set of all homomorphisms from A to B. If A and B are in a class $\mathbf{K}$ of τ-algebras, then we write $\mathbf{K}(A, B)$ instead of $\underline{\tau}(A, B)$. The symbol $x_1 \dots x_n w$ means that $x_1, \dots, x_n$ are exactly the variables appearing in the word w.

Regular and irregular identities play an essential role in this paper. Recall that the *regular* identities satisfied in algebras (or in a variety) are those having the same sets of variables on each side. A *strongly irregular* algebra (variety) is an algebra (variety) satisfying a *strongly irregular* identity $x \circ y = x$ for some derived operation $x \circ y$. The regularization $\widetilde{\mathbf{V}}$ of an irregular variety $\mathbf{V}$ of type τ is the variety defined by all the regular identities holding in $\mathbf{V}$. We use the fact that all algebras in the regularization $\widetilde{\mathbf{V}}$ of an irregular variety $\mathbf{V}$ of modes are Płonka sums of $\mathbf{V}$-algebras. (See e.g. [9, Chapter 4].) Recall that for a functor $F\colon I \to \underline{\tau}$, from a semilattice $(I, \vee)$ considered as a category to the category $\underline{\tau}$ of τ-algebras (A, Ω), the Płonka sum $IF = \coprod_{i \in I} A_i$ of the functor F (or of the algebras $iF = (A_i, \Omega)$ over I) is the algebra defined on the disjoint union $\bigcup_{i \in I} A_i$ with the Ω-algebra structure given by $\omega\colon A_{i_1} \times \dots \times A_{i_n} \to A_i$, $(a_{i_1}, \dots, a_{i_n}) \mapsto a_{i_1}\varphi_{i_1,i} \dots a_{i_n}\varphi_{i_n,i}\,\omega$, for each n-ary $\omega \in \Omega$, where $i = i_1 \vee \dots \vee i_n$ and, for $k = 1, \dots, n$, the mapping $\varphi_{i_k,i}\colon i_k F \to iF$ is the Ω-homomorphism $(i_k \to i)F$.

2. Modes

In the sense of [8, 9], *modes* are algebras in which each element forms a singleton subalgebra, and for which each operation is a homomorphism. For algebras (A, Ω) of a given type $\tau\colon \Omega \to \mathbb{N}$, these two properties are equivalent to satisfaction of the identity

$$x \dots x\psi = x$$

of *idempotence* for each operator ψ in Ω, and the identity

$$(x_{1,1} \dots x_{1,\psi\tau}\psi) \dots (x_{\phi\tau,1} \dots x_{\phi\tau,\psi\tau}\psi)\,\phi = (x_{1,1} \dots x_{\phi\tau,1}\phi) \dots (x_{1,\psi\tau} \dots x_{\phi\tau,\psi\tau}\phi)\,\psi$$

of *entropicity* for each pair ψ, ϕ of operators in Ω.

One of the main families of examples of modes is given by affine spaces over a commutative unital ring R (*affine R-spaces*), or, more generally, by subreducts (subalgebras of reducts) of affine spaces. Affine spaces are considered here as Mal'tsev modes, as explained in the monographs [8, 9]. In particular, if 2 is invertible in R, an affine R-space can be considered as the reduct $(A, \underline{R})$ of an R-module $(A, +, R)$, where $\underline{R}$ is the family of binary operations

$$\underline{r}\colon A^2 \to A; \qquad (x_1, x_2) \mapsto x_1 x_2 \underline{r} = x_1(1 - r) + x_2 r$$

for each $r \in R$. The class of all affine R-spaces is a variety (cf. also [2]) and is denoted by $\underline{\underline{R}}$.

An important class of subreducts of affine spaces is given by convex sets, defined as subreducts of affine $\mathbb{R}$-spaces, where $\mathbb{R}$ is the ring of real numbers. Convex sets are characterized as subsets of a real vector space closed under the operations $\underline{r}$ of weighted means coming from the open real unit interval $I^{o} =]0,1[$. Thus a convex set contains, along with any two of its points, the line segment joining them. The class $\mathbf{C}$ of convex sets, considered as such algebras $(C, \underline{I}^{o})$, generates the variety $\mathbf{B}$ of barycentric algebras, and is characterized as the subquasivariety of $\mathbf{B}$ defined by all the cancelation laws for the binary operations $\underline{r}$, where $r \in I^{o}$. (See [5].) The definition of real convex sets and barycentric algebras is easily generalized to the case of subfields F of the field $\mathbb{R}$. (See e.g. [9, Chapters 5, 7].)

Other well-known classes of modes are given by normal bands (semigroup modes), quasigroup modes, and many classes of groupoid modes.

3. Mode homomorphisms and submodes

Let $\tau\colon \Omega \to \mathbb{N}$ be a plural type. Let A and B be modes in a prevariety $\mathbf{K}$ of type τ. The set $\mathbf{K}(A,B)$ of homomorphisms from A to B is an algebra of type τ, with

$$f_1 \dots f_{\omega\tau}\omega\colon A \to B; \qquad a \mapsto af_1 \dots af_{\omega\tau}\omega$$

for a in A and f_i in $\mathbf{K}(A,B)$. As a subalgebra of the power B^A, the algebra $\mathbf{K}(A,B)$ lies in the prevariety $\mathbf{K}$. (See [9, Proposition 5.1, Corollary 5.2].)

Given a homomorphism $f\colon A \to B$, define its *graph* $\mathrm{gr}(f) = \{(a, af) \in A \times B : a \in A\}$. It is easy to check that $\mathrm{gr}(f)$ is a subalgebra of the direct product $A \times B$. Let $\mathrm{Gr}(A,B)$ be the set consisting of the graphs of all homomorphisms in $\mathbf{K}(A,B)$.

Lemma 3.1.
Let C be a subalgebra of $A \times B$. Then $C \in \mathrm{Gr}(A,B)$ if and only if the following two conditions are satisfied:

(a) $\{a \in A : (a,b) \in C\} = A$;

(b) *for all* $(a_1,b_1),(a_2,b_2) \in C$, $a_1 = a_2$ *implies* $b_1 = b_2$.

