DYADIC INTERVALS and DYADIC TRIANGLES

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REAL AFFINE SPACES

Given a vector space (a module) A over a field (a subring R of) $\mathbb{R}$.

An **affine space** A **over** R (or **affine** R**-space**) is the algebra

$$\Big(A, \sum_{i=1}^{n} x_i r_i \Big| \sum_{i=1}^{n} r_i = 1\Big).$$

In the case $2 \in R$ is invertible, this algebra is equivalent to

$$(A, \underline{R}),$$

where

$$\underline{R} = \{\underline{f} \mid f \in R\}$$

and

$$xy\underline{f} = \underline{f}(x, y) = x(1 - f) + yf.$$

REAL CONVEX SETS and BARYCENTRIC ALGEBRAS

Let R be a subfield of $\mathbb{R}$ and
$I^o :=]0,1[= (0,1) \subset R$.

Convex subsets of affine R-spaces are
I^o-subreducts $(A, \underline{I}^o)$ of R-spaces.

Real **polytopes** are finitely generated convex sets, real **polygons** are finitely generated convex subsets of R^2.

The class C of convex sets generates
the variety BA of **barycentric algebras**.

DYADIC CONVEX SETS

Consider the ring

$$\mathbb{D} = \mathbb{Z}[1/2] = \{m2^{-n} \mid m, n \in \mathbb{Z}\}$$

of dyadic rational numbers.

A **dyadic convex set** is the intersection of a real convex set with the space $\mathbb{D}^k$.

A **dyadic polytope** is the intersection of a real polytope and $\mathbb{D}^k$, with vertices in $\mathbb{D}^k$.

A **dyadic triangle** and **dyadic polygon** are (respectively) the intersection with $\mathbb{D}^2$ of a triangle or polygon in $\mathbb{R}^2$, with vertices in $\mathbb{D}^2$.

Dyadic intervals form the one-dimensional analogue.

REAL VERSUS DYADIC

• Real polytopes are barycentric algebras $(A, \underline{I}^o)$.

Dyadic polytopes are algebras $(A, \underline{\mathbb{D}}_1^o)$, where $\mathbb{D}_1^o =]0,1[\cap \mathbb{D}$.

Proposition Each dyadic polytope $(A, \underline{\mathbb{D}}_1^o)$ is equivalent to $(A, \cdot) = (A, \frac{1}{2}(x+y))$.

Note that the operation $\cdot$ is

idempotent: $x \cdot x = x$,
commutative: $x \cdot y = y \cdot x$,
entropic (medial): $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$.

Hence:
dyadic polytopes are **commutative binary modes** (or CB-modes).

MODES

An algebra (A, Ω) is a **mode** if it is

- **idempotent**:

  $$x \ldots x\omega = x,$$

  for each n-ary $\omega \in \Omega$, and

- **entropic**:

  $$(x_{11} \ldots x_{1n}\omega) \ldots (x_{m1} \ldots x_{mn}\omega)\,\varphi$$

  $$= (x_{11} \ldots x_{m1}\varphi) \ldots (x_{1n} \ldots x_{mn}\varphi)\,\omega.$$

  for all $\omega, \varphi \in \Omega$.

Affine R-spaces and barycentric algebras are modes.
Subreducts (subalgebras of reducts) of modes are modes

REAL VERSUS DYADIC, cont.

• All real intervals are isomorphic (to the interval $I = [0, 1] = S_1$). Each is generated by its ends.
All real triangles are isomorphic (to the simplex S_2). Each is generated by its vertices.

NOT TRUE for dyadic intervals and dyadic triangles.

Example The dyadic interval $[0, 3]$ is generated by no less than 3 elements. The minimal set of generators is given e.g. by the numbers $0, 2, 3$.

• The class of convex subsets of affine $\mathbb{R}$-spaces is characterized as the subquasivariety of cancellative barycentric algebras.

NOT TRUE for the class of convex dyadic subsets of affine $\mathbb{D}$-spaces.
(K. Matczak, A. Romanowska)

PROBLEMS

Which characteristic properties of real polytopes (in particular polygons) carry over to dyadic polytopes (polygons)?

Note that dyadic polygons are described using dyadic intervals and dyadic triangles.

