AUGMENTED QUASIGROUPS AND CHARACTER ALGEBRAS

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Abstract. The conjugacy classes of groups and quasigroups form association schemes, in which the relation products are defined by collapsing group or quasigroup multiplications. In previous work, sharp transitivity was used to identify association schemes, such as certain Johnson schemes, which cannot appear as quasigroup schemes. Thus quasigroup schemes only constitute a fragment of the full set of all association schemes. Nevertheless, the current paper shows that every association scheme is in fact obtained by collapsing a quasigroup multiplication. In a second application of a similar technique, character quasigroups are constructed for each finite group, as analogues of the character groups of abelian groups, to encode the multiplicative structure of group characters. As infrastructure for these and related results, three key unifying concepts in compact closed categories are established: augmented comagmas, augmented magmas, and augmented quasigroups, the latter serving to capture such diverse structures as groups and Heyting algebras.

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Contents

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1. Introduction

The ordinary character theory of finite groups emerged by the early twentieth century as an extension of discrete Fourier analysis and the character theory of abelian groups. Although it is now conventional to treat characters as traces of matrix representations or modules, the initial approach made use of the algebra of conjugacy classes [7, 11]. Continued by Kawada's work on character algebras [18], the approach developed into the theory of association schemes, also identified as algebraic combinatorics, and characterized as "group theory without groups" [3]. In particular, fusion algebras (as known, for example, in conformal field theory) were shown to correspond to character algebras [2].

When ordinary representation theory is further extended from groups to quasigroups (thus keeping the cancellativity, but no longer insisting on associativity [§2.3.1]), it transpires that the theories of modules and characters diverge [28]. Module theory studies vector space or abelian group objects in slice categories over a quasigroup, but the character theory continues to work with the original conjugacy class approach used for groups. The only change, forced by the potential absence of an identity element, is the need to replace inner automorphism group orbits on a group by multiplication group orbits on the direct square of a quasigroup [§2.3.2].

The association schemes obtained in this way from quasigroups are certainly rather special, and techniques such as sharp transitivity have been used to identify association schemes which cannot be quasigroup conjugacy class schemes [§2.3.3]. Thus it was long considered that quasigroup theory could only be relevant to a small part of algebraic combinatorics. A primary goal of the paper is to show that this is in fact quite far from the truth: Every association scheme, not just those coming from group or quasigroup conjugacy classes, actually lifts to a quasigroup [§5.2]. Since quasigroups are obtained from groups, namely as quotients by subgroups which are not necessarily normal (Appendix A), this result may be said to bring general association schemes back into groups.

Each abelian group A of finite order n has a character group $\widetilde{A}$ of the same order n, non-canonically isomorphic to A, where products in $\widetilde{A}$ determine the products of the characters. The second main goal of the paper is to find an analogue of the character group that works for the noncommutative case. Given an arbitrary group A of finite order n, we identify *character quasigroups* $\widetilde{A}$ of order n, such that products in $\widetilde{A}$ determine the products of the characters [§5.4]. In particular, the character group of an abelian group is its unique character quasigroup. While the character group $\widetilde{A}$ of a commutative group A determines A up to isomorphism, the character quasigroups $\widetilde{A}$ of a noncommutative group A cannot determine A uniquely. For example, the quaternion group Q_8 and dihedral group D_4 of order 8 share the same character quasigroups.

The third main goal of the paper is to bring some order into the abundance of different algebras that appear in algebraic combinatorics. Since these algebras usually come with a preferred basis, such bases should be identified in a canonical fashion. Working in the context of compact closed categories, modeled for instance by categories of finite-dimensional vector spaces over a given field, or by relations on sets, three levels of abstract structures are identified:

- augmented comagmas [Definition 3.14(c)];

- augmented magmas [Definition 3.6], and
- augmented quasigroups [Definition 4.1(f)].

Augmented comagmas comprise multisets and sets, as shown in §3.4. In turn, fusion algebras form augmented magmas [Theorem 3.30(b)]. Then augmented quasigroups embrace a wide range of structures, from quasigroups themselves [Example 4.14] in various categories, through group algebras [Example 4.2] and Heyting algebras [Example 4.5], to association schemes [Corollary 4.17], character algebras [Theorem 4.19], dual schemes [Corollary 4.20] and fusion algebras [Corollary 4.21].

2. Combinatorial algebras of algebraic combinatorics

Algebraic combinatorics makes use of a wide range of algebras, many of which are closely related, but differ in certain subtle details. For an excellent survey, see [4]. This chapter will introduce the structures forming the backbone of the paper. Together with other structures, they will subsequently fit in to the general compact-closed categorical approach introduced in the two following chapters.

2.1. **Character algebras and fusion algebras.**

2.1.1. *Character algebras.* These algebras were formalized by Kawada [2, Defn. 2.3], [3, §2.5], [18, §2].

Definition 2.1. A finite-dimensional, commutative, associative, unital algebra A over $\mathbb{C}$ is said to be a *character algebra* if it has a basis $\{1 = x_1, x_2, \dots, x_s\}$ equipped with an involution $x_i \mapsto x_{i'}$, and real structure constants p_{ij}^k with

$$x_i x_j = \sum_{k=1}^{s} p_{ij}^k x_k$$

for $1 \le i, j \le s$, such that:

(a) $\forall\ 1 \le i, j, k \le s\,,\ p_{i'j'}^{k'} = p_{ij}^k$;
(b) $\forall\ 1 \le i \le s\,,\ \exists\ \kappa_i > 0\,.\ \forall\ 1 \le j \le s\,,\ p_{ij}^1 = \delta_{ij'}\kappa_i$; and
(c) $A \to \mathbb{R} \colon x_i \mapsto \kappa_i$ is a representation of A.

A character algebra A is *of nonnegative type* whenever the structure constants p_{ij}^k are all nonnegative.

2.1.2. *Fusion algebras.* There are three flavors of these algebras, which were formalized in [2] as "fusion algebras at [the] algebraic level".

Definition 2.2. A finite-dimensional, commutative, associative, unital algebra A over $\mathbb{C}$ is a *fusion algebra* if it has a basis $\{1 = x_1, x_2, \ldots, x_s\}$ equipped with an involution $x_i \mapsto x_{i'}$, real structure constants N_{ij}^k with

$$x_i x_j = \sum_{k=1}^{s} N_{ij}^k x_k$$

for $1 \leq i, j \leq s$, and positive constants ν_i for $1 \leq i \leq s$, such that:

(a) $\forall\ 1 \leq i, j, k \leq s\ ,\ N_{i'j'}^{k'} = N_{ij}^k\ ;$
(b) $\forall\ 1 \leq i, j, k \leq s\ ,\ \forall\ \sigma \in \{i, j, k\}!\ ,\ N_{\sigma(i)\sigma(j)}^{\sigma(k)'} = N_{ij}^{k'}\ ;$
(c) $A \to \mathbb{R} \colon x_i \mapsto \sqrt{\nu_i}$ is a representation of A.

A fusion algebra A is *of nonnegative type* if its structure constants N_{ij}^k are all nonnegative. The algebra is *integral* if its structure constants N_{ij}^k are natural numbers.

2.2. **Association schemes and their duals.** Within the literature, various notions of "association scheme" and related concepts (such as coherent configurations) have appeared. Here, we follow [3, Ch. 2], but with notational conventions that are better suited to our algebraic context.[1]

2.2.1. *Association schemes.*

Definition 2.3. Let Q be a finite, nonempty set. A (*commutative*) *association scheme* (Q, Γ) on Q is a disjoint union partition

$$Q \times Q = C_1 + \ldots + C_s$$

or $\Gamma = \{C_1, \ldots, C_s\}$ of $Q \times Q$ such that the following axioms are satisfied:

(A1) $C_1 = \widehat{Q} = \{(x, x) \mid x \in Q\}\,;$
(A2) The converse of each relation in Γ belongs to $\Gamma\,;$
(A3) $\forall\, C_i \in \Gamma\,,\ \forall\, C_j \in \Gamma\,,\ \forall\, C_k \in \Gamma\,,\ \exists\, c_{ij}^k \in \mathbb{N}\,.\, \forall\, (x, y) \in C_k\,,$

$$\big|\{z \in Q \mid (x, z) \in C_i,\ (z, y) \in C_j\}\big| = c_{ij}^k\,;$$

(A4) $\forall\, 1 \leq i, j, k \leq s\,,\ c_{ij}^k = c_{ji}^k$.

Axiom (A4) is the *commutativity* of the scheme.

[1]Conversely, it will be apparent from the first paragraph of §5.2.1 that the original notational conventions of [3, §2.2] may be more natural in combinatorial contexts.

2.2.2. *The Bose-Mesner algebra.* Let (Q, Γ) be an association scheme, with $|Q| = n$. Consider complex $n \times n$-matrices with rows and columns indexed by Q. For $1 \le i \le s$, let A_i be the incidence matrix of C_i, defined by

$$[A_i]_{xy} = \begin{cases} 1 & \text{if } (x,y) \in C_i\,; \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in Q$. The *Bose-Mesner algebra* of the association scheme (Q, Γ) is the s-dimensional complex linear span of the linearly independent set $\{I_n = A_1, \dots, A_n\}$ of matrices [3, §2.3]. With respect to this spanning set as basis, the Bose-Mesner algebra forms a character algebra, with c_{ij}^k as its structure constants. We define the *valencies* n_i as the κ_i of Definition 2.1(b). Then for each x in Q, one has $n_i = \big|\{y \mid (x,y) \in C_i\}\big|$ for $1 \le i \le s$.

2.2.3. *Dual schemes.* As a character algebra, the s-dimensional Bose-Mesner algebra is commutative. It follows that the algebra has a basis $\{\frac{1}{n}J_n = E_1, \dots, E_s\}$ of orthogonal idempotents, where J_n is the $n \times n$ all-ones matrix, the incidence matrix of $Q \times Q$. Viewing $\{A_1, \dots, A_s\}$ as an algebraic implementation of the original association scheme (Q, Γ), it is convenient to describe $\widetilde{\Gamma} = \{nE_1, \dots, nE_s\}$ as constituting the *dual scheme* $(Q, \widetilde{\Gamma})$, even though the matrices nE_i are not incidence matrices of relations on Q in general, for $i > 1$. The set $\widetilde{\Gamma}$ is the basis of a character algebra structure, of nonnegative type, on (the underlying $\mathbb{C}$-space of) the Bose-Mesner algebra, using the entrywise or *Hadamard* product $\star$ of matrices. The structure constants $\tilde{c}_{ij}^k$ defined by

$$(nE_i) \star (nE_j) = \sum_{k=1}^{s} \tilde{c}_{ij}^k (nE_k) \tag{2.1}$$

are known as the *Krein parameters* [2, Ex. 2.3(b)], [3, 2,(3.12)]. The κ_i of Definition 2.1(b) are the traces f_i of the idempotents E_i in this case, known as *multiplicities*.