Proof. If $C \in \mathrm{Gr}(A,B)$, then C obviously satisfies conditions (a) and (b). Now let C be a subalgebra of $A \times B$ satisfying these two conditions. Let $\omega \in \Omega$ and $(a_1,b_1),\dots,(a_{\omega\tau},b_{\omega\tau}) \in C$. By defining $af = b$ for each $(a,b) \in C$, one obtains a function $f\colon A \to B$. Moreover, since

$$(a_1,b_1)\dots(a_{\omega\tau},b_{\omega\tau})\omega = (a_1 \dots a_{\omega\tau}\omega, b_1 \dots b_{\omega\tau}\omega),$$

and C is a subalgebra of $A \times B$, it follows that $a_1 \dots a_{\omega\tau}\omega f = b_1 \dots b_{\omega\tau}\omega = a_1 f \dots a_{\omega\tau} f \omega$. This implies that f is an Ω-homomorphism and $C = \mathrm{gr}(f)$. □

For each $\omega \in \Omega$, define an operation $\overline{\omega}$ on $\mathrm{Gr}(A,B)$ by

$$\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\overline{\omega} = \mathrm{gr}(f_1 \dots f_{\omega\tau}\omega), \tag{1}$$

for homomorphisms $f_1,\dots,f_{\omega\tau} \in \mathbf{K}(A,B)$. Obviously,

$$\mathrm{gr}(f_1 \dots f_{\omega\tau}\omega) = \{(a, af_1 \dots f_{\omega\tau}\omega) : a \in A\} = \{(a, af_1 \dots af_{\omega\tau}\omega) : a \in A\}. \tag{2}$$

Let $\overline{\Omega} = \{\overline{\omega} : \omega \in \Omega\}$. Then under the operations of $\overline{\Omega}$, the set $\mathrm{Gr}(A,B)$ is an algebra of the same type as $\mathbf{K}(A,B)$, with the following obvious property.

Corollary 3.2.

The algebras $\mathbf{K}(A,B)$ *of homomorphisms from* A *to* B *and the graph algebra* $\mathrm{Gr}(A,B)$ *are isomorphic,*

$$\mathbf{K}(A,B) \cong \mathrm{Gr}(A,B).$$

The algebra $\mathrm{Gr}(A,B)$ consists of certain subalgebras of the direct product $A \times B$. One of the very special properties of modes is that for any mode A, the set AT of its subalgebras again forms a mode. For $\omega \in \Omega$ and subalgebras $A_1, \dots, A_{\omega\tau}$ of A,

$$A_1 \dots A_{\omega\tau}\,\omega \overset{\text{df}}{=} \{a_1 \dots a_{\omega\tau}\,\omega : a_i \in A_i\} \tag{3}$$

is also a subalgebra of A. Under these operations, AT forms a mode satisfying all the regular linear identities true in A. The operations defined by (3) are called *complex operations*, and the algebra AT is called the (*total*) *submode algebra* of A.

A natural question arises as to how the graph algebra $\mathrm{Gr}(A,B)$ and submode algebra $(A \times B)T$ are related. First note the inclusion $\mathrm{Gr}(A,B) \subseteq (A \times B)T$. One can prove more.

Proposition 3.3.

Let $f_1, \dots, f_{\omega\tau} \in \mathbf{K}(A,B)$ *and* $\omega \in \Omega$. *Then* $\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\overline{\omega} \leq \mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\,\omega$.

Proof. First we show that $\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\overline{\omega} \subseteq \mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\,\omega$. This follows directly by the fact that

$$\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\overline{\omega} = \{(a, af_1 \dots af_{\omega\tau}\,\omega) : a \in A\} = \big\{(a \dots a\,\omega, af_1 \dots af_{\omega\tau}\,\omega) : a \in A\big\}$$

– see (1) and (2) – while

$$\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\,\omega = \big\{(a_1, a_1f_1)\dots(a_{\omega\tau}, a_{\omega\tau}f_{\omega\tau})\,\omega : a_i \in A\big\} = \big\{(a_1 \dots a_{\omega\tau}\,\omega, a_1f_1 \dots a_{\omega\tau}f_{\omega\tau}\,\omega) : a_i \in A\big\}. \tag{4}$$

Since both these algebras are subalgebras of $A \times B$, then $\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\overline{\omega}$ is a subalgebra of $\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\,\omega$. □

Example 3.4.

In general, the inclusion in Proposition 3.3 cannot be replaced by equality. Consider the $\underline{2^{-1}}$-reducts of two affine subspaces of the affine (real) plane $\mathbb{R}^2$: the x-axis A and the y-axis B. Consider the $\underline{2^{-1}}$-homomorphisms

$$f_1 : A \to B;\quad 0 \mapsto 0,\quad 1 \mapsto 1, \qquad\qquad f_2 : A \to B;\quad 0 \mapsto 0,\quad 1 \mapsto 2.$$

Observe that $\mathrm{gr}(f_1)$ is the line $y = x$ and $\mathrm{gr}(f_2)$ is the line $y = 2x$. Now $\mathrm{gr}(f_1)\mathrm{gr}(f_2)\underline{2^{-1}} = \mathbb{R}^2$, while $\mathrm{gr}(f_1)\mathrm{gr}(f_2)\overline{\underline{2^{-1}}}$ is the line $y = (3/2)x$. Note also that $\mathrm{gr}(f_1)\mathrm{gr}(f_2)\underline{2^{-1}}$ cannot be the graph of any $\underline{2^{-1}}$-homomorphism $h : A \to B$.

More generally, the graphs of non-trivial homomorphisms from the affine $\mathbb{R}$-space A to the affine $\mathbb{R}$-space B are just lines on the plane $\mathbb{R}^2$ not parallel to the y-axis. Applying the operation $\overline{\underline{r}}$, where $0 < r < 1$, to any two such distinct lines L_1 and L_2 produces another line, while applying the complex operation $\underline{r}$ produces the whole plane.

Lemma 3.5.

Let $\omega \in \Omega$ *and* $f_1, \dots, f_{\omega\tau} \in \mathbf{K}(A,B)$. *Then*

$$\mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\overline{\omega} = \mathrm{gr}(f_1)\dots\mathrm{gr}(f_{\omega\tau})\,\omega \tag{5}$$

if and only if for any $a_1, \dots, a_{\omega\tau} \in A$,

$$a_1f_1 \dots a_{\omega\tau}f_{\omega\tau}\,\omega = (a_1f_1 \dots a_1f_{\omega\tau}\,\omega)\dots(a_{\omega\tau}f_1 \dots a_{\omega\tau}f_{\omega\tau}\,\omega)\,\omega. \tag{6}$$

Proof. First note that by Proposition 3.3, the equation (5) is equivalent to

$$\mathrm{gr}(f_1)\ldots\mathrm{gr}(f_{\omega\tau})\overline{\omega} \supseteq \mathrm{gr}(f_1)\ldots\mathrm{gr}(f_{\omega\tau})\omega. \tag{7}$$

