ABSTRACT BARYCENTRIC ALGEBRAS

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Abstract. This paper presents a new approach to the study of (real) barycentric algebras, in particular convex subsets of real affine spaces. Barycentric algebras are cast in the setting of two-sorted algebras. The real unit interval indexing the set of basic operations of a barycentric algebra is replaced by an LΠ-algebra, the algebra of Łukasiewicz Product Logic. This allows one to define barycentric algebras abstractly, independently of the choice of the unit real interval. It reveals an unexpected connection between barycentric algebras and (fuzzy) logic. The new class of abstract barycentric algebras incorporates barycentric algebras over any linearly ordered field, the B-sets of G. M. Bergman, and E. G. Manes' if-then-else algebras over Boolean algebras.

1. Introduction

Real convex sets can be presented algebraically as sets with binary operations given by weighted means, the weights taken from the open unit interval in the real numbers. The class of convex sets is a quasivariety (defined by certain implications), and generates the variety (defined by identities) of so-called barycentric algebras. Both these classes have a well developed theory, a special case of the general theory of modes (idempotent and entropic algebras). See e.g. [21], [22], [25], [26], [27], [11], [28], [8].

However, in the specification of convex sets and barycentric algebras, the open unit interval itself has not hitherto been axiomatized. The current paper is intended to address this issue. We extend the open

1991 *Mathematics Subject Classification.* 52A01, 06A12, 08A68, 08A99, 03B52, 06D35, 20M99, 20N99.

Key words and phrases. affine space, barycentric algebra, mode, cancellative mode, LΠ-algebra, Łukasiewicz Product Logic, heterogeneous algebra, B-set, if-then-else algebra, rectangular algebra.

The paper was written within the framework of INTAS project No 03-51-4110. Part of the work on this paper was completed during a visit of the first author to the Department of Mathematics, Iowa State University, Ames, Iowa, during Summer 2006.

unit interval of operations to the closed interval, and consider barycentric algebras as two-sorted algebras. One sort corresponds to the set of elements of a traditional barycentric algebra, while the second sort corresponds to a certain algebra of fuzzy logic, the so-called ŁΠ-algebra ([5], [6], [19], [20]). The ŁΠ-algebras yield an appropriate axiomatization of the closed unit interval, and provide a broad extension of the class of traditional barycentric algebras. The new structures encompass two-sorted counterparts of barycentric algebras over any linearly ordered field, and include other algebras related to Boolean affine spaces, the B-sets of Bergman and Stokes ([1], [29], [30]), and "if-then-else" algebras ([18], [17]).

Following preliminary sections covering traditional barycentric algebras over the reals (Section 2) and ŁΠ-algebras (Section 3), abstract barycentric algebras are introduced as two-sorted algebras in Section 4. The two sorts are connected by a ternary operation assigning an element of the first sort to a triple consisting of two elements of the first sort and one of the second. The ternary operation satisfies axioms (identities) that correspond to the axioms of traditional real barycentric algebras. In this way the class $\mathcal{AB}$ of abstract barycentric algebras becomes a variety of two-sorted algebras. As this class contains algebras whose first sort is empty but whose second sort is nonempty, it follows by results of Goguen and Meseguer [7] that the class cannot be (categorically) equivalent to a variety of one-sorted algebras. Nevertheless, each barycentric algebra over a subfield of the real field has a two-sorted counterpart in $\mathcal{AB}$ (Section 5). As the axiomatization of abstract barycentric algebras requires the closed unit interval rather than the open one, in Section 4 we present the corresponding axiomatization of traditional barycentric algebras.

The closed unit interval of any subfield R of the field of real numbers can easily be described as an ŁΠ-algebra using truncations of the four basic operations of addition, subtraction, multiplication and division on R. There are other approaches to the axiomatization of the (closed) unit interval of reals. In [13], Jamison-Waldner introduced the notion of an *algebraic interval* (see also [25] and [27]), modelled on the structure of the real unit interval as a pair of commutative monoids connected by the involutary isomorphism $p \mapsto 1-p$. Although this notion proved useful for investigations concerning free modes on two generators (see [25] and [27]), and poses a number of interesting open problems, it does not capture all the essential structure of the unit interval. More specific algebraic intervals were considered by Kearnes [14], who investigated the structure of idempotent simple algebras (and among them simple modes). To describe simple modes, he needed non-trivial algebraic intervals (more than 2 elements) as structures closed under weighted means (and hence being commutative monoids), generating the group of the field. But again these intervals do not capture the structure we need.

As the two-element Boolean algebra is always a subalgebra of an interval LΠ-algebra defined on the unit interval of a linearly ordered field, our approach yields a new class of abstract barycentric algebras, namely *Boolean barycentric algebras*, whose second sort is a Boolean algebra. As one-sorted algebras, they are just binary reducts of Boolean affine spaces, equivalent to the B-sets of Bergman and Stokes (also including certain "if-then-else algebras"), and also equivalent to so-called *rectangular modes*, idempotent and entropic algebras with any number of binary rectangular band operations. These results are presented in Sections 6 and 7.

We usually summarize the necessary basic definitions and facts. For more information on modes, we refer the reader to [25] and [27]. Algebras of fuzzy logics are treated in [9], while general background on many-sorted algebras is available in [2], [10], [16] and [31].

As the algebras we investigate come from different "worlds," we use two different types of notation. When speaking about modes (in particular traditional barycentric algebras and affine spaces), we follow the conventions of [25] and [27], using postfix notation for operations and words. In dealing with algebras of logics, we use prefix notation. The translation from one language to another should be clear from the context. The expressions "term operation" and "derived operation" are used synonymously.

2. Convex sets and barycentric algebras

Let R be a subfield of the field $\mathbb{R}$ of real numbers. Let $I = [0, 1] \subset R$ be the closed unit interval of R. Let $I^o = (0, 1)$ be the corresponding open unit interval.