2.3. **Quasigroups.**

2.3.1. *Quasigroups and multiplication groups.* A *quasigroup*, written as Q, $(Q, \cdot)$, or $(Q, \cdot, /, \backslash)$, is a set Q equipped with three binary operations of multiplication (written as $\cdot$ or mere juxtaposition of arguments), *right division* $/$, and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{llll} \text{(IL)} & y\backslash(y \cdot x) = x\,; & \text{(IR)} & x = (x \cdot y)/y\,; \\ \text{(SL)} & y \cdot (y\backslash x) = x\,; & \text{(SR)} & x = (x/y) \cdot y\,. \end{array}$$

Note the left-right symmetry of these identities. For an element q of a quasigroup $(Q,\cdot)$, define the *left multiplication* $L.(q)$ or

$$L(q)\colon Q \to Q; x \mapsto q \cdot x$$

and *right multiplication* $R.(q)$ or

$$R(q)\colon Q \to Q; x \mapsto x \cdot q\,.$$

By (IL), the left multiplications are injective, while by (SL), they are surjective. Similarly, the right multiplications are bijective. Then the *multiplication group* $\operatorname{Mlt} Q$ or G of a quasigroup Q is the subgroup of the permutation group $Q!$ on Q that is generated by all the left and right multiplications. By (SL), it is apparent that G acts transitively on Q.

Example 2.4. Each group Q is a quasigroup, with $x/y = xy^{-1}$ and $x\backslash y = x^{-1}y$. While the multiplication satisfies the associative law, the divisions do not. The multiplication group G of a group Q is given by the exact sequence

$$\{1\} \longrightarrow Z(Q) \xrightarrow{D} Q \times Q \xrightarrow{T} G \longrightarrow \{1\}$$

with $D\colon z \mapsto (z,z)$ as the diagonal embedding of the center $Z(Q)$, and $T\colon (q,r) \mapsto L(q^{-1})R(r)$. Note that the inner automorphism group of Q is the image under T of the diagonal subgroup $\widehat{Q}$ of Q.

2.3.2. *Quasigroup conjugacy classes.* Let Q be a nonempty quasigroup, with multiplication group G. The group G acts on $Q \times Q$ with the *diagonal action*

$$g\colon (q_1, q_2)g \mapsto (q_1 g, q_2 g)$$

for q_1, q_2 in Q and g in G. The orbits of this action are defined as the (*quasigroup*) *conjugacy classes* of Q. Since G acts transitively on Q, one class is the diagonal $\widehat{Q} = C_1$, the relation $\{(q_1, q_2) \mid q_1 = q_2\}$ of equality on Q. There is a finite set

$$\Gamma = \{\widehat{Q} = C_1, C_2, \dots, C_s\} \tag{2.2}$$

of conjugacy classes, partitioning $Q \times Q$.

Example 2.5. In the context of Example 2.4, with e as the identity element of the group Q, define $C_i(e) = \{y \in Q \mid (e,y) \in C_i\}$ for $1 \le i \le s$. Then $\{C_i(e) \mid 1 \le i \le s\}$ is the set of group conjugacy classes of Q.

2.3.3. *Quasigroup schemes.* Let Q be a finite quasigroup. Then with Γ as in (2.2), (Q, Γ) forms an association scheme [15], [28, Th. 6.3]. Schemes of this type are known as *quasigroup conjugacy class schemes*, and as *group conjugacy class schemes* (or "group association schemes" [2, Ex. 2.2]) if Q is a group. It is known that certain association schemes cannot be implemented as quasigroup conjugacy class schemes [16], [28, §8.2]. An example—the Johnson scheme $J(5, 2)$—is discussed in §5.2.1.

2.3.4. *Character tables.* The two bases $\{A_1, \dots, A_s\}$ and $\{E_1, \dots, E_s\}$ of the Bose-Mesner algebra of the quasigroup conjugacy class scheme (Q, Γ) of a finite nonempty quasigroup Q stand in the mutually inverse relationships

$$A_i = \sum_{j=1}^{s} \xi_{ij} E_j \quad \text{and} \quad E_i = \sum_{j=1}^{s} \eta_{ij} A_j \, .$$

Then the character table Ψ of Q is the $s \times s$ matrix $[\psi_{ij}]$ with

$$\psi_{ij} = \frac{\sqrt{f_i}}{n_j} \, \xi_{ji} = \frac{n}{\sqrt{f_i}} \, \overline{\eta}_{ij} \, , \tag{2.3}$$

in terms of the valencies n_j of the scheme (Q, Γ) and multiplicities f_i of the dual scheme $(Q, \widetilde{\Gamma})$ [15], [28, Defn. 6.3].

The formula (2.3) was exhibited by Hoheisel for groups Q [11]. In the group case, the Krein parameters $\tilde{c}_{ij}^k$ in (2.1) give the multiplicities of the k-th irreducible character χ_k in the product $\chi_i \cdot \chi_j$ [2, Exs. 1.1, 3.1]. While these coefficients are usually considered as part of the structure of the character ring of the group Q, they will receive an interpretation in terms of quasigroups in §5.4.

3. Augmented magmas

3.1. Compact closed categories.

Definition 3.1. [6, 19] A symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is said to be a *compact closed category* if it has:

(a) a contravariant *duality functor* ${}^* \colon \mathbf{V} \to \mathbf{V}$;
(b) a natural transformation $\mathrm{ev}_A \colon A \otimes A^* \to \mathbf{1}$ of *evaluation*, and
(c) a natural transformation $\mathrm{coev}_A \colon \mathbf{1} \to A^* \otimes A$ of *coevaluation*

such that the composites

$$A \xrightarrow{1_A \otimes \mathrm{coev}} A \otimes A^* \otimes A \xrightarrow{\mathrm{ev} \otimes 1_A} A \tag{3.1}$$

and

$$A^* \xrightarrow{\mathrm{coev} \otimes 1_A} A^* \otimes A \otimes A^* \xrightarrow{1_A \otimes \mathrm{ev}} A^* \tag{3.2}$$

reduce to 1_A and 1_{A^*} respectively for each object A of $\mathbf{V}$.

The order of the tensor product factors appearing in Definition 3.1(b) is chosen to match our default preference for algebraic notation. For the following, compare [19, p.193].

Lemma 3.2. *There is a natural isomorphism with components*

$$\phi_{A,B,C} \colon \mathbf{V}(B \otimes A, C) \to \mathbf{V}(B, C \otimes A^*)$$

at objects A, B, C *of a compact closed category* $(\mathbf{V}, \otimes, \mathbf{1})$.

Example 3.3. Let R be a commutative, unital ring. Let $\underline{\underline{R}}$ be the category of finitely-generated free modules over R. Write RX for the free module over a finite set X. Then with $\mathbf{1} = R$, the free module over $\{1\}$, and with the usual tensor product of R-modules, we have a symmetric monoidal category $(\underline{\underline{R}}, \otimes, R)$. This category is compact closed, with $A^* = \underline{\underline{R}}(A, R)$ for an object A. The evaluation is given by

$$\mathrm{ev} \colon A \otimes A^* \to R; a \otimes \alpha \mapsto a\alpha \,,$$

the usual evaluation of functionals written with algebraic notation. If X is a finite set, then

$$\mathrm{coev} \colon R \to RX^* \otimes RX; 1 \mapsto \sum_{x \in X} \delta_x \otimes x \,,$$

with $y\delta_x = \delta_{yx}$ for $x, y \in X$, gives the coevaluation.

Example 3.4. If S is a commutative, unital semiring, for example the semiring $(\mathbb{N}, +, 0, \cdot, 1)$ of natural numbers, then the category $\underline{\underline{S}}$ of finitely-generated free semimodules over the semiring S is a compact closed category. Since subtraction was not needed in Example 3.3, all the notation and constructions of that example carry over. Let $\underline{\underline{\mathbb{N}}}$, in particular, denote the category of finitely generated free semimodules over $(\mathbb{N}, +, 0, \cdot, 1)$, the category of free commutative monoids.

Example 3.5. Consider the category **Rel** of relations between sets, with relation product $\circ$ as the composition of morphisms. There is a symmetric monoidal category structure $(\mathbf{Rel}, \otimes, \top)$, with the Cartesian product of sets taken as the tensor product. To match this notation, write an ordered pair (x, y) as $x \otimes y$, saving the usual Cartesian product notation for the specification of relations. Take $\top = \{0\}$.

The symmetric monoidal category $(\mathbf{Rel}, \otimes, \top)$ forms a compact closed category, with $A^* = A$ for each set A. The evaluation $\mathrm{ev}_A \colon A \otimes A^* \to \top$ is the relation

$$\{(a \otimes a, 0) \mid a \in A\} \,.$$

The coevaluation $\mathrm{coev}_A \colon \top \to A \otimes A^*$ is the converse relation

$$\{(0, a \otimes a) \mid a \in A\} .$$

Verification of the relation (3.1) reduces to the observation that the relation product of the relation $\{(a, a \otimes b \otimes b) \mid a, b \in A\}$ with the relation $\{(a \otimes a \otimes b, b) \mid a, b \in A\}$ is the identity relation $\{(a, a) \mid a \in A\}$ on A. Verification of (3.2) is similar.

3.2. **Augmented magmas.** Suppose $(\mathbf{V}, \otimes, \mathbf{1})$ is a compact closed category, with $\tau \colon A \otimes B \to B \otimes A; a \otimes b \mapsto b \otimes a$ for objects A, B of $\mathbf{V}$. Note that, although the category $\mathbf{V}$ need not be concrete, so that there may be no "elements" a of A or b of B, languages of Jay's type [13] enable one to employ definitions like this in a symbolic sense. Such conventions are used at various places in the paper.

Definition 3.6. An *augmented magma* $(A, \mu, \Delta, \varepsilon)$ in $(\mathbf{V}, \otimes, \mathbf{1})$ is an object A of $\mathbf{V}$, equipped with:

(a) a *multiplication* (*structure*) $\mu \colon A \otimes A \to A^*$;
(b) a *comultiplication* $\Delta \colon A \to A \otimes A$, and
(c) an *augmentation* $\varepsilon \colon A \to \mathbf{1}$

such that the diagram

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{\mathrm{coev}_A \otimes \mu} & A^* \otimes A \otimes A^* & \xrightarrow{1_{A^*} \otimes \Delta \otimes 1_{A^*}} & A^* \otimes A \otimes A \otimes A^* \\ {\scriptstyle \varepsilon \otimes \varepsilon} \downarrow & & & & \downarrow {\scriptstyle \tau \otimes \mathrm{ev}_A} \\ \mathbf{1} & & \xleftarrow[\mathrm{ev}_A]{} & & A \otimes A^* \end{array} \tag{3.3}$$

commutes.

Example 3.7. In any compact closed category $\mathbf{V}$, there is a *trivial* counital magma $(\mathbf{1}, \mu, \Delta, \varepsilon)$.