A typical element of the left-hand side of equation (7) has the form $(a, af_1\ldots f_{\omega\tau}\omega)$. By (4), a typical element of $\mathrm{gr}(f_1)\ldots\mathrm{gr}(f_{\omega\tau})\omega$ can be written as $(a_1\ldots a_{\omega\tau}\omega, a_1f_1\ldots a_{\omega\tau}f_{\omega\tau}\omega)$. Hence it belongs to the left-hand side precisely when

$$a_1f_1\ldots a_{\omega\tau}f_{\omega\tau}\omega = (a_1\ldots a_{\omega\tau}\omega)(f_1\ldots f_{\omega\tau}\omega).$$

Now observe that by the definition of $f_1\ldots f_{\omega\tau}\omega$, the homomorphic property of $f_1,\ldots,f_{\omega\tau}$, and the entropic law for ω, the following hold:

$$\begin{aligned}(a_1\ldots a_{\omega\tau}\omega)(f_1\ldots f_{\omega\tau}\omega) &= (a_1\ldots a_{\omega\tau}\omega)f_1\ldots(a_1\ldots a_{\omega\tau}\omega)f_{\omega\tau}\omega\\ &= (a_1f_1\ldots a_{\omega\tau}f_1\omega)\ldots(a_1f_{\omega\tau}\ldots a_{\omega\tau}f_{\omega\tau}\omega)\omega = (a_1f_1\ldots a_1f_{\omega\tau}\omega)\ldots(a_{\omega\tau}f_1\ldots a_{\omega\tau}f_{\omega\tau}\omega)\omega.\end{aligned}$$

Thus (7) is indeed equivalent to (6). □

A τ-mode (A,Ω) (or a variety of τ-modes) is called *diagonal* if the *diagonal identity*

$$(x_{1,1}\ldots x_{1,\omega\tau}\omega)\ldots(x_{\omega\tau,1}\ldots x_{\omega\tau,\omega\tau}\omega)\omega = x_{1,1}x_{2,2}\ldots x_{\omega\tau,\omega\tau}\omega$$

is satisfied for each $\omega\in\Omega$. (See e.g. [7], [9, § 5.2].)

Proposition 3.6.

Let V *be a diagonal subvariety of a prevariety* K *of* τ*-modes. Then for any non-empty algebras* $A\in\mathsf{K}$ *and* $B\in\mathsf{V}$, $\mathsf{K}(A,B)\leq(A\times B)T$.

Indeed, by Corollary 3.2, $\mathsf{K}(A,B)\cong\mathrm{Gr}(A,B)$. By Lemma 3.5, $\mathrm{Gr}(A,B)$ is a subalgebra of $(A\times B)T$.

In the case where the τ-modes A and B from K coincide, the algebra $\mathsf{K}(A,A)$ is the algebra of endomorphisms of A. It also carries the structure of a monoid $(\mathsf{K}(A,A),\cdot,1_A)$ under the composition $\cdot$ of mappings, with the identity mapping 1_A. It is easy to check that for an operation $\omega\in\Omega$ and $f_1,\ldots,f_{\omega\tau},h\in\mathsf{K}(A,A)$, the following distributive laws hold:

$$f_1\ldots f_{\omega\tau}\omega\cdot h = (f_1\cdot h)\ldots(f_{\omega\tau}\cdot h)\omega, \qquad h\cdot f_1\ldots f_{\omega\tau}\omega = (h\cdot f_1)\ldots(h\cdot f_{\omega\tau})\omega. \tag{8}$$

More generally, if φ is also an operator in Ω, and $g_1,\ldots,g_{\varphi\tau}\in\mathsf{K}(A,A)$, then

$$(f_1\ldots f_{\omega\tau}\omega)\cdot(g_1\ldots g_{\varphi\tau}\varphi) = \big((f_1\cdot g_1)\ldots(f_{\omega\tau}\cdot g_1)\omega\big)\ldots\big((f_1\cdot g_{\varphi\tau})\ldots(f_{\omega\tau}\cdot g_{\varphi\tau})\omega\big)\varphi. \tag{9}$$

Indeed,

$$\begin{aligned}x\big((f_1\ldots f_{\omega\tau}\omega)\cdot(g_1\ldots g_{\varphi\tau}\varphi)\big) &= (xf_1\ldots xf_{\omega\tau}\omega)(g_1\ldots g_{\varphi\tau}\varphi) = (xf_1\ldots xf_{\omega\tau}\omega)g_1\ldots(xf_1\ldots xf_{\omega\tau}\omega)g_{\varphi\tau}\varphi\\ &= \big(x(f_1\cdot g_1)\ldots x(f_{\omega\tau}\cdot g_1)\omega\big)\ldots\big(x(f_1\cdot g_{\varphi\tau})\ldots x(f_{\omega\tau}\cdot g_{\varphi\tau})\omega\big)\varphi\\ &= x\big((f_1\cdot g_1)\ldots(f_{\omega\tau}\cdot g_1)\omega\big)\ldots\big((f_1\cdot g_{\varphi\tau})\ldots(f_{\omega\tau}\cdot g_{\varphi\tau})\omega\big)\varphi\end{aligned}$$

for x in A. One obtains the algebra $(\mathsf{K}(A,A),\Omega,\cdot,1)$, in which the monoid operation and the Ω-operations are connected by the equations (8)–(9).

Entropic Ω-algebras with a monoid structure satisfying the identities (8) were investigated by algebraists in the former Soviet Union, where they were known as Ω*-rings*. The first essential paper on the topic was probably [3]. See also [11], and the references cited there.

Assume additionally that the algebra A is a diagonal mode, and consider idempotent endomorphisms $f_1, \dots, f_{\omega\tau}$ of A, i.e. endomorphisms $f_i\colon A \to A$ such that $f_i \cdot f_i = f_i$. Then by (9), the diagonal identities imply

$$\begin{aligned} f_1 \dots f_{\omega\tau}\,\omega \cdot f_1 \dots f_{\omega\tau}\,\omega &= \big((f_1 \cdot f_1) \dots (f_{\omega\tau} \cdot f_1)\,\omega\big) \dots \big((f_1 \cdot f_{\omega\tau}) \dots (f_{\omega\tau} \cdot f_{\omega\tau})\,\omega\big)\,\omega \\ &= \big((f_1 \cdot f_1) \dots (f_{\omega\tau} \cdot f_{\omega}\tau)\,\omega\big) = f_1 \dots f_{\omega\tau}\,\omega. \end{aligned}$$

It follows that the set of idempotent endomorphisms of A is closed under the operations Ω. For entropic and diagonal algebras, this fact was first observed in [4]. In [1], it was shown that for each algebra A in a variety $\mathbf{V}$, the set of idempotent endomorphisms of A is closed under the basic operations Ω precisely if the variety $\mathbf{V}$ is entropic and diagonal. In particular, this implies that the idempotent endomorphisms of a diagonal mode constitute a subalgebra of $\mathbf{K}(A, A)$, and hence of the algebra $(A \times A)T$.