Barycentric algebras over a subfield R of the field $\mathbb{R}$ may be defined as algebras $(A, \underline{I}^o)$ of type $\underline{I}^o \times \{2\}$, equipped with a binary operation

$$\underline{p} : A \times A \to A; (x, y) \mapsto xy\,\underline{p}$$

for each p in I^o, satisfying the identities

$$xx\,\underline{p} = x \tag{2.1}$$

of *idempotence* for each p in I^o, the identities

$$xy\,\underline{p} = yx\,\underline{1-p} \tag{2.2}$$

of *skew-commutativity* for each p in I^o, and the identities

$$xy\,\underline{p}\;z\,\underline{q} = x\;yz\,\underline{q/(p\circ q)}\;\underline{p\circ q} \tag{2.3}$$

of *skew-associativity* for each p, q in I^o. Here $p\circ q = p+q-pq$. Setting $p' := 1-p$, one obtains $p\circ q = (p'q')'$.

For a given subfield R, the barycentric algebras over R form a variety. This variety $\mathcal{B}$ is a variety of modes, idempotent and entropic algebras, as defined in [25], [26] and [27]. In particular, such a variety satisfies the *entropic* identities

$$xy\underline{p}\;zt\underline{p}\;\underline{q} = xz\underline{q}\;yt\underline{q}\;\underline{p} \tag{2.4}$$

for all $p, q \in I^o$. In other words, any two of the operations $\underline{p}$ and $\underline{q}$ "commute." The main models are provided by convex subsets of affine spaces over R, and $\underline{I}^o$-semilattices (algebras equivalent to semilattices, where the operations in $\underline{I}^o$ are associative, and any two of them are equal). Convex subsets of affine spaces over R are described as $\underline{I}^o$-subreducts of affine spaces $(A, \underline{R})$, see [27] and [25]. For a convex set C, the operations $\underline{p}$ are defined by

$$xy\underline{p} = x(1-p) + yp.$$

Among barycentric algebras, convex sets are characterized by cancellativity, i.e. they form the subquasivariety $\mathcal{C}$ of $\mathcal{B}$ defined by the cancellation laws

$$(xy\underline{p} = xz\underline{p}) \longrightarrow (y = z) \tag{2.5}$$

that hold for all $p \in I^o$. The quasivariety of convex sets and the variety of $\underline{I}^\circ$-semilattices are the only minimal quasivarieties of barycentric algebras. The variety of barycentric algebras may equivalently be defined as the class of homomorphic images of convex sets in $\mathcal{C}$.

Note that the latter definition of barycentric algebras may easily be extended to the case of subrings R of the ring $\mathbb{R}$. For example, if $R = \mathbb{D}$, the ring of dyadic rational numbers $n2^m$ with $m, n \in \mathbb{Z}$, one obtains the variety of "dyadic barycentric algebras" that in fact is equivalent to the variety of commutative binary (or groupoid) modes. See e.g. [27]. However, not all properties of barycentric algebras over a field extend to barycentric algebras over a ring that is not a field.

The structure of barycentric algebras over a subfield R of $\mathbb{R}$ is fully described in [26] and [27, Chapter 7]. In particular, all subdirectly irreducible convex sets are given by the bounded intervals $([0,1], \underline{I}^o)$, $((0,1], \underline{I}^o)$, $([0,1), \underline{I}^o)$, $((0,1), \underline{I}^o)$, and the closed unbounded intervals $([0,\infty), \underline{I}^o)$, $((-\infty,0], \underline{I}^o)$ and $(R, \underline{I}^o)$. The remaining subdirectly irreducibles are the Płonka sums of these algebras with the singleton

$\underline{I}^o$-algebra, as well as the 2-element $\underline{I}^o$-semilattice. A general barycentric algebra $(A, \underline{I}^o)$ embeds into a Płonka sum of convex sets over its $\underline{I}^o$-semilattice replica, the greatest $\underline{I}^o$-semilattice homomorphic image of $(A, \underline{I}^o)$ [27, Theorem 7.5.10].

3. LΠ-algebras

LΠ-algebras were introduced by F. Montagna (see [19] and [6]) as an algebraization of the so-called LΠ-*logic*. This logic results from the combination of Łukasiewicz and product logic, two of the main fuzzy logics (see [9]). Amongst several possible axiomatizations of the variety of LΠ-algebras, we here present one taken from [20].

First, define a *hoop* to be an algebra $(H, \star, \rightarrow, 1)$ such that $(H, \star, 1)$ is a commutative monoid, and the *implication* $\rightarrow$ satisfies the following identities:

(H1) $x \rightarrow x = 1$;
(H2) $x \rightarrow (y \rightarrow z) = (x \star y) \rightarrow z$;
(H3) $x \star (x \rightarrow y) = y \star (y \rightarrow x)$.

See e.g. [3]. A *Wajsberg algebra* is a hoop with a constant 0 satisfying the identities:

(W1) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;
(W2) $0 \rightarrow x = 1$.

Algebras equivalent to Wajsberg algebras have been studied under various names. (See [4] for a more complete list and historical summary.) For example, *bounded commutative* BCK-*algebras*, introduced by K. Isēki and S. Tanaka [12], and subsequently investigated by other authors, are dual to Wajsberg algebras, and can be defined using only the implication and one constant by a simpler set of axioms. Another equivalent version of Wajsberg algebras goes under the name of *MV-algebras*: the algebras of infinitely-valued Łukasiewicz logic. Let $(A, \star, \rightarrow, 0, 1)$ be a Wajsberg algebra. On the set A, define

$$\neg x := x \rightarrow 0$$

and

$$x \oplus y := \neg x \rightarrow y.$$

Then $(A, \oplus, \neg, 0, 1)$ is an MV-algebra. Abstractly, an *MV-algebra* is defined as an algebra $(A, \oplus, \neg, 0, 1)$ such that $(A, \oplus, 0)$ is a commutative monoid, and satisfying the following identities:

(MV1) $\neg\neg x = x$;
(MV2) $x \oplus \neg 0 = \neg 0$;
(MV3) $x \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$.