Example 3.8. Let R be a commutative unital ring. Consider the compact closed category $\underline{\underline{R}}$ defined in Example 3.3. Let G be a finite group. Consider the group algebra RG with the usual Hopf algebra structure $(RG, \nabla, \eta, \Delta, \varepsilon, S)$ [23, Ex. 1.6]. Define

$$\mu \colon RG \otimes RG \to RG^*; g \otimes h \mapsto [\delta_{gh} \colon x \mapsto \delta_{x,gh}] \tag{3.4}$$

with $g, h, x \in G$. Then $(RG, \mu, \Delta, \varepsilon)$ is an augmented magma in $\underline{\underline{R}}$. Indeed, for $g, h \in G$,

$$\begin{array}{ccccc} g \otimes h & \longmapsto & \sum_{x \in G} \delta_x \otimes x \otimes \delta_{gh} & \longmapsto & \sum_{x \in G} \delta_x \otimes x \otimes x \otimes \delta_{gh} \\ \downarrow & & & & \downarrow \\ 1 = \delta_{gh,gh} & & \longleftarrow & & \sum_{x \in G} \delta_{x,gh}(x \otimes \delta_x) \end{array}$$

gives the chase round the diagram (3.3).

In the case where R is the ring $\mathbb{Z}$ of integers, the counit ε of the Hopf algebra $(\mathbb{Z}G, \nabla, \eta, \Delta, \varepsilon, S)$ is the augmentation $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ of the ring $\mathbb{Z}G$. This is the reason for the use of the term "augmentation" in Definition 3.6(c).

3.3. **Hypermagmas.** In this section, it will be shown how magmas $(A, \cdot)$, and their set-valued analogues, appear as augmented magmas in the compact closed category $(\mathbf{Rel}, \otimes, \top)$ of relations on sets. The conventions of Example 3.5 are used. Set-valued magmas are defined as follows.

Definition 3.9. [21, Defn. 6.1(a)] A *hypermagma* $(A, \diamond)$ is a set A with a function

$$A \times A \to 2^A; (x, y) \mapsto x \diamond y$$

such that the product $x \diamond y$ is nonempty for all $x, y \in H$.

The specific goals are to show how the nonemptiness condition of Definition 3.9 is captured naturally by the augmented magma condition (3.3), and how augmented magmas in $(\mathbf{Rel}, \otimes, \top)$ serve equally well to capture both magmas and hypermagmas, despite the "type difference" whereby the product is element-valued in the former, and set-valued in the latter. We begin by setting up structure in $(\mathbf{Rel}, \otimes, \top)$.

Definition 3.10. Let A be a set with a function

$$A \times A \to 2^A; (x, y) \mapsto x \diamond y\,.$$

(a) Define a multiplication structure, or in this case a *multiplication relation* $\mu\colon A \otimes A \to A^*$, as

$$\{(x \otimes y, (z, 0)) \mid x, y, z \in A,\ z \in x \diamond y\}$$

or simply

$$\{(x \otimes y, z) \mid x, y, z \in A,\ z \in x \diamond y\}\,.$$

(b) Define a comultiplication $\Delta\colon A \to A \otimes A$ as the *diagonal* relation

$$\{(x, x \otimes x) \mid x \in A\}\,.$$

(c) Define augmentation $\varepsilon\colon A \to \top$ as the relation $\{(x, 0) \mid x \in A\}$.

This defines the structure $(A, \mu, \Delta, \varepsilon)$ in $(\mathbf{Rel}, \otimes, \top)$.

Proposition 3.11. *Suppose that A is a set with a function*

$$A \times A \to 2^A; (x, y) \mapsto x \diamond y\,.$$

Then $(A, \diamond)$ is a hypermagma if and only if the structure $(A, \mu, \Delta, \varepsilon)$ of Definition 3.10 is an augmented magma in $(\mathbf{Rel}, \otimes, \top)$.

Proof. Using the simplified version of the multiplication relation, the successive relations and relation products that constitute the augmented magma condition (3.3) are:

$$\begin{aligned}
\mathrm{coev}_A \otimes \mu &= \{(x \otimes y, a \otimes a \otimes z) \mid a, x, y, z \in A\,,\ z \in x \diamond y\}\,, \\
1_{A^*} \otimes \Delta \otimes 1_{A^*} &= \{(a \otimes b \otimes z, a \otimes b \otimes b \otimes z) \mid a, b, z \in A\}\,, \\
(\mathrm{coev}_A \otimes \mu) &\circ (1_{A^*} \otimes \Delta \otimes 1_{A^*}) \\
&= \{(x \otimes y, a \otimes a \otimes a \otimes z) \mid a, x, y, z \in A\,,\ z \in x \diamond y\}\,,
\end{aligned}$$

$$\begin{aligned}
\tau \otimes \mathrm{ev}_A &= \{(a \otimes b \otimes z \otimes z, b \otimes a) \mid a, b, z \in A\}\,, \\
(\mathrm{coev}_A \otimes \mu) &\circ (1_{A^*} \otimes \Delta \otimes 1_{A^*}) \circ (\tau \otimes \mathrm{ev}_A) \\
&= \{(x \otimes y, z \otimes z) \mid x, y, z \in A\,,\ z \in x \diamond y\}\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{ev}_A &= \{(z \otimes z, 0) \mid z \in A\}\,, \\
(\mathrm{coev}_A \otimes \mu) &\circ (1_{A^*} \otimes \Delta \otimes 1_{A^*}) \circ (\tau \otimes \mathrm{ev}_A) \circ (\mathrm{ev}_A) \\
&= \{(x \otimes y, 0) \mid x, y \in A\,,\ \exists\ z \in x \diamond y\}\,, \qquad (3.5)
\end{aligned}$$

and

$$\varepsilon \otimes \varepsilon = \{(x \otimes y, 0) \mid x, y \in A\}\,. \qquad (3.6)$$

Thus the agreement between (3.5) and (3.6), i.e., the commuting of (3.3) that makes $(A, \mu, \Delta, \varepsilon)$ into an augmented magma, is equivalent to the hypermagma condition that each $x \diamond y$ be nonempty. □

Example 3.12. A magma $(A, \cdot)$ may first be taken as a hypermagma $(x, y) \mapsto \{x \cdot y\}$, and then interpreted as an augmented magma in $(\mathbf{Rel}, \otimes, \top)$ according to the construction of Definition 3.10. Thus the multiplication relation is

$$\{(x \otimes y, x \cdot y) \mid x, y \in A\}\,,$$

which is just the function $A \times A \to A; (x, y) \mapsto x \cdot y$. It transpires that the augmented magma structure $(A, \mu, \Delta, \varepsilon)$ all lies in the symmetric monoidal category $(\mathbf{Set}, \times, \top)$, although the condition (3.3) moves into the compact closed structure of $(\mathbf{Rel}, \otimes, \top)$.

3.4. **Multisets.** Write $\mathbb{N}^+$ for the set of positive integers. A function $w\colon X \to \mathbb{N}^+$ is taken to represent a *multiset*, with X as its set of elements (compare [30, §I.1.5]). The image $w(x)$ of an element x of X is its *weight*. The multiset $w\colon X \to \mathbb{N}^+$ is *finite* if its domain X is finite. In that case, the number $|X|$ is known as the *tare weight* of the multiset, while $\sum_{x \in X} w(x)$ is known as its *gross weight*.

Example 3.13. Let (Q,Γ) be an association scheme.

(a) Assigning a weight $w(C_i) = |C_i|\big/|Q|$ to each class C_i yields a multiset $w\colon \Gamma \to \mathbb{N}^+$. In this case, $w(C_i) = n_i$, the usual valency of the relation C_i.
(b) Assigning a weight $w'(C_i) = |C_i|$ to each class C_i again yields a multiset $w\colon \Gamma \to \mathbb{N}^+$.
(c) Assigning a weight $w(E_i) = f_i = \mathrm{Tr}(E_i)$ to each idempotent E_i yields a multiset $w\colon \widetilde{\Gamma} \to \mathbb{N}^+$.

Consider the compact closed category $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$ that was introduced in Example 3.4, the category of finitely generated free commutative monoids or free semimodules over the semiring $\mathbb{N}$. The weight function $w\colon X \to \mathbb{N}^+$ of a multiset X, taken with enlarged codomain $\mathbb{N}$, extends freely to an $\underline{\underline{\mathbb{N}}}$-morphism

$$\varepsilon\colon \mathbb{N}X \to \mathbb{N}; x \mapsto w(x)\,, \tag{3.7}$$

an element of $\mathbb{N}X^*$. The goal is to find an abstract characterization of the structures $(\mathbb{N}X, \varepsilon)$ in $\underline{\underline{\mathbb{N}}}$ that are obtained in this fashion, with ε as in (3.7), from a finite multiset $w\colon X \to \mathbb{N}^+$.

The following definition works in the context of Example 3.4.

Definition 3.14. Let S be a commutative unital semiring. Consider an arbitrary object $A = SX$ of $\underline{\underline{S}}$, the free S-semimodule over a finite set X.

(a) The $\underline{\underline{S}}$-morphism
$$\Delta\colon A \to A \otimes A; x \mapsto x \otimes x$$
is known as the *diagonal comultiplication*.
(b) An $\underline{\underline{S}}$-morphism $\varepsilon\colon A \to S$ is described as an *augmentation*.
(c) The structure (A, Δ, ε) is called an *augmented comagma* in $\underline{\underline{S}}$.
(d) If $w\colon X \to \mathbb{N}^+$ is a multiset, then the augmented comagma $(\mathbb{N}X, \Delta, \varepsilon)$, with ε as in (3.7), is the *multiset augmented comagma* of $w\colon X \to \mathbb{N}^+$ in $\underline{\underline{N}}$.
(e) An element a of A is said to be *grouplike* if $a\Delta = a \otimes a$ and $a\varepsilon \neq 0$.
(f) In an augmented comagma (A, Δ, ε), write A_0 for the set of grouplike elements.
(g) An augmented comagma (A, Δ, ε) in $\underline{\underline{\mathbb{N}}}$ is described as being *multisetlike* if $A = \mathbb{N}A_0$.

Example 3.15. Let (A, Δ, ε) be an augmented comagma in $\underline{\underline{\mathbb{N}}}$ for which the semimodule A is nontrivial, but the augmentation is the zero morphism $\varepsilon\colon A \to \mathbb{N}$. Then A_0 is empty, so (A, Δ, ε) is not multisetlike.