4. The algebra of mode homomorphisms

As before, let $\tau\colon \Omega \to \mathbb{N}$ be a plural type, and let $\mathbf{K}$ be a prevariety of modes of type τ. Let A and B be modes in $\mathbf{K}$. The algebra $\mathbf{K}(A, B)$ of homomorphisms may be extended by the addition of homomorphisms from subalgebras of A to subalgebras of B.

For subalgebras X_1 and X_2 of A, let $X_1 \cap X_2$ denote the intersection of X_1 and X_2 in the lattice $(AT, +, \cap)$ of subalgebras of A. Similarly, let $Y_1 + Y_2$ denote the join in the lattice $(BT, +, \cap)$ of subalgebras Y_1 and Y_2 of B. Multiple intersections and joins are denoted by $\bigcap$ and $\sum$ respectively. In a semilattice $(S, \circ)$, an n-ary word $x_1 \dots x_n t$ (for positive n) is interpreted as the product $x_1 \circ \dots \circ x_n$. Thus the semilattices $(AT, \cap)$ and $(BT, +)$ may be considered as Ω-algebras, defining $X_1 \dots X_{\omega\tau}\,\omega = X_1 \cap \dots \cap X_{\omega\tau}$ in the former, and $Y_1 \dots Y_{\omega\tau}\,\omega = Y_1 + \dots + Y_{\omega\tau}$ in the latter.

Definition 4.1.
For modes A and B in a prevariety $\mathbf{K}$, let $\mathbf{K}_{\mathrm{ext}}(A, B)$ denote the set of all homomorphisms $f\colon X \to Y$ from subalgebras X of A to subalgebras Y of B. For each operation ω in Ω, and for elements $f_i\colon X_i \to Y_i$ of $\mathbf{K}_{\mathrm{ext}}(A, B)$, define

$$f_1 \dots f_{\omega\tau}\,\omega\colon \bigcap_{i=1}^{\omega\tau} X_i \to \sum_{i=1}^{\omega\tau} Y_i; \qquad x \mapsto xf_1 \dots xf_{\omega\tau}\,\omega. \tag{10}$$

The Ω-algebra $\mathbf{K}_{\mathrm{ext}}(A, B)$ will be called the *homomorphism algebra from A to B* or just the *homomorphism algebra*.

Remark 4.2.
(a) The set $\mathbf{K}_{\mathrm{ext}}(A, B)$ is the disjoint union of the sets $\mathbf{K}(X, Y)$, where X runs over all the elements of AT, and Y runs over all the elements of BT:

$$\mathbf{K}_{\mathrm{ext}}(A, B) = \dot{\bigcup}_{\substack{X \in AT \\ Y \in BT}} \mathbf{K}(X, Y).$$

(b) If the subalgebra Y of B is empty, while the subalgebra X of A is not, then the set $\mathbf{K}(X, Y)$ is empty. If the subalgebra X of A is empty, then $\mathbf{K}(\varnothing, Y)$ is the singleton consisting of the “empty” function $\varnothing \hookrightarrow Y$. Following this observation, we consider $\mathbf{K}_{\mathrm{ext}}(A, B)$ as the disjoint union of all the $\mathbf{K}(X, Y)$, where Y is non-empty if X is non-empty.

For the homomorphisms $f_i\colon X_i \to Y_i$ in the context of Definition 4.1, define

$$g_i\colon \bigcap_{j=1}^{\omega\tau} X_j \to \sum_{j=1}^{\omega\tau} Y_j; \qquad x \mapsto xf_i.$$

Note that the g_i are simultaneous restrictions of f_i to $\bigcap_{j=1}^{\omega\tau} X_j$ and expansions of f_i to $\sum_{j=1}^{\omega\tau} Y_j$. Then the map $f_1 \dots f_{\omega\tau}\,\omega$ of (10) is the homomorphism

$$g_1 \dots g_{\omega\tau}\,\omega \in \mathbf{K}\left(\bigcap_{i=1}^{\omega\tau} X_i, \sum_{i=1}^{\omega\tau} Y_i\right).$$

Theorem 4.3.
Let $\mathbf{V}$ *be a variety of* τ*-modes. Then for any algebras* A *and* B *in* $\mathbf{V}$*, the homomorphism algebra* $\mathbf{V}_{\text{ext}}(A,B)$ *satisfies all the regular identities holding in* $\mathbf{V}$.

Proof. Let $x_1 \dots x_n u = x_1 \dots x_n v$ be one of the regular identities holding in $\mathbf{V}$. Consider elements $f_i\colon X_i \to Y_i$ of $\mathbf{V}_{\text{ext}}(A,B)$. Note that

$$X_1 \dots X_n u = X_1 \cap \dots \cap X_n = X_1 \dots X_n v$$

in the Ω-semilattice obtained from $(AT, \cap)$, and

$$Y_1 \dots Y_n u = Y_1 + \dots + Y_n = Y_1 \dots Y_n v$$

in the Ω-semilattice obtained from $(BT, +)$. Set $X = X_1 \dots X_n u$ and $Y = Y_1 \dots Y_n u$. Then both $f_1 \dots f_n u$ and $f_1 \dots f_n v$ have domain X and codomain Y. For each element x of X, the satisfaction of $u = v$ by Y yields

$$x(f_1 \dots f_n u) = xf_1 \dots xf_n u = xf_1 \dots xf_n v = x(f_1 \dots f_n v).$$

Thus $\mathbf{V}_{\text{ext}}(A,B)$ satisfies $u = v$. □

Corollary 4.4.
Let $\mathbf{V}$ *be a variety of* τ*-modes. Then for algebras* A *and* B *in* $\mathbf{V}$*, the homomorphism algebra* $\mathbf{V}_{\text{ext}}(A,B)$ *lies in the regularization* $\widetilde{\mathbf{V}}$ *of* $\mathbf{V}$.

If algebras A and B are in a variety $\mathbf{V}$ satisfying an irregular identity, then this identity does not necessarily carry over to the homomorphism algebra $\mathbf{V}_{\text{ext}}(A,B)$.

Example 4.5.
For $\tau\colon \{\cdot\} \to \{2\}$ (the type of binars or magmas), consider the terminal algebra $\top$ of $\underline{\underline{\tau}}$, a singleton. Let $A = B = \top$. Then $\top T = \{\emptyset, \top\}$, and

$$\underline{\underline{\tau}}_{\text{ext}}(\top,\top) = \{1_\emptyset, 1_\top, j : \emptyset \hookrightarrow \top\}.$$

Regard $\top$ as a left zero semigroup, satisfying the irregular identity $x \cdot y = x$. Let $\mathbf{LZ}$ be the variety of left zero semigroups. Then $1_\emptyset \cdot 1_\top : \emptyset \cap \top \to \emptyset + \top$ is $j\colon \emptyset \to \top$, distinct from $1_\emptyset$. Thus $\mathbf{LZ}_{\text{ext}}(\top,\top)$ is not a left zero semigroup.