Defining

$$x \star y := \neg(\neg x \oplus \neg y)$$

and

$$x \to y := \neg(x \star \neg y) = \neg x \oplus y$$

in an MV-algebra, one obtains a Wajsberg algebra. Note that each Wajsberg algebra is necessarily a bounded distributive lattice, with the lattice operations given by

$$x \wedge y := x \star (x \to y)$$

and

$$x \vee y := (x \to y) \to y.$$

Now an ŁΠ-*algebra* is defined as a double hoop, an algebra

$$(A, \star, \to, \cdot_\pi, \to_\pi, 0, 1)\,,$$

with two hoop reducts, where $(A, \star, \to, 0, 1)$ is a Wajsberg algebra (or equivalently $(A, \oplus, \neg, 0, 1)$ is an MV-algebra), and $(A, \cdot_\pi, \to_\pi, 0, 1)$ is a hoop with 0, satisfying (W2) and the following identity:

(ŁΠ) $x \cdot_\pi (x \to_\pi y) = x \wedge y$.

See e.g. [20], [5], and also [6]. Each ŁΠ-algebra is necessarily a bounded distributive lattice. Moreover both implications are residuations. In particular, the product implication $x \to_\pi y$ is the largest z such that $x \cdot_\pi z \leq y$, so we have the adjunction

$$z \leq x \to_\pi y \text{ iff } x \cdot_\pi z \leq y.$$

Similarly the Łukasiewicz implication $x \to y$ is the largest z such that $x \star z \leq y$, and

$$z \leq x \to y \text{ iff } x \star z \leq y.$$

An example is provided by the two element Boolean algebra **2**, where $\vee = \oplus$ and $\wedge = \star = \cdot_\pi$; moreover $x \to y = x \to_\pi y$, and $\neg$ is the usual Boolean complement.

Another basic example is provided by the closed unit interval I of a subfield R of $\mathbb{R}$. The ŁΠ-algebra operations are defined on I as follows:

$$\begin{aligned} x \star y &:= 0 \vee (x + y - 1)\,; \\ x \to y &:= 1 \wedge (1 - x + y)\,; \\ \neg x &:= 1 - x\,; \\ x \oplus y &:= 1 \wedge (x + y)\,; \\ x \cdot_\pi y &:= x \cdot y\,; \\ x \to_\pi y &:= \ \mathbf{if}\ x \leq y\ \mathbf{then}\ 1\ \mathbf{else}\ y/x\,. \end{aligned}$$

Define

$$x \ominus y := 1 - (1 \wedge (1 - x + y))$$

and

$$x \oslash y := y \to_\pi x \,.$$

Then as noticed by Montagna [20], the operations $\oplus, \ominus, \cdot$ and $\oslash$ are truncations to I of the four basic operations of addition, subtraction, multiplication and division on R.

In similar fashion, one may define LΠ-algebra operations on the unit interval $U := \{x \in F \mid 0 \leq x \leq 1\}$ of each linearly ordered field F. Such an LΠ-algebra is called an *interval* LΠ-*algebra*.

By [19, Corollary 5.9], the following conditions on an LΠ-algebra A are equivalent:

(a) A is subdirectly irreducible;
(b) A is linearly ordered;
(c) A is simple.

Hence every LΠ-algebra is isomorphic to a subdirect product of linearly ordered LΠ-algebras. By [6, Theorem 7], every linearly ordered LΠ-algebra is isomorphic either to the interval algebra of a linearly ordered field, or to the algebra **2**. The variety of all LΠ-algebras is generated by the interval LΠ-algebra defined on $I \subset \mathbb{R}$ ([19, Theorem 5.7]).

Note that for a given subfield R of $\mathbb{R}$ and $I^o \subset R$, the convex barycentric algebra $(I, \underline{I}^o)$ can be described as a reduct of the LΠ-algebra I. The operations $\underline{p}$ of the reduct are defined by

$$xy\underline{p} = (x \cdot_\pi \neg p) \oplus (y \cdot_\pi p).$$

4. Abstract barycentric algebras

In the definition and axioms of barycentric algebras, all the necessary algebraic operations from I^o are actually operations of LΠ-algebras. However, to make use of this structure, we need the closed unit interval I rather than the open one, so we have to extend the set $\underline{I}^o$ of operations to the set $\underline{I}$. This yields algebras $(A, \underline{I})$, where the set $\underline{I}^o$ of operations is extended by adding two projections $xy\underline{0} := x$ and $xy\underline{1} := y$. (Cf. [28], where the author investigates barycentic algebras under the name of "convexors," and uses the closed unit interval instead of the open one. The closed interval was also used by S. P. Gudder [8].) The two projection operations are obviously idempotent, skew-commutative, and associative. However, skew-associativity is not always defined. This inconvenience will be overcome in the definition given later, where barycentric algebras are specified as two-sorted algebras. The definition is motivated by the observation above.

First note, however, that the class of barycentric algebras $(A, \underline{I})$ has the same subdirectly irreducible members as the class $\mathcal{B}$ defined earlier. The representation as subalgebras of Płonka sums of convex sets remains valid for the reduct $(A, \underline{I}^o)$. In the case where all the operations $\underline{p}$ for $p \in I^o$ are equal, the algebra $(A, \underline{I})$ has the semilattice operation $\cdot = \underline{p}$ and the two projections $\underline{0}$ and $\underline{1}$ as its basic operations.