Lemma 3.16. *Suppose that S is a commutative unital semiring, and that $a = \sum_{x\in I} e_x x$ is a grouplike element of an augmented comagma structure $(SX, \Delta, \varepsilon)$ over a set X. Then $\{e_x \mid x \in I\}$ is a set of mutually orthogonal idempotents in S.*

Proof. The condition $a\Delta = a \otimes a$ translates to $a\Delta =$

$$\sum_{x\in I} e_x(x \otimes x) = \sum_{x\in I} e_x x \otimes \sum_{y\in I} e_y y = \sum_{x\in I}\sum_{y\in I} e_x e_y (x \otimes y)\,. \tag{3.8}$$

Equating coefficients of $x \otimes y$ in (3.8) for $x \neq y$ yields $e_x e_y = 0$. Equating coefficients of $x \otimes y$ in (3.8) for $y = x$ yields $e_x = e_x e_x$. □

Proposition 3.17. *Let (A, Δ, ε) be an augmented comagma in $\underline{\underline{\mathbb{N}}}$. Then the following two conditions are equivalent:*

(a) *The augmented comagma (A, Δ, ε) is multisetlike.*
(b) *The augmentation ε restricts to a multiset $\varepsilon\colon A_0 \to \mathbb{N}^+$ whose multiset augmented comagma in $\underline{\underline{N}}$ is (A, Δ, ε).*

Proof. The implication (b) ⇒ (a) is immediate by definition. For the converse, suppose that (A, Δ, ε) is a multisetlike augmented comagma, with $A = \mathbb{N}X$ for a finite generating set X. Let $a = \sum_{x\in X} e_x x$ be a grouplike element of A. Then by Lemma 3.16, $\{e_x \mid x \in X\}$ is a set of orthogonal idempotents. Since $\varepsilon(a) = \sum_{x\in X} e_x \varepsilon(x)$ is nonzero, there is an element x of X for which e_x is a nonzero idempotent of $\mathbb{N}$, namely 1. For $x \neq y \in X$, the orthogonality $e_x e_y = 0$ implies that $e_y = 0$, so that $a = x$. Thus $A_0 \subseteq X$. Since $A = \mathbb{N}A_0$, it follows that $A_0 = X$. The augmentation restricts appropriately, by the second part of Definition 3.14(e). □

Definition 3.18. Let (A, Δ, ε) be a multisetlike augmented comagma in $\underline{\underline{\mathbb{N}}}$. If $\varepsilon = \sum_{a\in A_0} \delta_a$, then (A, Δ, ε) is *setlike.*

Note that in a setlike multiset, the weight of each element is 1. Thus the results of this section enable us to conflate multisetlike objects with finite multisets, and setlike objects with finite sets.

3.5. **Lifting and covering.** Working in the category $\underline{\underline{N}}$, suppose an object $(\underline{\underline{N}}Q, \Delta, \varepsilon_Q)$ is setlike and an object $(\underline{\underline{N}}X, \Delta, \varepsilon_X)$ is multisetlike.

Definition 3.19. The multisetlike object $(\underline{\underline{N}}X, \Delta, \varepsilon_X)$ *lifts* to the setlike object $(\underline{\underline{N}}Q, \Delta, \varepsilon_Q)$, or the latter *covers* the former, if there is a surjective *covering* function $f\colon Q \to X$ for which the relation

$$\varepsilon_X = \sum_{x\in X}\Big(\sum_{q\in Q} qf\delta_x\Big)\delta_x \tag{3.9}$$

holds.

For an element x of X, the weight $\varepsilon_X(x)$ assigned by the formula (3.9) to an element x of X is the cardinality $|f^{-1}\{x\}|$ of the preimage $f^{-1}\{x\}$ of x in Q. Indeed, the function $f\delta_x$ is the characteristic function of the preimage as a subset of Q. In a reinforced interpretation of the First Isomorphism Theorem for sets [30, Th. O.3.3.1], the multiset X is identified as a quotient of the set Q by an equivalence relation on Q.

3.6. **Weighted magmas.** The following definition relates to notation from [9, §1], which will reappear more fully in §4.3.

Definition 3.20. Let $w\colon X \to \mathbb{N}^+$ be a finite multiset. Then a *weighted magma* structure (X, w, α) on X is given by a *multiplication function*

$$\alpha\colon X \times X \times X \to \mathbb{N}; (a, b, x) \mapsto \alpha_x(a, b) \tag{3.10}$$

such that the condition

$$\forall\ a, b \in X\ ,\ \sum_{x\in X} \alpha_x(a, b) = w(a)w(b) \tag{3.11}$$

is satisfied. The multiset terminology of §3.4 (gross weight, etc.) carries over to weighted magmas.

Proposition 3.21. *Let $(A, \cdot)$ be a finite magma, with multiplication also written simply as juxtaposition. Construe the set A as the multiset $\varepsilon_A\colon A \to \{1\}$. Then the magma structure $(A, \cdot)$ is equivalent to the weighted magma structure $(A, \varepsilon_A, \delta)$, where*

$$\delta\colon A \times A \times A \to \mathbb{N}; (a, b, x) \mapsto \delta_{x,ab}$$

is the multiplication function.

Proof. Note that for elements x, a, b of A, the equation $a \cdot b = x$ holds in $(A, \cdot)$ if and only if the equation $\delta_x(a, b) = 1$ holds in (A, w, δ). $\square$

3.6.1. *Rescaling.* The content of this paragraph is inspired by [9, p.163], where the same idea appears in a slightly different context.

Definition 3.22. Let (X, w, α) be a weighted magma structure on a set X. Let k be a positive integer. Then

$$w'\colon X \to \mathbb{N}^+; x \mapsto kw(x)$$

and

$$\alpha'\colon X \times X \times X \to \mathbb{N}; (a, b, x) \mapsto k^2\alpha_x(a, d)$$

constitute a *rescaling* of (X, w, α).

Proposition 3.23. *Let (X, w, α) be a weighted magma structure on a set X. Let k be a positive integer. Then the rescaled version (X, w', α') of (X, w, α) is a weighted magma.*

Proof. Multiplying the weighted magma condition (3.11) for (X, w, α) yields

$$\sum_{x\in X} \alpha'_x(a,b) = \sum_{x\in X} k^2\alpha_x(a,b) = kw(a)kw(b) = w'(a)w'(b)\,,$$

which is the weighted magma condition for (X, w', α'). □

3.6.2. *Weighted magmas and augmented magmas.* The multiplication function (3.10) of a weighted magma extends freely to an $\underline{\underline{\mathbb{N}}}$-morphism

$$\alpha\colon \mathbb{N}X \otimes \mathbb{N}X \otimes \mathbb{N}X \to \mathbb{N};\, a\otimes b\otimes x \mapsto \alpha_x(a,b)\,,$$

which in turn corresponds to the $\underline{\underline{\mathbb{N}}}$-morphism

$$\alpha\phi_{\mathbb{N}X,\mathbb{N}X\otimes\mathbb{N}X,\mathbb{N}}\colon \mathbb{N}X \otimes \mathbb{N}X \to \mathbb{N}X^*;\, a\otimes b \mapsto [x \mapsto \alpha_x(a,b)]$$

according to Lemma 3.2.

Proposition 3.24. *A weighted magma* (X, w, α) *is equivalent to an augmented magma* $(\mathbb{N}X, \alpha\phi_{\mathbb{N}X,\mathbb{N}X\otimes\mathbb{N}X,\mathbb{N}}, \Delta, \varepsilon)$ *in the category* $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$, *where* $(\mathbb{N}X, \Delta, \varepsilon)$ *is a multisetlike augmented comagma.*

Proof. Suppose that (X, w, α) is a weighted magma. Then for $a, b \in X$,

$$\begin{array}{ccc}
a\otimes b & \longmapsto & \sum_{x\in X}\delta_x\otimes x\otimes[x\mapsto\alpha_x(a,b)] \\
\Big\downarrow & & \Big\downarrow \\
 & & \sum_{x\in X}\delta_x\otimes x\otimes x\otimes[x\mapsto\alpha_x(a,b)] \\
 & & \Big\downarrow \\
w(a)w(b)=\sum_{x\in X}\alpha_x(a,b) & \longleftarrow & \sum_{x\in X}\alpha_x(a,b)(x\otimes\delta_x)
\end{array}$$

gives the chase round the diagram (3.3) that verifies the augmented comagma condition. Conversely, suppose that $(\mathbb{N}X, \alpha\phi_{\mathbb{N}X,\mathbb{N}X\otimes\mathbb{N}X,\mathbb{N}}, \Delta, \varepsilon)$ is an augmented magma, where $(\mathbb{N}X, \Delta, \varepsilon)$ is a multisetlike augmented comagma. The augmentation of the latter structure restricts to yield a multiset $w\colon X \to \mathbb{N}^+$. Then the equality in the bottom left of the commuting chase establishes that (X, w, α) is a weighted magma. □

3.6.3. *Covering magmas.* The following provides a first reformulation of the *amalgamation* concept from [9, pp.142–3]. The concept will be revisited in §5.1.

Proposition 3.25. *Let* $(A, \cdot)$ *be a magma, yielding the weighted magma structure* $(A, \varepsilon_A, \delta)$ *of Proposition 3.21. Let* $f\colon (A, \Delta, \varepsilon_A) \to (X, \Delta, w)$ *be a covering. Then*

$$\alpha\colon X\times X\times X \to \mathbb{N};\, (x,y,z) \mapsto \big|\{(a,b)\in f^{-1}\{x\}\times f^{-1}\{y\} \mid (ab)f = z\}\big|$$

is the multiplication function of a weighted magma (X, w, α).

Proof. The weighted magma condition (3.11) amounts to

$$\sum_{z\in X} \alpha_z(x,y) = w(x)w(y) = |f^{-1}\{x\} \times f^{-1}\{y\}| .$$

It holds since the left hand side counts the number of elements of $f^{-1}\{x\} \times f^{-1}\{y\}$ whose magma product in $(A, \cdot)$ lies in some arbitrary preimage $f^{-1}\{z\}$ of an element z of X. $\square$

Definition 3.26. In the context of Proposition 3.25, the weighted magma is said to *lift* to, or be *covered* by, or be an *amalgamation* of, the magma $(A, \cdot)$.

3.7. **Association schemes as augmented magmas.**

Theorem 3.27. *Let* (Q, Γ) *be an association scheme. Then a weighted magma structure* (Γ, w, α) *is given by*

$$\alpha\colon \Gamma \times \Gamma \times \Gamma \to \mathbb{N}; (C_i, C_j, C_k) \mapsto c_{ij}^k n_k \tag{3.12}$$

on the multiset $w\colon \Gamma \to \mathbb{N}^+; C_i \mapsto n_i = |C_i| \big/ |Q|$ *of Example 3.13(a).*

Proof. For $1 \le i \le s$, let A_i be the incidence matrix of the subset C_i of $Q \times Q$. For C_i, C_j in Γ, the equation

$$A_i A_j = \sum_{k=1}^{s} c_{ij}^k A_k \tag{3.13}$$

holds by virtue of Definition 2.3(A3). An application of the Bose-Mesner algebra representation $A \mapsto AE_1$ to (3.13) then yields

$$n_i n_j = \sum_{k=1}^{s} c_{ij}^k n_k \quad \text{ or } \quad w(C_i)w(C_j) = \sum_{k=1}^{s} \alpha_{C_k}(C_i, C_j) ,$$

verifying the weighted magma condition (3.11). $\square$

Corollary 3.28. *Let* (Q, Γ) *be an association scheme of order* $|Q|$. *Then a weighted magma structure* (Γ, w', α') *is given by*

$$\alpha'\colon \Gamma \times \Gamma \times \Gamma \to \mathbb{N}; (C_i, C_j, C_k) \mapsto |Q| \cdot |C_k| c_{ij}^k \tag{3.14}$$

on the multiset $w'\colon \Gamma \to \mathbb{N}^+; C_i \mapsto |C_i|$ *of Example 3.13(b).*

Proof. The weighted magma structure here is obtained, according to Proposition 3.23, as the weighted magma structure of Theorem 3.27 rescaled by a factor of $|Q|$. $\square$

Application of Proposition 3.24 to the weighted magmas constructed in Theorem 3.27 and Corollary 3.28 yields the following.