Example 4.6.
Let A be a non-empty left zero semigroup. Then the defining identity $x \cdot y = x$ does not even need to hold in $\mathbf{LZ}_{\text{ext}}(A,A)$ for homomorphisms $f_i\colon X_i \to Y_i$ with non-empty domains. Indeed, $f_1 \cdot f_2\colon X_1 \cap X_2 \to Y_1 + Y_2$ equal to $f_1\colon X_1 \to Y_1$ would imply $X_1 \subseteq X_2$ and $Y_2 \subseteq Y_1$, which is not necessarily satisfied.

Corollary 4.7.
Let A *and* B *be algebras in an irregular variety* $\mathbf{V}$ *of* τ*-modes. Let* $X \in AT$ *and* $Y \in BT$*. Then the set* $\mathbf{V}(X,Y)$ *of homomorphisms from* X *to* Y *is a subalgebra of the homomorphism algebra* $\mathbf{V}_{\text{ext}}(A,B)$*, and lies in* $\mathbf{V}$.

5. The structure of homomorphism algebras

For τ-modes A and B, the semilattice

$$(S(A,B), \vee) = (AT, \cap) \times (BT, +)$$

is a join-semilattice ordered by $(X_0, Y_0) \leq (X_1, Y_1)$ iff $X_0 \geq X_1$ and $Y_0 \leq Y_1$. The operation $\vee$ is defined by

$$(X_0, Y_0) \vee (X_1, Y_1) = (X_0 \cap X_1, Y_0 + Y_1).$$

The semilattice $(S(A,B),\vee)$ may also be considered as an Ω-semilattice $(S(A,B),\Omega)$ with Ω-operations defined by

$$(X_1,Y_1)\dots(X_n,Y_n)\,\omega=(X_1\cap\dots\cap X_n,Y_1+\dots+Y_n)$$

for each (n-ary) $\omega\in\Omega$.

Proposition 5.1.

Let A and B lie in a prevariety $\mathbf{K}$ of τ-modes. Consider the boundary map

$$\partial\colon \mathbf{K}_{\mathrm{ext}}(A,B)\to S(A,B)=AT\times BT;\qquad (f\colon X\to Y)\mapsto (X,Y).$$

Then the following hold:

(a) *For X in AT and Y in BT, we have $\partial^{-1}\{(X,Y)\}=\mathbf{K}(X,Y)$.*

(b) *The map ∂ is a τ-homomorphism.*

(c) *The image*

$$\partial(\mathbf{K}_{\mathrm{ext}}(A,B))=\big\{(X,Y)\in AT\times BT : Y=\varnothing \text{ implies } X=\varnothing\big\}$$

of the boundary map forms a subsemilattice of $(S(A,B),\vee)$.

Proof. The first statement is immediate. The second statement follows since

$$f_1\dots f_n\,\omega\partial=\Big(\bigcap X_i,\sum Y_i\Big)=f_1\partial\dots f_n\partial\,\omega$$

for n-ary $\omega\in\Omega$ and $f_i\colon X_i\to Y_i$ in $\mathbf{K}_{\mathrm{ext}}(A,B)$. The third statement is also immediate, recalling that $AT\times BT$ is a τ-semilattice. □

Write $\widetilde{S}(A,B)$ for the semilattice $\partial(\mathbf{K}_{\mathrm{ext}}(A,B))$ of Proposition 5.1 (c). Recall that the *semilattice replica* of an algebra is its largest semilattice quotient [10, § IV.2.1].

Proposition 5.2.

Let $\mathbf{V}$ be an irregular variety of τ-modes, with A and B in $\mathbf{V}$.

(a) *For a subalgebra X of A and a subalgebra Y of B, the subalgebra $\mathbf{V}(X,Y)$ of the homomorphism algebra $\mathbf{V}_{\mathrm{ext}}(A,B)$ has no non-trivial semilattice quotients.*

(b) *The algebra $\big(\widetilde{S}(A,B),\vee\big)$ is the semilattice replica of $\mathbf{V}_{\mathrm{ext}}(A,B)$.*

Proof. By [9, Corollary 5.2], Corollary 4.7, and Proposition 5.1, the classes $\partial^{-1}\{(X,Y)\}$, as members of $\mathbf{V}$, are incapable of further decomposition. The statements (a) and (b) follow. □

Definition 5.3.

Consider the semilattice $\big(\widetilde{S}(A,B),\vee\big)$ as a poset category. The functor $\widetilde{F}_{A,B}\colon\big(\widetilde{S}(A,B),\vee\big)\to\underline{\underline{\tau}}$ taking a morphism $(X_0,Y_0)\to(X_1,Y_1)$ to the Ω-homomorphism

$$\underline{\underline{\tau}}(X_0,Y_0)\to\underline{\underline{\tau}}(X_1,Y_1);\qquad \big(f\colon X_0\to Y_0,\ x\mapsto xf\big)\mapsto\big(f{\restriction}_{X_1}\colon X_1\to Y_1,\ x\mapsto xf\big)\tag{11}$$

is known as the *restriction-expansion functor*.

Note that (11) both restricts a function $f\colon X_0\to Y_0$ to the subset X_1 of its domain X_0, while at the same time expanding the codomain from Y_0 to the superset Y_1.

Theorem 5.4.

Let A and B be members of a prevariety $\mathbf{K}$ of τ-modes. Then the homomorphism algebra $\mathbf{K}_{\mathrm{ext}}(A,B)$ is the Płonka sum of the restriction-expansion functor

$$\widetilde{F}_{A,B}\colon (\widetilde{S}(A,B),\vee)\to \underline{\underline{\tau}}$$

of Definition 5.3.

Proof. Taking into account Proposition 5.1, it is enough to show that the Ω-operations are defined on the sum according to the definition of a Płonka sum. We omit the standard proof using the fact that for homomorphisms $f_i\colon X_i\to Y_i$ as in Definition 4.1, the map $f_1\dots f_{\omega\tau}\,\omega$ is the homomorphism $g_1\dots g_{\omega\tau}\,\omega$. □

See [9, §§ 2.3, 4.2] for the definition and properties of directed colimits of algebras over a semilattice.

Corollary 5.5.