An *abstract barycentric algebra* is a two-sorted algebra $(A, J, F \sqcup \{t\})$ — or simply (A, J) — with two sorts A and J, the set

$$F = \{\oplus, \neg, \cdot_\pi, \to_\pi, 0, 1\}$$

of operations defined on J with values in J, and one ternary operation

$$t : A \times A \times J \to A; (x, y, p) \mapsto xy\underline{p} =: \underline{p}(y, x)$$

such that:

(A) (J, F) is an LΠ-algebra;

(B) the operation t satisfies the following identities for $x, y \in A$ and $p \in J$:

$$\underline{0}\,(x, y) = y = \underline{1}(y, x)\,; \tag{4.1}$$

$$\underline{p}\,(x, x) = x\,; \tag{4.2}$$

$$\underline{p}\,(x, y) = \underline{\neg p}\,(y, x)\,; \tag{4.3}$$

$$\underline{p}\,(x, \underline{q}\,(y, z)) = \underline{p \circ q}\,(\underline{(p \circ q \to_\pi q)}(x, y), z)\,. \tag{4.4}$$

The derived operation $\circ$ is defined by $p \circ q := \neg((\neg p) \cdot_\pi (\neg q))$.

Let $\mathcal{AB}$ denote the class of abstract barycentric algebras.

Example 4.1. Basic examples are obtained from real barycentric algebras $(A, \underline{I})$. First note that skew-associativity holds for all $p, q \in I$. This can easily be checked if one of p or q is 0 or 1. Now note that

$$p \circ q = (p'q')' = q1\underline{p}.$$

Hence for $p, q \in I^o$, one has $q < q1\underline{p}$. It follows that

$$(p \circ q) \to_\pi q = q/(p \circ q).$$

Moreover,

$$(0 \circ 0) \to_\pi 0 = 0 \to_\pi 0 = 1,$$
$$(1 \circ 1) \to_\pi 1 = (0 \circ 1) \to_\pi 1 = 1 \to_\pi 1 = 1$$

and

$$(1 \circ 0) \to_\pi 0 = 1 \to_\pi 0 = 0/1 = 0.$$

This implies skew-associativity in the remaining cases. Note that skew-associativity for $(A, \underline{I})$ may be written as

$$xy\underline{p}\,z\underline{q} = x\ yz\underline{(p \circ q \to_\pi q)}\ \underline{p \circ q}. \tag{4.5}$$

The corresponding abstract barycentric algebra has two sorts A and $J = I$. Moreover, (I, F) is an interval LΠ-algebra as defined in Section 3, and the ternary operation t is given by

$$t : A \times A \times I \to A; (x, y, p) \mapsto xy\underline{p} =: \underline{p}(y, x).$$

Example 4.2. Consider an abstract barycentric algebra $(A, J, F \sqcup \{t\})$, where $J = \{0, 1\}$ and (J, F) is a two element Boolean algebra $\mathbf{2}$. Note that $\mathbf{2}$ may be considered as the interval LΠ-algebra of the integral domain $\mathbb{Z}$ of integers. The two basic operations of the single-sorted algebra $(A, \underline{J})$ are given by $xy\underline{0} = x$ and $xy\underline{1} = y$. This gives A the structure of an algebra with one left-zero and one right-zero operation. Note that each abstract barycentric algebra $(A, J, F \sqcup \{t\})$ has the subalgebra $(A, \{0, 1\}, F \sqcup \{t\})$ that will be denoted as $(A, \mathbf{2})$.

5. Concrete and abstract barycentric algebras

Basic concepts for many-sorted algebras, such as subalgebras, direct products, and homomorphic images, are defined in the usual way, "componentwise." (See e.g. [2], [7], [31].) We will first use these concepts to analyze the structure of abstract barycentric algebras obtained from real barycentric algebras (cf. Example 4.1.) For a barycentric algebra $(A, \underline{I})$, its two-sorted counterpart will be called its *two-sorted* or *abstract companion*, or just the *companion*. Similarly, if an abstract barycentric algebra (A, J), with $J = I$, is a companion of a one-sorted barycentric algebra $(A, \underline{I})$, then $(A, \underline{I})$ will be called the (*one-sorted*) *companion* of (A, I).

The direct product $\prod_{k \in K}(A_k, I_k)$ of abstract barycentric algebras (A_k, I_k), where $I_k = I$ for each $k \in K$, is the algebra $(\prod A_k, \prod I_k)$ with the operations defined componentwise. Let $\widehat{I}$ be the subset of $\prod I_k$ consisting of functions f such that $f(k) = p$ for all $k \in K$ and some $p \in I$. Obviously, $(\widehat{I}, F) \leq \prod(I_k, F)$, and $(\widehat{I}, F)$ is isomorphic with (I, F). It is also easy to see that $(\prod A_k, \widehat{I})$ is a subalgebra of $(\prod A_k, \prod I_k)$. The algebra $(\prod A_k, \widehat{I})$ has the barycentric algebra $(\prod A_k, \underline{I})$ as a companion. The companions of subalgebras $(B, \underline{I})$ of the latter algebra are precisely the subalgebras $(B, \widehat{I})$ of the algebra $(\prod A_k, \widehat{I})$.

The companions of convex sets $(A, \underline{I})$ over R are easy to describe. First recall that the class $\mathcal{C}$ of convex sets is equal to the class $\mathsf{SP}(R)$ of subalgebras of powers of the convex set R. (See e.g. [27, Chapter 7], where the relevant results — especially Corollary 7.2.4 and Lemma 7.6.3 — are formulated for $\underline{I}^o$-algebras, but hold equally well for $\underline{I}$-algebras.) Thus each convex set is a subalgebra, say $(C, \underline{I})$, of the algebra $(R^k, \underline{I})$ for some cardinal k. The latter algebra has the abstract companion $(R^k, \widehat{I})$, and its subalgebra C has the companion $(C, \widehat{I})$. This shows the following.