Corollary 3.29. *Two augmented magma structures* $(\mathbb{N}\Gamma, \mu, \Delta, \varepsilon)$ *in the compact closed category* $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$ *are furnished by each association scheme* (Q, Γ)*.*

(a) *The first has multiplication structure*

$$\mu\colon \mathbb{N}\Gamma \otimes \mathbb{N}\Gamma \to \mathbb{N}\Gamma^*; C_i \otimes C_j \mapsto \left[C_k \mapsto c_{ij}^k n_k\right]$$

and an augmentation

$$\varepsilon\colon \mathbb{N}\Gamma \to \mathbb{N}; C_i \mapsto n_i \tag{3.15}$$

known as the valency augmentation*.*

(b) *The second has multiplication structure*

$$\mu\colon \mathbb{N}\Gamma \otimes \mathbb{N}\Gamma \to \mathbb{N}\Gamma^*; C_i \otimes C_j \mapsto \left[C_k \mapsto |Q| \cdot |C_k| c_{ij}^k\right]$$

and an augmentation

$$\varepsilon\colon \mathbb{N}\Gamma \to \mathbb{N}; C_i \mapsto |C_i| \tag{3.16}$$

known as the relational augmentation*.*

3.8. **Character and fusion algebras as augmented magmas.** We work in the compact closed category $(\underline{\underline{\mathbb{C}}}, \otimes, \mathbb{C})$.

Theorem 3.30. (a) *Let* A *be a character algebra, with basis* $\{x_1, \dots, x_s\}$*. Then an augmented magma structure* $(A, \mu, \Delta, \varepsilon)$ *in* $(\underline{\underline{\mathbb{C}}}, \otimes, \mathbb{C})$ *is given by the coproduct*

$$\Delta\colon A \to A \otimes A; x_i \mapsto x_i \otimes x_i\,,$$

the multiplication structure

$$\mu\colon A \otimes A \to A^*; x_i \otimes x_j \mapsto \left[x_k \mapsto p_{ij}^k \kappa_k\right],$$

and an augmentation

$$\varepsilon\colon A \to \mathbb{C}; x_i \mapsto \kappa_i \tag{3.17}$$

known as the representation augmentation*.*

(b) *Suppose that* A *is a fusion algebra, with basis* $\{x_1, \dots, x_s\}$*. Then an augmented magma structure* $(A, \mu, \Delta, \varepsilon)$ *in* $(\underline{\underline{\mathbb{C}}}, \otimes, \mathbb{C})$ *is given by the coproduct*

$$\Delta\colon A \to A \otimes A; x_i \mapsto x_i \otimes x_i\,,$$

the multiplication structure

$$\mu\colon A \otimes A \to A^*; x_i \otimes x_j \mapsto \left[x_k \mapsto N_{ij}^k \sqrt{\nu_k}\right],$$

and an augmentation

$$\varepsilon\colon A \to \mathbb{C}; x_i \mapsto \sqrt{\nu_i} \tag{3.18}$$

known as the representation augmentation*.*

Proof. (a) The augmented magma condition (3.3) is verified by the chase

$$\begin{array}{ccc}
x_i \otimes x_j & \xmapsto{\mathrm{coev}_A \otimes \mu} & \sum_{h=1}^{s} \delta_{x_h} \otimes x_h \otimes \left[x_k \mapsto p_{ij}^k \kappa_k\right] \\
& & \big\downarrow {\scriptstyle 1_{A^*} \otimes \Delta \otimes 1_{A^*}} \\
\big\downarrow {\scriptstyle \varepsilon \otimes \varepsilon} & & \sum_{h=1}^{s} \delta_{x_h} \otimes x_h \otimes x_h \otimes \left[x_k \mapsto p_{ij}^k \kappa_k\right] \\
& & \big\downarrow {\scriptstyle \tau \otimes \mathrm{ev}_A} \\
\kappa_i \kappa_j = \sum_{h=1}^{s} p_{ij}^h \kappa_h & \xleftarrow[\mathrm{ev}_A]{} & \sum_{h=1}^{s} p_{ij}^h \kappa_h (x_h \otimes \delta_{x_h})
\end{array}$$

in which the commuting, namely the equality in the lower left hand corner, follows when the equation

$$x_i x_j = \sum_{h=1}^{s} p_{ij}^h x_h$$

is mapped according to the representation augmentation (3.17), as in Definition 2.1(c).

The proof of (b) is similar, replacing the structure constants p_{ij}^k with their analogues N_{ij}^k, and the κ_i with $\sqrt{\nu_i}$. In place of Definition 2.1(c), Definition 2.2(c) is used. □

4. Augmented quasigroups

4.1. **Prequasigroups and augmented quasigroups.** Suppose that $(\mathbf{V}, \otimes, \mathbf{1})$ is a compact closed category. Define

$$\tau_{13} \colon A_3 \otimes A_2 \otimes A_1 \to A_1 \otimes A_2 \otimes A_3; a_3 \otimes a_2 \otimes a_1 \mapsto a_1 \otimes a_2 \otimes a_3$$

and

$$\tau_{23} \colon A_1 \otimes A_3 \otimes A_2 \to A_1 \otimes A_2 \otimes A_3; a_1 \otimes a_3 \otimes a_2 \mapsto a_1 \otimes a_2 \otimes a_3$$

(recalling the remarks at the beginning of §3.2), along with

$$\tau_{13}^* \colon \mathbf{V}(A_1 \otimes A_2 \otimes A_3, \mathbf{1}) \to \mathbf{V}(A_3 \otimes A_2 \otimes A_1, \mathbf{1}); \theta \mapsto \tau_{13}\theta$$

and

$$\tau_{23}^* \colon \mathbf{V}(A_1 \otimes A_2 \otimes A_3, \mathbf{1}) \to \mathbf{V}(A_1 \otimes A_3 \otimes A_2, \mathbf{1}); \theta \mapsto \tau_{23}\theta$$

for objects A_1, A_2, A_3 of $\mathbf{V}$. In conjunction with the above, we will also use the notation of Lemma 3.2.

Definition 4.1. Let $(A, \mu, \Delta, \varepsilon)$ be an augmented magma in $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) The $\mathbf{V}$-morphism $\rho\colon A \otimes A \to A^*$ given by

$$\rho = \mu\phi^{-1}_{A,A\otimes A,\mathbf{1}}\tau^*_{13}\phi_{A,A\otimes A,\mathbf{1}}$$

is called the *right division* (*structure*) on the augmented magma $(A, \mu, \Delta, \varepsilon)$.

(b) The $\mathbf{V}$-morphism $\lambda\colon A \otimes A \to A^*$ given by

$$\lambda = \mu\phi^{-1}_{A,A\otimes A,\mathbf{1}}\tau^*_{23}\phi_{A,A\otimes A,\mathbf{1}}$$

is called the *left division* (*structure*) on the augmented magma $(A, \mu, \Delta, \varepsilon)$.

(c) The structure $(A, \mu, \rho, \lambda, \Delta, \varepsilon)$ is called the (*augmented*) *pre-quasigroup* on the augmented magma $(A, \mu, \Delta, \varepsilon)$.

(d) The structures $(A, \mu, \Delta, \varepsilon)$ or $(A, \mu, \rho, \Delta, \varepsilon)$ are described as (*augmented*) *right quasigroups* if $(A, \rho, \Delta, \varepsilon)$ is an augmented magma.

(e) The structures $(A, \mu, \Delta, \varepsilon)$ or $(A, \mu, \lambda, \Delta, \varepsilon)$ are described as (*augmented*) *left quasigroups* if $(A, \lambda, \Delta, \varepsilon)$ is an augmented magma.

(f) The structures $(A, \mu, \Delta, \varepsilon)$ or $(A, \mu, \rho, \lambda, \Delta, \varepsilon)$ are described as (*augmented*) *quasigroups* if both $(A, \rho, \Delta, \varepsilon)$ and $(A, \lambda, \Delta, \varepsilon)$ are augmented magmas.

Example 4.2. Consider the augmented magma $(RG, \mu, \Delta, \varepsilon)$ built in Example 3.8 from the group algebra RG of a finite group G over a commutative unital ring R, with

$$\mu\colon x \otimes y \mapsto \delta_{xy}$$

as its multiplication structure (3.4) written in an abbreviated form. Since $\mu\phi^{-1}_{RG,RG\otimes RG,R}\colon x \otimes y \otimes z \mapsto \delta_{z,xy}$ and

$$\mu\,\phi^{-1}_{RG,RG\otimes RG,R}\,\tau^*_{13}\colon z \otimes y \otimes x \mapsto \delta_{z,xy} = \delta_{x,zy^{-1}}\,,$$

one obtains the right division structure

$$\mu\,\phi^{-1}_{RG,RG\otimes RG,R}\,\tau^*_{13}\,\phi_{RG,RG\otimes RG,R} = \rho\colon z \otimes y \mapsto \delta_{zy^{-1}}\,.$$

Similarly, the left division structure is obtained as $\lambda\colon x \otimes z \mapsto \delta_{x^{-1}z}$. Now in Example 3.8, it turned out that verification of the augmented magma condition (3.3) for the multiplication structure (3.4) did not involve the associativity of the group multiplication, so a comparable verification works for the right and left division structures. In other words, the augmented magma $(RG, \mu, \Delta, \varepsilon)$ is actually an augmented quasigroup in $(\underline{\underline{R}}, \otimes, R)$.

4.2. **Marty quasigroups.** The term "hypergroup" has been used in a number of different contexts over the years, with a number of different interpretations that have generally involved a concept of inversion, for example in the "postulate of the inverse" in [33, p.78]. Here, we return closer to the spirit of the original definition of Marty [24, p.89], but without any associativity requirement for $(A, \diamond)$.

Definition 4.3. Let $(A, \diamond, \swarrow, \searrow)$ be a set A with three hypermagma structures $(A, \diamond)$, $(A, \swarrow)$, and $(A, \searrow)$. Then $(A, \diamond, \swarrow, \searrow)$ is a *Marty quasigroup* if and only if the conditions

$$z \in x \diamond y \quad \Leftrightarrow \quad x \in z \swarrow y \quad \Leftrightarrow \quad y \in x \searrow z$$

hold for elements x, y, z of A.

Example 4.4. A *residuated magma* is defined as a partially ordered algebra $(A, \leq, \cdot, /, \backslash)$ such that $(A, \leq)$ is a poset, and the three binary operations $\cdot, /, \backslash$ satisfy the *residuation property*

$$x \cdot y \leq z \quad \Leftrightarrow \quad x \leq z/y \quad \Leftrightarrow \quad y \leq x \backslash z \tag{4.1}$$

[14, 26]. Define the *up-set* $\uparrow a = \{x \in A \mid a \leq x\}$ and the *down-set* $\downarrow a = \{x \in A \mid x \leq a\}$ for an element a of the poset $(A, \leq)$. Then the specifications

$$x \diamond y = \uparrow(x \cdot y)\,, \quad z \swarrow y = \downarrow(z/y)\,, \quad x \searrow z = \downarrow(x \backslash z)$$

yield a Marty quasigroup $(A, \diamond, \swarrow, \searrow)$.