For algebras A and B in a prevariety $\mathbf{K}$ of τ-modes, the algebra $\mathbf{K}(A,B)$ is the directed colimit of the subalgebras $\mathbf{K}(X,Y)$ of $\mathbf{K}_{\mathrm{ext}}(A,B)$ over the semilattice $\widetilde{S}(A,B)$.

Corollary 5.6.

Let $\mathbf{V}$ be an irregular variety of τ-modes, with A and B in $\mathbf{V}$. Then the homomorphism algebra $\mathbf{V}_{\mathrm{ext}}(A,B)$ is a Płonka sum of $\mathbf{V}$-algebras over its semilattice replica $\big(\widetilde{S}(A,B),\vee\big)$.

Proof. Indeed, by Proposition 5.2 and Corollary 4.7, $\mathbf{V}_{\mathrm{ext}}(A,B)$ is the Płonka sum of the fibres $\mathbf{V}(X,Y)$, which are members of $\mathbf{V}$, over the semilattice replica $\big(\widetilde{S}(A,B),\vee\big)$ by the functor $\widetilde{F}_{A,B}$. □

Remark 5.7.

If $\mathbf{V}$ is an irregular variety of τ-modes, then $\mathbf{V}$ is defined by a set of regular identities and one irregular identity of the form $x\circ y = x$, where $x\circ y$ is a derived binary operation involving both x and y. (See e.g. [9, § 4.3].) Then in the Płonka sum of Corollary 5.6, for each ordered pair $(X,Y)\le(X',Y')$ in $\big(\widetilde{S}(A,B),\vee\big)$, the Płonka homomorphism $\mathbf{V}(X,Y)\to\mathbf{V}(X',Y')$ is given by

$$(f\colon X\to Y)\mapsto (f\colon X\to Y)\circ(b\colon X'\to Y'),$$

where

$$(f\colon X\to Y)\circ(b\colon X'\to Y')\colon\ X\cap X'\to Y+Y',\qquad x\mapsto xf\circ xb = xf,$$

and $b\colon X'\to Y'$ is any element of $\mathrm{V}(X',Y')$. In particular, note that $(X',Y')\in\widetilde{S}(A,B)$ implies $\mathbf{V}(X',Y')\neq\emptyset$.

Corollary 5.6 does not extend to regular varieties of modes.

Example 5.8.

Let $\mathbf{B}$ be the variety of barycentric algebras over the ring $\mathbb{R}$ of real numbers (see [9, Chapters 5, 7].) Let $\top$ be a singleton algebra in the variety $\mathbf{B}$, a terminal object of the category $\mathbf{B}$. Consider the barycentric algebra $(I,\underline{I}^{\mathrm{o}})$, where $I=[0,1]$ is the closed unit interval of $\mathbb{R}$. Then the algebra $\mathbf{B}(\top,I)$ of homomorphisms from $\top$ to $(I,\underline{I}^{\mathrm{o}})$ is isomorphic to $(I,\underline{I}^{\mathrm{o}})$, which has a 3-element semilattice replica.

The algebra $\mathbf{B}_{\mathrm{ext}}(\top,I)$ is more complicated. Though it is still a Płonka sum of barycentric algebras (by Theorem 5.4), the algebra $\partial(\mathbf{B}_{\mathrm{ext}}(\top,I))$ is not the semilattice replica of $\mathbf{B}_{\mathrm{ext}}(\top,I)$, and the summands of the Płonka sum are not necessarily indecomposable. Indeed, if Y is a non-empty subinterval of I, then the fibre $\mathbf{B}(\top,Y)$ is isomorphic to $(Y,\underline{I}^{\mathrm{o}})$. If Y is not an open interval, then the fibre has a non-trivial semilattice replica.

Example 5.8 shows that the homomorphism algebra $\mathbf{K}_{\mathrm{ext}}(A,B)$ from a mode A to a mode B in a prevariety $\mathbf{K}$ may often be a very large object. Hence it may be useful to study various smaller subobjects.

Consider the semilattice $\big(\widetilde{S}(A,B),\vee\big)$. It has $(\emptyset, B)$ as its largest element, and $(\emptyset,\emptyset)$ or $(A,\{b\})$, $b\in B$, as minimal elements. The semilattice $S(A,B)$ has additional elements of the form $(X,\emptyset)$, for non-empty subalgebras X of A, and a smallest element $(A,\emptyset)$.

Suppose that B is non-empty. For a fixed non-empty subalgebra Y of B, the elements (X,Y), where X runs over all subalgebras of A, form a subsemilattice of $\big(\widetilde{S}(A,B),\vee\big)$ isomorphic to the semilattice $(AT,\cap)$, with a largest element $(\emptyset,Y)$ and smallest element (A,Y). Now fix a subalgebra X of A. Then the elements (X,Y), where Y runs over all non-empty subalgebras of B, form a subsemilattice of $\big(\widetilde{S}(A,B),\vee\big)$ isomorphic with a subsemilattice of $(BT,+)$ with the largest element (X,B), and $(X,\{b\})$, $b\in B$, as minimal elements.

Proposition 5.9.

Suppose that A and B are members of a prevariety $\mathbf{K}$ of τ-modes.

(a) *The set*

$$\bigcup_{Y\in BT}\mathbf{K}(\emptyset,Y)$$

forms a subalgebra of $\mathbf{K}_{\mathrm{ext}}(A,B)$ isomorphic to the Ω-semilattice equivalent to $(BT,+)$.

(b) *Let $X\in AT$ and $y\in B$. Then $\mathbf{K}(X,\{y\})=\big\{X\to\{y\}\big\}$. Moreover, the set*

$$\bigcup_{X'\in AT}\mathbf{K}(X',\{y\})$$

forms a subalgebra of $\mathbf{K}_{\mathrm{ext}}(A,B)$ isomorphic to the Ω-semilattice equivalent to $(AT,\cap)$.

Remark 5.10.

Suppose that $B\neq\emptyset$. If $x\in A$ and $\emptyset\neq Y\in BT$, then $\mathbf{K}(\{x\},Y)$ is isomorphic to Y. By [9, Theorem 7.5.3], the algebra Y has the semilattice of principal walls (or equivalently, the semilattice of principal sinks) as its semilattice replica. It follows that when the algebra Y has non-trivial principal walls (or non-trivial principal sinks), it is a non-trivial semilattice sum of algebraically open subalgebras.

Similarly as in Example 5.8, for a fixed $x\in A$, the union of the fibres $\mathbf{K}(\{x\},Y)$ forms a Płonka sum isomorphic to the Płonka sum of the subalgebras Y of B over the (join) semilattice of non-empty subalgebras of B.

Corollary 5.11.