Problems:
Are all dyadic intervals finitely generated?
Are all dyadic triangles finitely generated?

Problem: Classify all dyadic intervals and all dyadic triangles up to isomorphism.

Isomorphisms of dyadic polytopes are described as restrictions of automorphisms of the affine dyadic spaces, members of the affine group $GA(n, \mathbb{D})$.

RELATED PROBLEMS

Problem: Are all finitely generated subgroupoids of the groupoid $(\mathbb{D}, \cdot)$ intervals?

Problem: Are all finitely generated subgroupoids of the groupoid $(\mathbb{D}^2, \cdot)$ polygons?

Problem: Characterise finitely generated subgroupoids of the groupoid $(\mathbb{D}^2, \cdot)$ which are triangles.

DYADIC INTERVALS

The isomorphism classes of dyadic intervals are determined by the orbits of $GL(1, \mathbb{D})$ on the set of nonzero dyadic numbers.

THEOREM [K. Matczak, A. Romanowska, J. D.H. Smith] Each interval of $\mathbb{D}$ is isomorphic to some interval $[0, k]$ (is **of type** k), where k is an odd positive integer. Two such intervals are isomorphic precisely when their right hand ends are equal.

The interval $[0, 1]$ is generated by its ends.
For each positive integer k, and each integer r, the intervals $[0, k]$, $[0, k2^r]$ and $[d, d + k2^r]$ are isomorphic.

THEOREM Each dyadic interval of type $k > 1$ is minimally generated by three integers $0, z, k$, where $0 < z < k$ and $\gcd\{z, k\} = 1$, e.g. by $0, 2^n, k$, where $n = \lfloor \log_2 k \rfloor$.

DYADIC INTERVALS, cont.

The unit interval $\mathbb{D}_1$ contains infinitely many subintervals of each type k for $k \in 2\mathbb{N}+1$, and infinitely many finitely generated subgroupoids which are not necessarily intervals.

Example The elements $0, 3/16, 6/16, 9/16$ generate a subgroupoid of $\mathbb{D}_1$ which does not coincides with the interval $[0, 9/16]$. Note however, that this subgroupoid is isomorphic to the interval $[0, 3]$.

Lemma If G is a subgroupoid of $(\mathbb{D}, \cdot)$ (finitely) generated by dyadic numbers $g_1 < \cdots < g_r$, then G is isomorphic to a subgroupoid generated by some integers $0 = z_1 < z_2 < \cdots < z_r$. If one of z_i is one, then G coincides with the interval $[0, z_r]$.

Proposition Assume that G has at least three generators z_i. Then $\text{gcd}\{z_2, \ldots, z_r\} = 2^n$ for some natural number n if and only if G coincides with the interval $[0, z_r]$.

Lemma If $g := \text{gcd}\{z_2, \ldots, z_r\} \neq 1$, then G is isomorphic to the subgroupoid H of $\mathbb{D}$ generated by $0 = z_1/g, z_2/g, \ldots, z_r/g$ with $\text{gcd}\{z_2/g, \ldots, z_r/g\} = 1$.

Theorem Each finitely generated subgroupoid of $(\mathbb{D}, \cdot)$ is isomorphic to some interval of $\mathbb{D}$.

Corollary [A. Mucka, K. Matczak, A. Romanowska] A subgroupoid of $(\mathbb{D}, \cdot)$ is isomorphic to an interval of $\mathbb{D}$ if and only if it is finitely generated.

DYADIC TRIANGLES AND THEIR BOUNDARY TYPES

The types m, n, k of sides of a triangle determine its **boundary type** (m, n, k).

Theorem [K. Matczak, A. Romanowska, J. D. H. Smith] The triangles of *right type* (i.e.with shorter side parallel to the coordinate axes) are determined uniquely up to isomorphism by its boundary type.

If the types of shorter sides are m and j, then the hypotenuse is of type gcd$\{m, j\}$.

The boundary type does not determine a general dyadic triangle.