Proposition 5.1. *Each convex set $(C, \underline{I})$ over a field R has a companion. If $(C, \underline{I})$ is a subalgebra of $(R^k, \underline{I})$, then its companion $(C, \widehat{I})$ is a subalgebra of $(R^k, \widehat{I}) \leq (R^k, I^k) = (R, I)^k$.*

Each $\underline{I}^o$-semilattice $(A, \underline{I}^o)$ is term equivalent to the semilattice $(A, \cdot)$, where $\cdot = \underline{p}$ for each $p \in I^o$. The corresponding I-algebra counterpart $(A, \underline{I})$ is term equivalent to $(A, \cdot, \underline{0}, \underline{1})$. Its abstract companion (A, I) satisfies the identities

$$\underline{p}(x, y) = \underline{q}(x, y)$$

for all $(p, q) \in I^o \times I^o$, or equivalently

$$t(x, y, p) = t(x, y, q).$$

A congruence relation θ of an abstract barycentric algebra (A, J) is a pair (θ_1, θ_2) of equivalence relations on A and J, respectively, such that $(\theta_1, \theta_2) \leq (A, J)^2$. In particular, θ_2 is a congruence of (J, F).

Proposition 5.2. *Let $\theta = (\theta_1, \theta_2)$ be a congruence of an abstract barycentric algebra (A, I). Let $\theta_2 = \hat{I}$, the equality relation on I. Then θ is a congruence of (A, I) precisely if θ_1 is a congruence of its companion $(A, \underline{I})$.*

Proof. Since $\theta_2 = \hat{I}$, it follows that $(p, p') \in \theta_2$ iff $p = p'$. Hence θ is a congruence precisely if

$$(a_1, a_1') \in \theta_1,\ (a_2, a_2') \in \theta_1 \text{ implies } (a_1 a_2 \underline{p}, a_1' a_2' \underline{p}) \in \theta_1.$$

This means that θ_1 is a congruence of $(A, \underline{I})$. □

Proposition 5.2 shows that all homomorphic images of real convex sets have abstract companions. As each real barycentric algebra is a homomorphic image of a convex set, this implies the following corollary.

Corollary 5.3. *Each barycentric algebra $(A, \underline{I})$ (a homomorphic image of a convex set) has an abstract companion (a homomorphic image of a companion of this convex set).*

Another corollary yields a class of subdirectly irreducible algebras.

Corollary 5.4. *An abstract barycentric algebra (A, I), with $|A| > 1$, is subdirectly irreducible iff its companion $(A, \underline{I})$ is subdirectly irreducible.*

Proof. It is enough to note that (I, F) has only two congruences, $\hat{I}$ and the improper congruence $I \times I$, and obviously for each congruence θ_1 of $(A, \underline{I})$, the congruence $(\theta_1, \hat{I}) \leq (\theta_1, I \times I)$. □

Lemma 5.5. *Let $\theta = (\theta_1, \theta_2)$ be a congruence of a barycentric algebra (A, J). Let $\theta_1 = \hat{A}$, the equality relation on A. Then A consists precisely of one element.*

Proof. Since $\theta_1 = \hat{A}$, it follows that $(a, a') \in \theta_1$ iff $a = a'$. Hence θ is a congruence precisely if

$$(a_1 a_2 \underline{p}, a_1 a_2 \underline{p'}) \in \theta_1$$

for all $p, p' \in I$. But $\theta_1 = \hat{A}$ implies that

$$a_1 a_2 \underline{p} = a_1 a_2 \underline{p'}.$$

In particular $\underline{0} = \underline{1}$ and $x = xy\underline{0} = xy\underline{1} = y$, whence $|A| = 1$. □

Corollary 5.6. *An abstract barycentric algebra (A, I) with $|A| = 1$ is subdirectly irreducible (and indeed simple).*

6. Boolean affine spaces and barycentric algebras

Let R be a commutative ring. Affine spaces over R are characterized algebraically in [25] and [27]. In particular, if R is a Boolean ring B, we speak of *Boolean affine spaces* or *affine B-spaces.* For a given Boolean ring B, the class $\underline{\underline{B}}$ of affine B-spaces is a variety, characterized as equivalent to the variety of all Mal'cev modes $(A, P, \underline{B})$ with the ternary Mal'cev operation P and binary operations $\underline{p}$, for $p \in B$, satisfying the identities:

$$xy\underline{0} = x = yx\underline{1}\,; \tag{6.1}$$

$$xy\underline{p}\, xy\underline{q}\, \underline{r} = xy\underline{pq\underline{r}}\,; \tag{6.2}$$

$$(xy\underline{p}\, xy\underline{q}\, xy\underline{r})P = xy\underline{pqrP}\,. \tag{6.3}$$

This is an immediate corollary of [27, Theorems 6.3.3 and 6.3.4]. See also [25]. Other identities true in all Boolean affine spaces include:

$$yxyP = xy\underline{2} = x\,; \tag{6.4}$$

$$x\, xy\underline{p}\, \underline{q} = xy\underline{pq}\,. \tag{6.5}$$

In each Boolean affine space $(A, P, \underline{B})$, the basic operations are defined by

$$xyzP = x + y + z \text{ and } xy\underline{p} = x(1 + p) + yp$$

for each $p \in B$.

Proposition 6.1. *Each Boolean affine space $(A, P, \underline{B})$ satisfies the identities of skew-commutativity and skew-associativity for all $p, q \in B$.*

Proof. The skew-commutativity follows directly by

$$xy\underline{p} = x(1+p) + yp = yx\underline{(1+p)}.$$

To show that the skew-associativity (4.5) holds as well, first note that in a Boolean ring B one has

$$p \circ q = p + q - pq = p + q + pq = p \vee q$$

and

$$p \to q = p \to_\pi q = p' \vee q.$$

Now for $p, q \in B$,

$$p \circ q \to q = (p \circ q)' \vee q = (p \vee q)' \vee q = (p'q') \vee q = p' \vee q = p \to q.$$

This implies that when p and q are elements of a Boolean ring B, the identity of skew-associativity can be written as follows:

$$xy\underline{p}\, z\underline{q} = x\ yz\underline{(p' \vee q)}\ \underline{p \vee q}. \tag{6.6}$$