Suppose that A and B are members of a prevariety $\mathbf{K}$ of τ-modes. Then the homomorphism algebra $\mathbf{K}_{\mathrm{ext}}(A,B)$ contains subalgebras isomorphic to non-empty subalgebras of B, and to Ω-semilattices equivalent to $(AT,\cap)$ or $(BT,+)$.

If the algebras A and B coincide, the two sets AT and BT coincide as well, and instead of two semilattices $(AT,\cap)$ and $(BT,+)$, one obtains the lattice $(AT,+,\cap)$.

6. Surjection algebras

Let $\mathbf{K}_{\mathrm{sur}}(A,B)$ denote the subset of $\mathbf{K}_{\mathrm{ext}}(A,B)$ consisting of all surjective homomorphisms $f\colon X\to Y$ from $X\in AT$ onto $Y=Xf\in BT$. By the definition of Ω in $\mathbf{K}_{\mathrm{ext}}(A,B)$, for each ($n$-ary) $\omega\in\Omega$ and $f_i\colon X_i\to Y_i=X_if$,

$$f_1\ldots f_n\,\omega\colon \bigcap_{i=1}^{n}X_i\to\sum_{i=1}^{n}Y_i;\qquad x\mapsto xf_1\ldots xf_n\,\omega. \tag{12}$$

Instead of (12), consider the corestriction

$$f_1\ldots f_n\,\omega_{\mathrm{c}}\colon \bigcap_{i=1}^{n}X_i\to\left(\bigcap_{i=1}^{n}X_i\right)(f_1\ldots f_n\,\omega);\qquad x\mapsto xf_1\ldots xf_n\,\omega\,.$$

Note the following obvious lemma.

Lemma 6.1.
For each (n-ary) $\omega \in \Omega$ and $f_i\colon X_i \to Y_i = X_i f_i$ in $\mathbf{K}_{\mathrm{sur}}(A,B)$,

$$\left(\bigcap_{i=1}^{n} X_i\right)(f_1 \dots f_n \omega_c) \leq \sum_{i=1}^{n} Y_i.$$

The following example shows that the inequality in Lemma 6.1 cannot be replaced by equality in general.

Example 6.2.
Consider $\mathbb{R}$ as a barycentric algebra, and the two closed subintervals $[0,2]$ and $[1,3]$ of $\mathbb{R}$ as subalgebras X_1 and X_2. Consider the (uniquely defined) barycentric algebra isomorphisms

$$f_1\colon X_1 \to Y_1 = [4,5], \qquad f_2\colon X_2 \to Y_2 = [6,7].$$

Then $Y_1 + Y_2 = [4,7]$, while $(X_1 \cap X_2)(f_1 f_2 \underline{1/2}) = [21/4, 23/4]$.

Let $\Omega_c = \{\omega_c : \omega \in \Omega\}$. Under the operations Ω_c, the set $\mathbf{K}_{\mathrm{sur}}(A,B)$ becomes a well defined τ-algebra, the *surjection algebra from A to B*, or more briefly the *surjection algebra* if the context is clear.

Now instead of surjections $f\colon X \to Y$, one may consider their graphs $\mathrm{gr}(f) = \{(x, xf) : x \in X\}$, subalgebras of the direct product $X \times Y$. There is a one-to-one correspondence between the surjections in $\mathbf{K}_{\mathrm{ext}}(A,B)$ and their graphs. Note that there is no such correspondence between general homomorphisms of $\mathbf{K}_{\mathrm{ext}}(A,B)$ and their graphs.

Let $\mathrm{Gr}_{\mathrm{sur}}(A,B)$ be the set consisting of the graphs of the surjections in $\mathbf{K}_{\mathrm{sur}}(A,B)$. Note the following corollary of Lemma 3.1.

Lemma 6.3.
Let C be a subalgebra of $A \times B$. Then $C \in \mathrm{Gr}_{\mathrm{sur}}(A,B)$ if and only if for all $(a_1,b_1),(a_2,b_2) \in C$, $a_1 = a_2$ implies $b_1 = b_2$.

For each (n-ary) $\omega \in \Omega$ and any $f_i\colon X_i \to Y_i = X_i f_i$ in $\mathbf{K}_{\mathrm{sur}}(A,B)$, define an operation $\overline{\omega}_c$ on $\mathrm{Gr}_{\mathrm{sur}}(A,B)$, as follows:

$$\mathrm{gr}(f_1)\dots\mathrm{gr}(f_n)\overline{\omega}_c = \mathrm{gr}(f_1 \dots f_n \omega_c).$$

Similarly as in the case of Lemma 3.1, one has

$$\mathrm{gr}(f_1 \dots f_n \omega_c) = \{(x, xf_1 \dots f_n \omega) : x \in X\} = \big\{(x, xf_1 \dots xf_n \omega) : x \in X\big\},$$

where $X = \bigcap_{i=1}^{n} X_i$. Let $\overline{\Omega}_c = \{\overline{\omega}_c : \omega \in \Omega\}$. Then under the operations of $\overline{\Omega}_c$, the set $\mathrm{Gr}_{\mathrm{sur}}(A,B)$ is an algebra of the same type as $\mathbf{K}_{\mathrm{sur}}(A,B)$, with the following obvious property.

Corollary 6.4.
The algebra $\mathbf{K}_{\mathrm{sur}}(A,B)$ of surjective homomorphisms from A to B is isomorphic to the algebra $\mathrm{Gr}_{\mathrm{sur}}(A,B)$ of their graphs.

Let us return to the homomorphism algebra $\mathbf{K}_{\mathrm{ext}}(A,B)$. Define the following relation δ on $\mathbf{K}_{\mathrm{ext}}(A,B)$:

$$(f_1, f_2) \in \delta \qquad \Longleftrightarrow \qquad \mathrm{gr}(f_1) = \mathrm{gr}(f_2).$$

Lemma 6.5.
The relation δ is a congruence relation of the homomorphism algebra $\mathbf{K}_{\mathrm{ext}}(A,B)$.