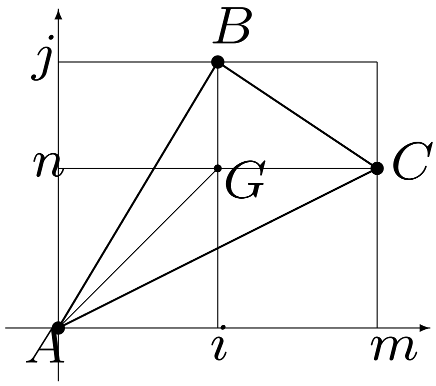
Proposition There are infinitely many pairwise non-isomorphic triangles of boundary type $(1, 1, 1)$.

There are triangles in $\mathbb{D}$ not isomorphic to right triangles.

TYPES OF DYADIC TRIANGLES

Automorphisms of the dyadic plane $\mathbb{D}^2$ are described as elements of the affine group $GA(2, \mathbb{D})$. These automorphisms transform any of the triangles in the plane $\mathbb{D}^2$ into an isomorphic triangle.

Lemma Each dyadic triangle is isomorphic to a triangle ABC contained in the first quadrant, with A located at the origin. Moreover, the vertices B and C may be chosen so that they have integral coordinates.



A point $A = (p2^q, u2^v)$ of $\mathbb{D}^2$, where p, u, q and v are integers, with p and u being odd, is said to be **axial** if

$$\text{gcd}\{p, u\} \in \{p, u, 1\} .$$

Lemma A $\mathbb{D}$-module automorphism of the plane $\mathbb{D}^2$ transforms A into a point on one of the axes if and only if A is axial.

A classification of dyadic triangles depends on the existence of axial vertices.

THEOREM Each dyadic triangle, located as in the lemma, belongs to one of three **basic types** (with m, n, j positive integers):

- triangles isomorphic to right triangles $T_{0,j,m,0}$ (with vertices $(0, 0), (m, 0)$ and $(0, j)$);
- triangles isomorphic to triangles $T_{i,j,m,0}$ (with vertices $(0, 0), (i, j), (m, 0)$);
- triangles in which neither B nor C is axial.

Definition Let i, j, m, n be non-negative integers such that $0 \leq i < m$, $0 \leq n < j$, and with odd gcd$\{i, m\}$ and gcd$\{j, n\}$. The quadruple (i, j, m, n) is said to be an **encoding quadruple** if it satisfies the following conditions:

- if $i = n = 0$, then j and m are odd and $j \leq m$;
- if one of i and n is zero, then this is n, and in this case $i \leq m/2$, $j > 1$ and is odd, and moreover gcd$\{i, j\} \neq j$;
- if none of i, n is zero, then $j \leq m$, gcd$\{i, j\} \notin \{i, j, 1\}$ and gcd$\{m, n\} \notin \{m, n, 1\}$.

Definition For each encoding quadruple (i, j, m, n), we define a **representative triangle** $T_{i,j,m,n}$ as a dyadic triangle ABC located in the dyadic plane with vertices $A = (0, 0), B = (i, j), C = (m, n)$, as in Figure.

Theorem Each dyadic triangle is isomorphic to some representative triangle $T_{i,j,m,n}$. Two dyadic triangles are isomorphic if and only if they are both isomorphic to the same representative triangle $T_{i,j,m,n}$.

Definition If a dyadic triangle is isomorphic to some representative triangle $T_{i,j,m,n}$, then we will say that it is of **triangle type** (i, j, m, n).

Proposition (a) If a triple (r, s, t) of odd positive integers satisfies the condition

$$\text{gcd}\{r, s\} = \text{gcd}\{r, t\} = \text{gcd}\{s, t\}, \qquad (1)$$

then there is a dyadic triangle of boundary type (r, s, t).

(b) If T is a representative triangle $T_{i,j,m,n}$ with the types of sides r, s and t, then r, s, t satisfy the condition (1), and moreover

$$\text{gcd}\{r, s\} = \text{gcd}\{i, j, m - i, j - n\}. \qquad (2)$$

If (r, s, t) is a boundary type of a dyadic triangle, then the condition (1) holds precisely when each of the three types r, s, t is a linear combinations of the other two.

FINITE GENERATION

The classification of dyadic triangles given above provided a basis for proving the following.

THEOREM Each dyadic triangle is finitely generated.

COROLLARY Each dyadic polygon is finitely generated.