And indeed in a Boolean affine space we have:

$$\begin{aligned} xy\underline{p}\, z\underline{q} &= (x(1+p) + yp)(1+q) + zq \\ &= x(1+p+q+pq) + yp(1+q) + zq \\ &= x(p \vee q)' + ypq' + zq \\ &= x(1 + (p \vee q)) + y(1 + (p' \vee q)) + z(p' \vee q)(p \vee q) \\ &= x\ yz\underline{(p' \vee q)}\ \underline{p \vee q}. \end{aligned}$$

Hence skew-associativity holds for all $p, q \in B$. □

Note the following consequences of the skew-associativity (6.6):

$$xy\underline{p}\, z\underline{p} = xz\underline{p}, \tag{6.7}$$

$$x\, yz\underline{p}\, \underline{p} = xz\underline{p}. \tag{6.8}$$

Indeed,

$$xy\underline{p}\, z\underline{p} = x\, yz\underline{(p' \vee p)}\, \underline{(p \vee p)} = xz\underline{p}$$

and

$$\begin{aligned} x\, yz\underline{p}\, \underline{p} &= yz\underline{p}\, x\, \underline{p'} = yz\underline{p}\, x\, \underline{(1+p)} \\ &= y\, zx\underline{(1+p)}\, \underline{1} = y\, zx\underline{p'}\, \underline{1} = zx\underline{p'} = xz\underline{p}. \end{aligned}$$

This means that each of the operations $\underline{p}$ is in fact a rectangular band operation. An entropic algebra (A, Ω) with a rectangular band operation $\cdot$ commuting with the operations in Ω decomposes as a direct product $(A_1, \Omega) \times (A_2, \Omega)$ such that (A_1, Ω) satisfies the identity $x \cdot y = x$ (i.e. $(A_1, \cdot)$ is a left-zero band), and (A_2, Ω) satisfies the identity $x \cdot y = y$ (i.e. $(A_2, \cdot)$ is a right-zero band) [27, Theorem 1.3.2].

It follows that each of the operations $\underline{p}$ for $p \in B - \{0, 1\}$ decomposes the algebra $(A, \underline{B})$ as a product $(A_1, \underline{B}) \times (A_2, \underline{B})$ such that $(A_1, \underline{B})$ satisfies $xy\underline{p} = x$ and $(A_2, \underline{B})$ satisfies $xy\underline{p} = y$. In particular, if B is a finite 2^n-element Boolean algebra, then $(A, \underline{B})$ is a direct product of $2^{2^n - 2}$ *projection* algebras $(A_i, \underline{B})$, where each of the operations $\underline{p}$ is a left- or a right-zero operation.

The binary reducts $(A, \underline{B})$ of Boolean affine spaces $(A, P, \underline{B})$ are B-sets in the sense of G. Bergman [1] and T. Stokes [29]. As shown by T. Stokes, for a given B, the variety of B-sets consists of the subalgebras of such reducts. This variety is defined by the following identities:

(1) $\underline{0}(x, y) = y\,;$
(2) $\underline{p}(x, x) = x\,;$
(3) $\underline{\neg p}(x, y) = \underline{p}(y, x)\,;$
(4) $\underline{pq}(x, y) = \underline{p}(\underline{q}(x, y), y)\,;$
(5) $\underline{p}(\underline{p}(x, y), z) = \underline{p}(x, z)\,;$
(6) $\underline{p}(x, \underline{p}(y, z)) = \underline{p}(x, z)\,.$

Note that the first three identities are just the identities (4.1), (4.2) and (4.3) from the definition of abstract barycentric algebras. The fourth identity is the identity (6.5) written in prefix notation, while the fifth and sixth identities are (6.7) and (6.8) written in prefix notation. Note also that any two operations $\underline{p}$ and $\underline{q}$ of an abstractly defined B-set are mutually entropic. (See [30].)

Bergman's motivation for investigating B-sets came from sheaf theory. But the algebras also served as a foundation for the notion of "if-then-else" (see McCarthy [18] and Manes [17], and also [29] and [30]). In [18], $\underline{p}(x, y)$ is viewed as the statement "**if** p **then** x **else** y". We return to Manes' approach in Example 6.4.

Corollary 6.2. *Let B be a Boolean ring. Then B-sets (considered as binary subreducts of affine B-spaces) can be viewed as abstract barycentric algebras.*

Proof. Each subalgebra $(C, \underline{B})$ of the reduct $(A, \underline{B})$ of a given B-space $(A, P, \underline{B})$ can be considered as a two-sorted algebra (C, B), where B is a Boolean algebra and $t : C \times C \times B : (x, y, p) \mapsto \underline{p}(y, x) = xy\underline{p}$. All the axioms of abstract barycentric algebras are satisfied. □

Corollary 6.2 justifies the name of *Boolean barycentric algebras* for B-sets.

Example 6.3. One of the standard examples of B-sets is given by a Cartesian product $\prod_{k \in K} A_k$ of sets A_k, with the Boolean algebra $B = \mathcal{P}(K)$ of all subsets of the index set K. The B-action is defined for all $p \in B$ by

$$\underline{p}(k)(a,b) = \textbf{if}\, k \in p\, \textbf{then}\, a(k)\, \textbf{else}\, b(k)$$

[30]. Such a B-set has an abstract companion

$$(\prod_{k\in K} A_k, B) = (\prod_{k\in K} A_k, \mathcal{P}(K)) \cong (\prod_{k\in K} A_k, \mathbf{2}^K)\,.$$

It is easy to see that the latter algebra is isomorphic to the product $\prod_{k\in K}(A_k, \mathbf{2})$ of the two-sorted barycentric algebras from Example 4.2.