Proof. It is clear that δ is an equivalence relation. To show that it is a congruence relation, let $\omega \in \Omega$ be n-ary, and let $(f_i\colon X_i \to Y_i)$ and $(h_i\colon X_i' \to Y_i')$, for $i = 1, \dots, n$, be in $\mathsf{K}_{\text{ext}}(A, B)$. Assume that for $i = 1, \dots, n$, we have $(f_i, h_i) \in \delta$. This means that $X_i = X_i'$ and $xf_i = xh_i$ for each $x \in X_i$. Recall that

$$f_1 \dots f_n \omega\colon \bigcap_{i=1}^{n} X_i \to \sum_{i=1}^{n} Y_i; \qquad x \mapsto xf_1 \dots xf_n \omega.$$

Similarly,

$$h_1 \dots h_n \omega\colon \bigcap_{i=1}^{n} X_i \to \sum_{i=1}^{n} Y_i'; \qquad x \mapsto xh_1 \dots xh_n \omega.$$

By (2), it follows that

$$\begin{aligned} \operatorname{gr}(f_1 \dots f_n \omega) &= \left\{(x, xf_1 \dots f_n \omega) : x \in \bigcap_{i=1}^{n} X_i\right\} = \left\{(x, xf_1 \dots xf_n \omega) : x \in \bigcap_{i=1}^{n} X_i\right\} \\ &= \left\{(x, xh_1 \dots xh_n \omega) : x \in \bigcap_{i=1}^{n} X_i\right\} = \left\{(x, xh_1 \dots h_n \omega) : x \in \bigcap_{i=1}^{n} X_i\right\} = \operatorname{gr}(h_1 \dots h_n \omega). \end{aligned}$$ □

Theorem 6.6.
Let A and B be members of a prevariety K of τ-modes. Then the surjection algebra $\mathsf{K}_{\text{sur}}(A, B)$ is isomorphic to a quotient of the homomorphism algebra $\mathsf{K}_{\text{ext}}(A, B)$. More precisely, $\mathsf{K}_{\text{sur}}(A, B) = \mathsf{K}_{\text{ext}}(A, B)^{\delta}$.

Proof. By Lemma 6.5, it is sufficient to note that each δ-class contains precisely one element which is surjective. □

The structure of $\mathsf{K}_{\text{sur}}(A, B)$ may be described similarly as the structure of $\mathsf{K}_{\text{ext}}(A, B)$. In particular, if K is an irregular variety V of τ-modes, then $\mathsf{V}_{\text{sur}}(A, B)$, as a quotient of $\mathsf{V}_{\text{ext}}(A, B)$, also belongs to the regularization $\widetilde{\mathsf{V}}$ of V. Hence, it is also a Płonka sum of V-algebras. To describe the structure of $\mathsf{K}_{\text{sur}}(A, B)$ in more detail, consider again the functor $\widetilde{F}_{A,B}$. Note that $\widetilde{S}(A, B)$ contains a subsemilattice, consisting of all pairs (X, B) for $X \in AT$, isomorphic to the semilattice $(AT, \cap)$. Let us identify these two semilattices. Then the functor $\widetilde{F}_{A,B}$ restricts to the functor

$$\overline{F}_{A,B}\colon (AT, \vee) = (AT, \cap) \to \underline{\tau}$$

providing the Płonka sum of the summands $\mathsf{K}(X, B)$ over the semilattice $(AT, \cap)$. Evidently, this Płonka sum $\coprod_{X \leq A} \mathsf{K}(X, B)$ forms a subalgebra of $\mathsf{K}_{\text{ext}}(A, B)$.

Lemma 6.7.
The surjection algebra $\mathsf{K}_{\text{sur}}(A, B)$ and the algebra $\coprod_{X \leq A} \mathsf{K}(X, B)$ are isomorphic. Moreover, for each $X \in AT$, the corresponding Płonka fibre $\mathsf{K}_{\text{sur}}(X)$ of $\mathsf{K}_{\text{sur}}(A, B)$ consists of all the surjective homomorphisms in $\mathsf{K}(X, B)$.

Proof. First note that for $f\colon X \to Y$ in $\mathsf{K}_{\text{ext}}(A, B)$, the δ-class f^{δ} of f contains precisely one surjection $f^{s}\colon X \to Xf$, $x \to xf$, belonging to $\mathsf{K}(X, Xf)$, and precisely one element of $\mathsf{K}(X, B)$, namely, the homomorphism $f^{B}\colon X \to B$, $x \mapsto xf$. In fact,

$$f^{\delta} = \{h : X \to Y \mid Xh \leq Y,\, xh = xf\}.$$

Define the mapping

$$\varphi\colon \coprod_{X \leq A} \mathsf{K}(X, B) \to \mathsf{K}_{\text{sur}}(A, B); \qquad (f\colon X \to B) \mapsto (f^{s}\colon X \to Xf).$$

The mapping φ is bijective. Consider n-ary $\omega \in \Omega$, $X_i \in AT$ and $f_i\colon X_i \to Y_i$, $1 \leq i \leq n$, in $\coprod_{X \leq A} \mathsf{K}(X, B)$. Then, by the definitions of the operations ω in $\mathsf{K}_{\text{ext}}(A, B)$ and ω_{c} in $\mathsf{K}_{\text{sur}}(A, B)$, one has

$$(f_1 \dots f_n \omega)\varphi = (f_1 \dots f_n \omega)^{s} = f_1^{s} \dots f_n^{s} \omega_{c} = f_1\varphi \dots f_n\varphi \omega_{c}.$$

Hence φ is an isomorphism. The last statement of the lemma is immediate. □

Corollary 6.8.

Let A and B be algebras in an irregular variety $\mathbf{V}$ of τ-modes. Then the surjection algebra $\mathbf{V}_{\text{sur}}(A, B)$ is isomorphic to the Płonka sum of $\mathbf{V}$-algebras $\mathbf{V}(X, B)$ for $X \in AT$ over the semilattice $(AT, \cap)$.

7. Conclusion and future work

For modes A, B from a given prevariety $\mathbf{V}$, the respective algebras $\mathbf{V}_{\text{ext}}(A, B)$ of homomorphisms and $\mathbf{V}_{\text{sur}}(A, B)$ of surjective homomorphisms, from subalgebras of A to subalgebras of B, have been constructed, with the latter as a quotient of the former (Theorem 6.6). Their Płonka sum structure has been determined (cf. Theorem 5.4), most notably for the case where $\mathbf{V}$ is an irregular variety (Corollaries 5.6 and 6.8). Algebras, semilattices, and lattices that are representable as subalgebras of $\mathbf{V}_{\text{ext}}(A, B)$ have been identified (Corollary 5.11).

For a variety $\mathbf{V}$ of modes of given type $\tau\colon \Omega \to \mathbb{N}$, future work should address the following problems.

Problem 7.1.

Characterize those $\mathbf{V}$-algebras which may be realized as homomorphism algebras $\mathbf{V}(A, B)$ for members A, B of $\mathbf{V}$.

Problem 7.2.

Which (Ω-)semilattices may be realized as semilattice replicas of extended homomorphism algebras $\mathbf{V}_{\text{ext}}(A, B)$ for members A, B of $\mathbf{V}$?

Problem 7.3.

Characterize those algebras from the regularization of $\mathbf{V}$ which may be realized in the form of extended homomorphism algebras $\mathbf{V}_{\text{ext}}(A, B)$ for members A, B of $\mathbf{V}$.

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