Example 4.5. As a special case of Example 4.4, consider a *Heyting algebra* $(A, \wedge, \rightarrow)$, a meet semilattice $(A, \wedge)$ with

$$x \wedge y \leq z \quad \Leftrightarrow \quad x \leq y \rightarrow z \quad \Leftrightarrow \quad y \leq x \rightarrow z \tag{4.2}$$

as a residuation [17, §I.1.10]. By comparison of (4.2) with (4.1), the specifications

$$x \diamond y = \uparrow(x \wedge y)\,, \quad z \swarrow y = \downarrow(y \rightarrow z)\,, \quad x \searrow z = \downarrow(x \rightarrow z) \tag{4.3}$$

yield a Marty quasigroup $(A, \diamond, \swarrow, \searrow)$.

The following theorem demonstrates that Marty quasigroups are equivalent to augmented quasigroups in $(\mathbf{Rel}, \otimes, \top)$.

Theorem 4.6. *Suppose that $(A, \diamond)$ is a hypermagma in the sense of Definition 3.9. Let $(A, \mu, \Delta, \varepsilon)$ be the corresponding augmented magma in $(\mathbf{Rel}, \otimes, \top)$ provided by Proposition 3.11. Then let $(A, \mu, \rho, \lambda, \Delta, \varepsilon)$ be the corresponding augmented prequasigroup constructed according to Definition 4.1(c).*

(a) *If $(A, \diamond, \nwarrow, \nearrow)$ is a Marty quasigroup, the respective augmented magmas provided by Proposition 3.11 from $(A, \nwarrow)$ and $(A, \nearrow)$ are $(A, \rho, \Delta, \varepsilon)$ and $(A, \lambda, \Delta, \varepsilon)$, so that $(A, \mu, \rho, \lambda, \Delta, \varepsilon)$ is an augmented quasigroup in $(\mathbf{Rel}, \otimes, \top)$.*

(b) *If $(A, \mu, \rho, \lambda, \Delta, \varepsilon)$ is an augmented quasigroup in the sense of Definition 4.1(f), then $(A, \diamond)$ extends to a Marty quasigroup $(A, \diamond, \nwarrow, \nearrow)$, where the respective augmented magmas provided by Proposition 3.11 from $(A, \nwarrow)$ and $(A, \nearrow)$ are $(A, \rho, \Delta, \varepsilon)$ and $(A, \lambda, \Delta, \varepsilon)$.*

Proof. For an object A of the compact closed category $(\mathbf{Rel}, \otimes, \top)$, the natural isomorphism component $\phi_{A,A\otimes A,\top}$ of Lemma 3.2 acts as

$$\alpha = \{x \otimes y \otimes z \mid x \otimes y \otimes z \in \alpha\} \mapsto \{(x \otimes y, z) \mid x \otimes y \otimes z \in \alpha\},$$
$$\{x \otimes y \otimes z \mid (x \otimes y, z) \in \beta\} \mapsto \{(x \otimes y, z) \mid (x \otimes y, z) \in \beta\} = \beta$$

in the simplified notation that identifies a relation $\xi \in X^*$, for a set X, with the subset (unary relation) $\{x \in X \mid (x, 0) \in \xi\}$ of X.

Under the main hypothesis of the proposition, the multiplication relation of $(A, \mu, \Delta, \varepsilon)$ is

$$\mu = \{(x \otimes y, z) \mid x, y, z \in A,\ z \in x \diamond y\}$$

in the simplified notation. Thus

$$\mu\, \phi^{-1}_{A,A\otimes A,\top} = \{x \otimes y \otimes z \mid x, y, z \in A,\ z \in x \diamond y\},$$
$$\mu\, \phi^{-1}_{A,A\otimes A,\top}\, \tau^*_{13} = \{z \otimes y \otimes x \mid x, y, z \in A,\ z \in x \diamond y\}, \text{ and}$$

$$\mu\, \phi^{-1}_{A,A\otimes A,\top}\, \tau^*_{13}\, \phi_{A,A\otimes A,\top} = \{(z \otimes y, x) \mid x, y, z \in A,\ z \in x \diamond y\} \tag{4.4}$$

in the simplified notation.

(a) If $(A, \diamond, \nwarrow, \nearrow)$ is a Marty quasigroup, then (4.4) reduces to

$$\rho = \{(z \otimes y, x) \mid z, y, x \in A,\ x \in z \nwarrow y\},$$

so that $(A, \rho, \Delta, \varepsilon)$ is the augmented magma corresponding to $(A, \nwarrow)$ via Proposition 3.11. The corresponding left-sided statement follows similarly.

(b) If $(A, \rho, \Delta, \varepsilon)$ is an augmented magma, with

$$\rho = \{(z \otimes y, x) \mid x, y, z \in A,\ z \in x \diamond y\},$$

define

$$z \nwarrow y = \{x \in A \mid (z \otimes y, x) \in \rho\}$$

for z, y in A. Then $z \in x \diamond y$ if and only if $x \in z \nwarrow y$. A similar consideration on the left-hand side yields a set-valued operation $\nearrow$ such that $(A, \diamond, \nearrow\!\!\!\!\nwarrow, \nwarrow\!\!\!\!\searrow)$ is a Marty quasigroup, and by our construction the two respective augmented magmas provided by Proposition 3.11 from $(A, \nearrow\!\!\!\!\nwarrow)$ and $(A, \nwarrow\!\!\!\!\searrow)$ are $(A, \rho, \Delta, \varepsilon)$ and $(A, \lambda, \Delta, \varepsilon)$. □

Corollary 4.7. *With the considerations of Example 3.12, quasigroups* $(A, \cdot, /, \backslash)$ *are equivalent to augmented quasigroups* $(A, \cdot, /, \backslash, \Delta, \varepsilon)$*, for which all the structure actually lies in* $(\mathbf{Set}, \times, \top)$*.*

4.3. **Weighted quasigroups.** Weighted quasigroups were originally defined in [9, §1]. Since the conventions used for that definition do not completely match the usual general quasigroup conventions, the definition as given here is slightly different: our $\beta_x(a, b)$ corresponds to the $\delta_x(b, a)$ in [9].[2]

Definition 4.8. Let (X, w, α) be a weighted magma, equipped with multiplication function (3.10).

(a) Define the *right division function*

$$\beta\colon X \times X \times X \to \mathbb{N}; (c, d, y) \mapsto \alpha_c(y, d)\,, \tag{4.5}$$

and write $(a, b, x)\beta = \beta_x(a, b) = \alpha_a(x, b)$ for a, b, x in X.

(b) Define the *left division function*

$$\gamma\colon X \times X \times X \to \mathbb{N}; (c, d, y) \mapsto \alpha_d(c, y)\,, \tag{4.6}$$

and write $(a, b, x)\gamma = \gamma_x(a, b) = \alpha_b(a, x)$ for a, b, x in X.

(c) The structure $(X, w, \alpha, \beta, \gamma)$ is called a *weighted prequasigroup.*

The left and right division functions of a weighted prequasigroup $(X, w, \alpha, \beta, \gamma)$ are already specified by the multiplication function. Thus it may suffice to refer to (X, w, α) alone as the prequasigroup structure.

Definition 4.9. A weighted prequasigroup $(X, w, \alpha, \beta, \gamma)$ is a *weighted quasigroup* if (X, w, β) and (X, w, γ) are weighted magmas.

In computational terms, the conditions from Definition 4.8 may be summarized as follows.

Proposition 4.10. *A weighted magma* (X, w, α) *extends to a weighted quasigroup* $(X, w, \alpha, \beta, \gamma)$ *if and only if the conditions*

$$\sum_{x \in X} \alpha_x(a, b) = \sum_{x \in X} \alpha_a(x, b) = \sum_{x \in X} \alpha_b(a, x) \tag{4.7}$$

are satisfied for all a, b *in* X*.*

[2]The latter notation was more appropriate in the commutative case that was important to the authors of [9]. It enabled them to avoid the kind of reversal of argument order that is seen in the middle equation of (4.3) connected with the commutative meet operation of a Heyting algebra.

Proof. Since (X, w, α) is a weighted magma, the first sum in (4.7) is $w(a)w(b)$. Equality with the second sum in (4.7) then corresponds to (X, w, β) being a weighted magma, while equality with the third sum corresponds to (X, w, γ) being a weighted magma. □

Corollary 4.11. *If a weighted magma (X, w, α) extends to a weighted quasigroup, then so does each rescaled version (X, w', α').*

Proof. In the notation of Proposition 3.23, the verification for (X, w', α') is obtained on multiplying (4.7) throughout by a factor of k^2. □

Proposition 4.12. *Suppose that $(A, \cdot, /, \backslash)$ is a finite quasigroup. Take the set A as the setlike multiset $\varepsilon_A \colon X \to \{1\}$. Then the quasigroup structure is equivalent to a weighted quasigroup structure $(X, \varepsilon_A, \delta, \delta', {}'\delta)$, where*

$$\delta \colon X \times X \times X \to \mathbb{N}; (a, b, x) \mapsto \delta_{x, a\cdot b}$$

is the multiplication function,

$$\delta' \colon X \times X \times X \to \mathbb{N}; (a, b, x) \mapsto \delta_{x, a/b}$$

is the right division function, and

$${}'\delta \colon X \times X \times X \to \mathbb{N}; (a, b, x) \mapsto \delta_{x, a\backslash b}$$

is the left division function.

Proof. As in the proof of Proposition 3.21, note that for elements x, a, b of A, the equation $a \cdot b = x$ holds in $(A, \cdot)$ if and only if the equation $\delta_x(a, b) = 1$ holds in $(A, \varepsilon_A, \delta)$.

In the quasigroup $(A, \cdot, /, \backslash)$, the equation $x = a/b$ holds if and only if $x \cdot b = a$. The latter condition translates to $\delta_a(x, b) = 1$ in the weighted magma $(A, \varepsilon_A, \alpha)$, and thus to $\delta'_x(a, b) = 1$ in the weighted prequasigroup $(A, \varepsilon_A, \delta, \delta', {}'\delta)$. Hence $(A, \varepsilon_A, \delta')$ is a weighted magma.

Similarly, in the quasigroup $(A, \cdot, /, \backslash)$, the equation $x = a\backslash b$ holds if and only if $a \cdot x = b$. In the weighted magma $(A, \varepsilon_A, \delta)$, the latter condition translates to $\delta_b(a, x) = 1$, and thus to ${}'\delta_x(a, b) = 1$ in the weighted prequasigroup $(A, \varepsilon_A, \delta, \delta', {}'\delta)$. Hence $(A, \varepsilon_A, {}'\delta)$ is a weighted magma. □

We conclude this section with the analogue of Proposition 3.24 for weighted quasigroups.