Example 6.4. Consider a family of I-semilattices $(A_k, \underline{I})$ for $k \in K$, each term equivalent to $(A_k, \cdot, \underline{0}\,, \underline{1}\,)$. The corresponding abstract companions are (A_k, I). Let $B = \mathbf{2}^K$. Then the direct product

$$\prod_{k\in K} (A_k, I) \cong \Big(\prod_{k\in K} A_k, I^K\Big)$$

contains the B-set

$$\Big(\prod_{k\in K} A_k, B\Big)$$

as a subalgebra. Moreover $t(a,b,p) = t(a,b,q)$ for all $p,q \in I^o$, so that $\underline{p} = \underline{q}$ is in fact a semilattice operation $\cdot$ on $\prod_{k\in K} A_k$, entropic with the operations $\underline{p}$ for $p \in B$. It follows that $(\prod_{k\in K} A_k, \underline{B}, \cdot)$ and its subalgebras are in fact semilattice modes (see [15]) or entropic modals (see [27, Chapter 10]). On the other hand, Stokes [29] observed that when semilattices (bounded below) are interpreted as join semilattices, such algebras are equivalent to the *if-then-else algebras* of Manes [17].

7. Rectangular algebras

We will show that B-sets are also equivalent to certain modes obtained in a totally different way. Call a mode $(A, (p_k)_{k\in K})$ a *rectangular K-mode* when each of the operations p_k is a rectangular band operation. In particular, each operation $p \in \{p_k \mid k \in K\}$ is idempotent and satisfies the identities

$$x\,yzp\,p = xyp\,z\,p = xzp\,. \tag{7.1}$$

Moreover, these operations are mutually entropic. Note that the class $\mathcal{RE}_K$ of rectangular K-modes is a variety. If the set K is finite, these algebras are *binary rectangular algebras* as considered in [23] and [32]. Many properties of these algebras carry over to rectangular modes with infinitely many operations. It is easy to see that B-sets are rectangular modes. We will show that rectangular modes are equivalent to B-sets.

Lemma 7.1. *The binary derived operations of a rectangular K-mode $(A, (p_k)_{k\in K})$ are all rectangular band operations.*

Proof. First note that each operator $\cdot = p_k$, for $k \in K$, determines four binary operations: $x\cdot y, y\cdot x$ and two projections $(x\cdot y)\cdot(y\cdot x) = x =: xy0$ and $(y\cdot x)\cdot(x\cdot y) = y =: xy1$. Now assume that $|K| > 1$. We will show that for any three (derived) binary rectangular band operations p, q, r, their composition

$$x \circ y := xyp\, xyq\, r \tag{7.2}$$

is again a rectangular band operation. i.e. it satisfies the identities

$$(x \circ y) \circ z = x \circ (y \circ z) = x \circ z.$$

Let us calculate that, indeed, the first and last expressions form a true identity. The proof that the first and the second one are equal is similar.

$$\begin{aligned}
(x \circ y) \circ z &= [(x \circ y)zp]\,[(x \circ y)zq]\,r \\
&= [(xyp\, xyq\, r)zp]\,[(xyp\, xyq\, r)zq]\,r \\
&= [(xyp\, zp)\,(xyq\, zp)r]\,[(xyp\, zq)\,(xyq\, zq)\,r)]\,r \quad \text{(by entropicity)} \\
&= (xyp\, zp)\,(xyq\, zq)\,r \quad \text{(by rectangularity)} \\
&= xzp\, xzq\, r = x \circ z.
\end{aligned}$$

It follows that indeed each binary derived operation is a rectangular band operation. □

Lemma 7.1 also follows from the fact that the identities defining general rectangular modes are in fact hyperidentities. (See [23].)

Lemma 7.2. *The binary derived operations of a rectangular K-mode $(A, (p_k)_{k\in K})$ form a Boolean algebra generated by the basic operations.*

Proof. Let B be the set of all (derived) rectangular band operations of $(A, (p_k)_{k\in K})$. For $p, q \in B$, define

$$xyp' := yxp$$

and

$$xy(pq) := x\, xyp\, q.$$

Note that p' and pq are also rectangular band operations, and that by the entropic law, $pq = qp$. This multiplication on B is known to be associative. Indeed, $xy(p \cdot qr) = x\,(xyp)\,(qr) = x\,(x\, xyp\, q)\,r = x(xy(pq))r = xy(pq \cdot r)$. As evidently $pp = p$, it follows that the multiplication just defined is a semilattice operation. Now define

$$xy(p \vee q) := xy(p'q')' = xyp\, yq.$$

As in the case of multiplication, the operation $\vee$ is also a semilattice operation. Moreover

$$xy(p \vee pq) = (xyp)\,(xyp\,yq)\,p = x\,(xyq\,yp)\,p = xyp.$$

Similarly, $xy(p(p \vee q)) = xyp$. This shows that $(B, \cdot, \vee)$ is a lattice. It is easy to check that the projections 0 and 1 are bounds of this lattice, and that $p \vee p' = 1$ and $pp' = 0$. Distributivity is checked in similar fashion:

$$\begin{aligned} xyp(q \vee r) &= x\,xyp\,(q \vee r) = (x\,xyp\,q)\,(xyp)\,r \\ &= (x\,xyp\,q)\,[(x\,xyp\,q)\,y\,p]\,r && \text{(by rectangularity} \\ &= (x\,xyp\,q)\,y\,(pr) = xy(pq)\,y\,(pr) = xy(pq \vee pr). \end{aligned}$$

It follows that $(B, \cdot, \vee, ', 0, 1)$ is a Boolean algebra, generated by the operations p_k for $k \in K$. □

Proposition 7.3. *Each rectangular K-mode $(A, (p_k)_{k\in K})$ is term equivalent to the B-set $(A, \underline{B})$, where B is the Boolean algebra of rectangular band operations of $(A, (p_k)_{k\in K})$.*

Proof. The proof follows directly by the two lemmas above. □

The lattice $L(\mathcal{M}_R)$ of subvarieties of the variety $\mathcal{M}_R$ of modules over a (unital) commutative ring R is dually isomorphic to the lattice of ideals of the ring R, and isomorphic to the lattice $L(\underline{\underline{R}})$ of subvarieties of the variety $\underline{\underline{R}}$ of affine R-spaces [27, Section 5.3]. In particular, these isomorphisms obtain for each variety $\underline{\underline{B}}$ of Boolean affine spaces over a Boolean ring B.

Proposition 7.4. *For a given Boolean ring B, the lattice of subvarieties of the variety $\underline{\underline{B}}$ of affine B-spaces and the lattice of subvarieties of the variety $\mathcal{BS}$ of B-sets are isomorphic, i.e.*

$$L(\underline{\underline{B}}) \cong L(\mathcal{BS}).$$

Proof. Each variety of algebras is generated by the free algebra over a countably infinite set X of generators. The free affine B-space XB consists of linear combinations $\sum_{i=1}^{n} x_i p_i$ with $p_i \in B$ and $\sum_{i=1}^{n} p_i = 1$ [27, Chapter 6]. On the other hand, it is an immediate consequence of [24, Lemmas 6.1, 6.2] that the free algebra over X in the (quasi)variety $\mathcal{BS}$ of subreducts $(S, \underline{B})$ of affine B-spaces $(A, \underline{B}, P)$ is isomorphic to the B-subreduct of XB generated by X, and consists of linear combinations with coefficients in the free $\underline{\underline{B}}$-algebra on two generators. Equivalently, as independently shown by Bergman [1], the free algebra consists of those linear combinations satisfying the additional condition that $p_i p_j = 0$ whenever $i \neq j$. Note that the free affine B-space

$\{0,1\}B$ on two free generators 0 and 1 is built on the ring B, and the free $\mathcal{BS}$-algebra on two generators 0 and 1 is just the B-subreduct of $\{0,1\}B$ generated by 0 and 1.

Now it suffices to assign to each subvariety $\underline{\underline{B'}}$ of the variety $\underline{\underline{B}}$ (generated by the free algebra XB') the subvariety $\mathcal{B}'\mathcal{S}$ of $\mathcal{BS}$ (generated by the B'-subreduct of XB'). This is a lattice isomorphism. □

Note that when B is a free Boolean ring over a set X, the free B-set $X\mathcal{BS}$ does not satisfy any identities that are not consequences of the defining identities of B-sets or the axioms of Boolean algebras. The same is true for the variety $\mathcal{RE}_X$ of rectangular X-modes. In such a case, Corollary 7.3 implies the following.

Corollary 7.5. *Let B be the free Boolean ring over $\{p_k \mid k \in K\}$. Let $\mathcal{BS}_K$ be the variety of B-sets, and let $\mathcal{RE}_K$ be the variety of rectangular K-modes $(A,(p_k)_{k\in K})$. Then the varieties $\mathcal{BS}_K$ and $\mathcal{RE}_K$ are equivalent. Moreover*

$$L(\mathcal{BS}_{\mathcal{K}}) \cong L(\mathcal{RE}_K).$$

Note that each subvariety of $\mathcal{RE}_K$ is determined by an ideal of B.

As easy corollaries of Lemmas 4.1, 4.2, 4.4 of [32], one obtains that each mode in the variety $\mathcal{RE}_n$ of rectangular modes with a finite number n of basic operations embeds as a subreduct into an affine space over the *affinization ring*

$$R = \mathbf{Z}[X_1,\dots,X_n]/\langle X_i(1-X_i) \mid i=1,\dots,n\rangle \cong \mathbf{Z}^{2^n}$$

of the variety $\mathcal{RE}_n$ [27, Chapter 7]. The operations p_k are defined as the affine space operations $xyp_k = xy\underline{X}_k$. The ring $\mathbf{Z}^{2^n}$ contains the Boolean ring $\mathbf{2}^{2^n}$ as a subreduct, and the corresponding rectangular mode embeds into a module over this ring.

In the more general setting considered here, an embedding of rectangular modes as subreducts into affine B-modules follows from the general theory of modes, much as in the case of rectangular modes with finitely many basic operations. The affinization ring of the variety $\mathcal{RE}_K$ is calculated in standard fashion [27, Section 7.1] as the ring $R(\mathcal{RE}_K) = \mathbf{Z}[\{X_k\}]/\langle\{X_k(1-X_k)\} \mid k \in K\rangle \cong \mathbf{Z}^{2^{|K|}}$. Then the structure of rectangular modes is described by the following lemma.

Lemma 7.6. *Each rectangular mode $(A,(p_k)_{k\in K})$ is a subdirect product of projection rectangular modes.*

Proof. For each $k \in K$, define two binary relations θ_k^1 and θ_k^2 on A by

$$\theta_k^1 := \{(a,b) \mid abp_k = b\} \text{ and } \theta_k^2 := \{(a,b) \mid abp_k = a\}.$$

These two relations form a pair of factor congruences on $(A, (p_k)_{k\in K})$ [27, Section 1.3], so that $A \cong A/\theta_k^1 \times A/\theta_k^2$. In particular, $\theta_k^1 \wedge \theta_k^2 = \widehat{A}$, the equality relation on A.

Now for each function $f : K \to \{1, 2\}$, define

$$\theta_f := \prod_{k\in K} \theta_k^{f(k)}.$$

It is easy to see that each $A_f := A/\theta_f$ is a projection mode. Let $\mathsf{K} := \{f : K \to \{1, 2\}\}$. Evidently,

$$\prod_{f\in\mathsf{K}} \theta_f = \widehat{A}.$$

It follows that $(A, (p_k)_{k\in K})$ is a subdirect product of the algebras $(A_f, (p_k)_{k\in K})$. □

Corollary 7.7. *Each rectangular mode embeds as a subreduct into an affine Boolean space.*

The proof of Corollary 7.7 is very similar to the corresponding proof for the case where the number of rectangular operations is finite [32]. One represents a rectangular mode A as a subdirect product of projection rectangular algebras, as in Lemma 7.6. Then one embeds these projection algebras into corresponding modules, and finally one embeds the product of projection algebras into the product of the corresponding modules. Note that this embedding uses only the operations $\underline{X}$, so that in fact the embedding is into a module over the Boolean subring of the ring $R(\mathcal{RE}_K)$. In particular, this provides an alternative proof of the Stokes embedding theorem.

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