Proposition 4.13. *Let $(X, w, \alpha, \beta, \gamma)$ be a weighted quasigroup. Then there is an equivalent augmented quasigroup $(\mathbb{N}X, \mu, \rho, \lambda, \Delta, \varepsilon)$ in the category $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$, where $(\mathbb{N}X, \Delta, \varepsilon)$ is a multisetlike augmented comagma.*

Proof. Consider the augmented magma $(\mathbb{N}X, \mu, \Delta, \varepsilon)$ equivalent to the weighted magma (X, w, α) that is furnished by Proposition 3.24. Since its multiplication is $\mu = \alpha\phi_{\mathbb{N}X, \mathbb{N}X\otimes\mathbb{N}X, \mathbb{N}}$, or more simply $\mu = \alpha\phi$ without the suffices, we have $\mu\phi^{-1} = \alpha \colon x \otimes b \otimes a \mapsto \alpha_a(x, b)$, whence

$$\rho = \mu\phi^{-1}\tau_{13}\phi \colon a \otimes b \otimes x \mapsto \alpha_x(a, b) = \beta_x(a, b)$$

by Definition 4.1(a) and (4.5). By Proposition 3.24, $(\mathbb{N}X, \rho, \Delta, \varepsilon)$ is an augmented magma, equivalent to the weighted magma (X, w, β). In similar fashion, $(\mathbb{N}X, \lambda, \Delta, \varepsilon)$ is the augmented magma equivalent to the weighted magma (X, w, γ), and so $(\mathbb{N}X, \mu, \rho, \lambda, \Delta, \varepsilon)$ is an augmented quasigroup. □

Example 4.14. Suppose that $(A, \cdot, /, \backslash)$ is a finite quasigroup. Then Propositions 4.12 and 4.13 yield a corresponding augmented quasigroup $(\mathbb{N}A, \mu, \rho, \lambda, \Delta, \varepsilon)$ in the category $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$, with

$$\mu \colon a \otimes b \mapsto [x \mapsto \delta_{x, a\cdot b}]$$

as the multiplication structure. On the other hand, Corollary 4.7 yields an augmented quasigroup structure $(A, \cdot, /, \backslash, \Delta, \varepsilon)$ in the subcategory $(\mathbf{Set}, \times, \top)$ of $(\mathbf{Rel}, \otimes, \top)$, with

$$\mu \colon a \otimes b \mapsto a \cdot b$$

as the multiplication structure.

4.4. **The weighted quasigroups of an association scheme.** Given an association scheme (Q, Γ), Theorem 3.27 built a weighted magma structure (Γ, w, α), with multiplication function

$$\alpha \colon \Gamma \times \Gamma \times \Gamma \to \mathbb{N}; (C_i, C_j, C_k) \mapsto c_{ij}^k n_k \tag{4.8}$$

and the valency augmentation. In turn, Corollary 3.28 built a weighted magma structure (Γ, w', α'), with multiplication function

$$\alpha' \colon \Gamma \times \Gamma \times \Gamma \to \mathbb{N}; (C_i, C_j, C_k) \mapsto |Q| \cdot |C_k| c_{ij}^k \tag{4.9}$$

and the relational augmentation.

Theorem 4.16, the main result of this section, shows that (Γ, w, α) and (Γ, w', α') are actually weighted quasigroups. The theorem is based on the following combinatorial observation.

Lemma 4.15. [3, Prop. 2.2.2(v)] *Let (Q, Γ) be an association scheme. Then*

$$\sum_{k=1}^{s} c_{ki}^j = \sum_{k=1}^{s} c_{ik}^j = n_i \tag{4.10}$$

for $1 \leq i, j \leq s$.

Theorem 4.16. *If (Q,Γ) is an association scheme, then the weighted magmas (Γ, w, α) of Theorem 3.27 and (Γ, w', α') of Corollary 3.28 are weighted quasigroups.*

Proof. Corresponding to the multiplication function (4.8), the weighted magma (Γ, w, α) has the right division function

$$\beta\colon \Gamma\times\Gamma\times\Gamma \to \mathbb{N}; (C_i, C_j, C_k) \mapsto c_{kj}^i n_i$$

and the left division function

$$\gamma\colon \Gamma\times\Gamma\times\Gamma \to \mathbb{N}; (C_i, C_j, C_k) \mapsto c_{ik}^j n_j\,.$$

Given the commutativity $c_{ij}^k = c_{ji}^k$ for all $1 \le i,j,k \le s$, verification of the weighted magma condition for the right and left divisions of (Γ, w, α) reduces to the verification of

$$\sum_{k=1}^{s} c_{ik}^j n_j = n_i n_j$$

for all $1 \le i,j \le s$, which follows by Lemma 4.15. Then the statement for (Γ, w', α') is a direct application of Corollary 4.11. □

Corollary 3.29(a) thus yields the following.

Corollary 4.17. *An association scheme (Q,Γ) furnishes an augmented quasigroup structure $(\mathbb{N}\Gamma, \mu, \rho, \lambda, \Delta, \varepsilon)$ in the compact closed category $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$, with the valency augmentation $\varepsilon\colon \mathbb{N}\Gamma \to \mathbb{N}; C_i \mapsto n_i$. The multiplication structure is*

$$\mu\colon \mathbb{N}\Gamma\otimes\mathbb{N}\Gamma \to \mathbb{N}\Gamma^*; C_i\otimes C_j \mapsto \left[C_k \mapsto c_{ij}^k n_k\right].$$

Then the right division structure is

$$\rho\colon \mathbb{N}\Gamma\otimes\mathbb{N}\Gamma \to \mathbb{N}\Gamma^*; C_i\otimes C_j \mapsto \left[C_k \mapsto c_{kj}^i n_i\right],$$

while

$$\lambda\colon \mathbb{N}\Gamma\otimes\mathbb{N}\Gamma \to \mathbb{N}\Gamma^*; C_i\otimes C_j \mapsto \left[C_k \mapsto c_{ik}^j n_j\right]$$

is the left division structure.

4.5. **The augmented quasigroup of a character algebra.** This paragraph provides a generalization of Corollary 4.17. Let A be a character algebra, with the notation of Definition 2.1. We begin with an algebraic analogue of the combinatorial Lemma 4.15.

Lemma 4.18. [3, Prop. 2.5.1] *The relation $p_{jh}^i \kappa_i = p_{j'i}^h \kappa_h$ holds for $1 \le h,i,j \le s$.*

Theorem 4.19. *A character algebra A furnishes an augmented quasigroup $(A, \mu, \rho, \lambda, \Delta, \varepsilon)$ in the compact closed category $(\underline{\underline{\mathbb{C}}}, \otimes, \mathbb{C})$, with representation augmentation*

$$\varepsilon \colon A \to \mathbb{C}; x_i \mapsto \kappa_i \, .$$

As in Theorem 3.30(a), the multiplication structure is

$$\mu \colon A \otimes A \to A^*; x_i \otimes x_j \mapsto \left[x_k \mapsto p_{ij}^k \kappa_k\right] .$$

The right division structure is

$$\rho \colon A \otimes A \to A^*; x_i \otimes x_j \mapsto \left[x_k \mapsto p_{kj}^i \kappa_i\right] ,$$

while

$$\lambda \colon A \otimes A \to A^*; x_i \otimes x_j \mapsto \left[x_k \mapsto p_{ik}^j \kappa_j\right]$$

is the left division structure.

Proof. By Theorem 3.30(a), $(\mathbb{C}A, \mu, \Delta, \varepsilon)$ forms an augmented magma. Then the augmented magma condition (3.3) for the left division is verified by the chase

$$\begin{array}{ccc}
x_i \otimes x_j & \xmapsto{\mathrm{coev}_A \otimes \rho} & \sum_{h=1}^s \delta_{x_h} \otimes x_h \otimes \left[x_k \mapsto p_{kj}^i \kappa_i\right] \\
 & & \Big\downarrow {\scriptstyle 1_{A^*} \otimes \Delta \otimes 1_{A^*}} \\
{\scriptstyle \varepsilon \otimes \varepsilon} \Big\downarrow & & \sum_{h=1}^s \delta_{x_h} \otimes x_h \otimes x_h \otimes \left[x_k \mapsto p_{kj}^i \kappa_i\right] \\
 & & \Big\downarrow {\scriptstyle \tau \otimes \mathrm{ev}_A} \\
\kappa_i \kappa_j = \sum_{h=1}^s p_{hj}^i \kappa_i & \xleftarrow[\mathrm{ev}_A]{} & \sum_{h=1}^s p_{hj}^i \kappa_i (x_h \otimes \delta_{x_h})
\end{array}$$

in which the commuting, namely the equality in the lower left hand corner, follows from the computations

$$\kappa_i \kappa_j = \kappa_{j'} \kappa_i = \sum_{h=1}^s p_{j'i}^h \kappa_h = \sum_{h=1}^s p_{jh}^i \kappa_i = \sum_{h=1}^s p_{hj}^i \kappa_i$$

using [3, §2.5(d)] and Lemma 4.18. There is a comparable verification for the right division. □

4.5.1. *The augmented quasigroup of a dual scheme.* We consider the dual to Corollary 4.17 as a first application of Theorem 4.19. Let $\mathbb{R}^+$ denote the subsemiring $[0, \infty[$ of $\mathbb{R}$.

Corollary 4.20. *An association scheme (Q, Γ), with (non-negative) Krein parameters $\{\tilde{c}_{i,j}^k \mid 1 \leq i, j, k \leq s\}$, furnishes an augmented quasigroup $(\mathbb{R}^+\widetilde{\Gamma}, \mu, \rho, \lambda, \Delta, \varepsilon)$ in the compact closed category $(\underline{\underline{\mathbb{R}^+}}, \otimes, \mathbb{R}^+)$,* *with representation augmentation*

$$\varepsilon\colon \mathbb{R}^+\widetilde{\Gamma} \to \mathbb{R}^+; nE_i \mapsto f_i\,.$$

The multiplication structure is

$$\mu\colon \mathbb{R}^+\widetilde{\Gamma} \otimes \mathbb{R}^+\widetilde{\Gamma} \to \mathbb{R}^+\widetilde{\Gamma}^*; nE_i \otimes nE_j \mapsto \left[nE_k \mapsto \tilde{c}_{ij}^k f_k\right]$$

as suitably restricted from Theorem 3.30(a). Then the right division structure is

$$\rho\colon \mathbb{R}^+\widetilde{\Gamma} \otimes \mathbb{R}^+\widetilde{\Gamma} \to \mathbb{R}^+\widetilde{\Gamma}^*; nE_i \otimes nE_j \mapsto \left[nE_k \mapsto \tilde{c}_{kj}^i f_i\right],$$

while

$$\lambda\colon \mathbb{R}^+\widetilde{\Gamma} \otimes \mathbb{R}^+\widetilde{\Gamma} \to \mathbb{R}^+\widetilde{\Gamma}^*; nE_i \otimes nE_j \mapsto \left[nE_k \mapsto \tilde{c}_{ik}^j f_j\right]$$

is the left division structure.

4.5.2. *The augmented quasigroup of a fusion algebra.* We now present a second application of Theorem 4.19.

Corollary 4.21. *Let A be a fusion algebra with basis $X = \{x_1, \dots, x_s\}$, structure constants N_{ij}^k, and representation $x_i \mapsto \sqrt{\nu_i}$.*

(*a*) *In general, take $\mathbb{K} = \mathbb{C}$.*
(*b*) *If A is of nonnegative type, take $\mathbb{K} = \mathbb{R}^+$ (notation of §4.5.1).*
(*c*) *If A is of integral type, take $\mathbb{K} = \mathbb{N}$.*

Then A furnishes an augmented quasigroup $(\mathbb{K}X, \mu, \rho, \lambda, \Delta, \varepsilon)$ in the compact closed category $(\underline{\underline{\mathbb{K}}}, \otimes, \mathbb{K})$, with representation augmentation

$$\varepsilon\colon \mathbb{K}X \to \mathbb{K}; x_i \mapsto \nu_i\,.$$

The multiplication structure is

$$\mu\colon \mathbb{K}X \otimes \mathbb{K}X \to \mathbb{K}X^*; x_i \otimes x_j \mapsto \left[x_k \mapsto \sqrt{\nu_i\nu_j\nu_k}N_{ij}^k\right].$$

The right division structure is

$$\rho\colon \mathbb{K}X \otimes \mathbb{K}X \to \mathbb{K}X^*; x_i \otimes x_j \mapsto \left[x_k \mapsto \sqrt{\nu_i\nu_j\nu_k}N_{kj}^i\right],$$

while

$$\lambda\colon \mathbb{K}X \otimes \mathbb{K}X \to \mathbb{K}X^*; x_i \otimes x_j \mapsto \left[x_k \mapsto \sqrt{\nu_i\nu_j\nu_k}N_{ik}^j\right]$$

is the left division structure.

Proof. By [2, Th. 3.1], the fusion algebra yields a character algebra with structure constants

$$p_{ij}^k = \sqrt{\frac{\nu_i\nu_j}{\nu_k}}N_{ij}^k$$

for $1 \le i, j, k \le s$, and $\kappa_i = \nu_i$ for $1 \le i \le s$. □

5. Quasigroup lifts

5.1. **Covering quasigroups.** The amalgamation concept of §3.6.3 is now extended from magmas to quasigroups [9, pp.142–3]. While the following result corresponds to [9, Prop. 2], the statement and proof are made more transparent by the use of algebraic (quasigroup-theoretical) rather than combinatorial (Latin square) techniques.

Proposition 5.1. *Let $(A,\cdot,/,\backslash)$ be a quasigroup, yielding the weighted quasigroup structure $(A,\varepsilon_A,\delta,\delta',{}'\delta)$ presented in Proposition 4.12. Let $f\colon(A,\Delta,\varepsilon_A)\to(X,\Delta,\varepsilon_X)$ be a covering. Then the multiplication function*

$$\alpha\colon X\times X\times X\to\mathbb{N};\quad (x,y,z)\mapsto\left|\{(a,b)\in f^{-1}\{x\}\times f^{-1}\{y\}\mid (ab)f=z\}\right| \tag{5.1}$$

of Proposition 3.25 combines with the right division function

$$\beta\colon X\times X\times X\to\mathbb{N};\quad (x,y,z)\mapsto\left|\{(a,b)\in f^{-1}\{x\}\times f^{-1}\{y\}\mid (a/b)f=z\}\right|$$

and the left division function

$$\gamma\colon X\times X\times X\to\mathbb{N};\quad (x,y,z)\mapsto\left|\{(a,b)\in f^{-1}\{x\}\times f^{-1}\{y\}\mid (a\backslash b)f=z\}\right|$$

to yield a weighted quasigroup $(X,\varepsilon_X,\alpha,\beta,\gamma)$.

Proof. The weighted magma condition (3.11) for the multiplication function α was verified in the proof of Proposition 3.25.

By (4.5) from Definition 4.8(a), the right division function β that accompanies the multiplication function α is $\beta_z(x,y)=\alpha_x(z,y)=$

$$\begin{aligned}
&\left|\{(c,b)\in f^{-1}\{z\}\times f^{-1}\{y\}\mid (cb)f=x\}\right|\\
=&\left|\{(c,b,a)\in f^{-1}\{z\}\times f^{-1}\{y\}\times f^{-1}\{x\}\mid cb=a\}\right|\\
=&\left|\{(c,b,a)\in f^{-1}\{z\}\times f^{-1}\{y\}\times f^{-1}\{x\}\mid c=a/b\}\right|\\
=&\left|\{(a,b)\in f^{-1}\{x\}\times f^{-1}\{y\}\mid (a/b)f=z\}\right|
\end{aligned}$$

for x,y,z in X. The weighted magma condition for this right division function now follows by applying Proposition 3.25 to the magma $(A,/)$. Treatment of the left division function γ is similar. □

Definition 5.2. In the context of Proposition 5.1, the weighted quasigroup $(X,\varepsilon_X,\alpha,\beta,\gamma)$ is variously said to *lift* to, or be *covered* by, or be an *amalgamation* of, the quasigroup $(A,\cdot,/,\backslash)$.

5.2. **Lifting association schemes to quasigroups.** The following converse of Proposition 5.1 appeared in [8, Th. 1], [9, Th. 1].

Theorem 5.3. *Suppose that (X, w, α) is a weighted quasigroup, of gross weight W. Then (X, w, α) lifts to a quasigroup structure on a set Q of cardinality W.*

Corollary 5.4. *Let (Q, Γ) be an association scheme. Then the weighted quasigroups (Γ, w, α) and (Γ, w', α') of Theorem 4.16 lift to respective quasigroup structures on the sets Q and Q^2.*

5.2.1. *The Petersen graph and the Johnson scheme $J(5, 2)$.* On a base set B of finite positive cardinality v, and for a parameter $k \leq \lfloor v/2 \rfloor$, the *Johnson scheme* $J(v, k)$ consists of the set Q of k-element subsets of B together with the partition $\Gamma = \{C_1, \dots, C_{k+1}\}$ of Q, where

$$C_i = \{(S, T) \in Q^2 \mid |S \cap T| = k - i + 1\}$$

for $1 \leq i \leq k + 1$ [3, §3.2]. It is the case $J(5, 2)$ which is of interest here, corresponding to the Petersen graph, as the Kneser graph $KG_{5,2}$. Specifically, the 10-element set Q may be taken as the vertex set of the Petersen graph, and a vertex pair (S, T) lies in C_i if and only if S and T are at distance $i - 1$ in the graph.

The Bose-Mesner algebra of the scheme is given by

$J(5,2)$	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$6 \cdot C_1 + 3 \cdot C_2 + 4 \cdot C_3$	$2 \cdot C_2 + 2 \cdot C_3$
C_3	C_3	$2 \cdot C_2 + 2 \cdot C_3$	$3 \cdot C_1 + C_2$

with nontrivial valencies $n_2 = 6$ and $n_3 = 3$. Then, although the scheme $J(5, 2)$ is known not to be the conjugacy class scheme of a quasigroup [16, Cor. 4.3] [28, Cor. 8.3], Corollary 5.4 implies that the weighted quasigroup (Γ, w, α) of Theorem 4.16, namely

(5.2)

Γ	C_1	C_2	C_3
C_1	C_1	$6 \cdot C_2$	$3 \cdot C_3$
C_2	$6 \cdot C_2$	$6 \cdot C_1 + 18 \cdot C_2 + 12 \cdot C_3$	$12 \cdot C_2 + 6 \cdot C_3$
C_3	$3 \cdot C_3$	$12 \cdot C_2 + 6 \cdot C_3$	$3 \cdot C_1 + 6 \cdot C_2$

does lift to a quasigroup structure on the set Q.

The lifting will be written as

$$t \mapsto C_1,\ o_1, \dots, o_6 \mapsto C_2,\ n_1, n_2, n_3 \mapsto C_3\,,$$

where the symbols for the elements of Q stand for sets having *t*wo, *o*ne, or *n*o elements in their intersection with a given reference set.

The multiplication table of one possible lift is

(5.3)

Q	t	o_1	o_2	o_3	o_4	o_5	o_6	n_1	n_2	n_3
t	t	o_1	o_2	o_3	o_4	o_5	o_6	n_1	n_2	n_3
o_1	o_1	o_4	n_1	n_2	t	o_6	o_2	o_5	n_3	o_3
o_2	o_2	n_1	o_5	n_3	o_6	t	o_4	n_2	o_3	o_1
o_3	o_3	n_2	n_3	o_6	o_2	o_4	t	o_1	o_5	n_1
o_4	o_4	t	o_3	o_2	o_1	n_2	n_1	n_3	o_6	o_5
o_5	o_5	o_6	t	o_4	n_1	o_2	n_3	o_3	o_1	n_2
o_6	o_6	o_1	o_2	t	n_2	n_3	o_3	o_5	n_1	o_4
n_1	n_1	o_2	n_2	o_1	n_3	o_3	o_5	t	o_4	o_6
n_2	n_2	n_3	o_4	o_5	o_3	n_1	o_1	o_6	o_2	t
n_3	n_3	o_3	o_6	n_1	o_5	o_1	n_2	o_4	t	o_2

.

Note that the table (5.3) was constructed manually, rather than with the algorithm implicit in the proof of Theorem 5.3.

5.2.2. *Group conjugacy class schemes.*

Example 5.5. Suppose that Q is a group, with group conjugacy class scheme (Q, Γ). Then the function $f \colon Q \to \Gamma; q \mapsto q^Q$, which maps a group element to its group conjugacy class, lifts the association scheme weighted quasigroup (Γ, w, α) of Theorem 4.16, taken with the valency augmentation, to the (quasi)group Q.

Remark 5.6. Two distinct groups may have the same character table, and thus the same abstract group conjugacy class association scheme [28, Th. 6.7]. The most elementary example consists of the dihedral group D_4 and quaternion group Q_8 of order 8. Examples of this type provide natural illustrations of a feature of Theorem 5.3, that a given weighted quasigroup (X, w, α), of gross weight W, may lift to distinct quasigroup structures of order W.

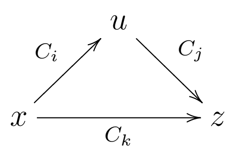
5.3. **Quasigroup conjugacy class schemes.** Example 5.5 noted that a group conjugacy class scheme lifts to the corresponding group. The following result presents the analogue for the quasigroup conjugacy class schemes of arbitrary finite quasigroups, as presented in §2.3.3.

Theorem 5.7. *Let Q be a nonempty quasigroup of finite order n, with quasigroup conjugacy class scheme (Q, Γ). Then the weighted quasigroup (Γ, w', α') of Theorem 4.16, with the relational augmentation, lifts to the quasigroup Q^2.*

Proof. For conjugacy classes C_i, C_j (with $1 \leq i, j \leq s$), consider the rectangle in the multiplication table of Q^2 comprising the table rows labeled from C_i and the table columns labeled from C_j. This rectangle has $|C_i| \cdot |C_j| = n^2 n_i n_j$ elements. For a conjugacy class C_k, it will be shown that $|Q| \cdot |C_k| c_{ij}^k$ of these elements lie in C_k, thus confirming that the weighted magma (Γ, w', α') of Theorem 4.16 lifts to Q^2. In essence, the combinatorial associative relation products $C_i \circ C_j$ in the conjugacy class association scheme are correlated with the algebraic nonassociative quasigroup multiplications in the $C_i \times C_j$ portion of the multiplication table of Q^2.

First, consider the number $|Q| \cdot |C_k| c_{ij}^k$. The initial factor $|Q|$ (or n) counts elements y of Q. The remaining factor $|C_k| c_{ij}^k$ counts *triangles*, configurations

(5.4)



with $(x, z) \in C_k$, $(x, u) \in C_i$, and $(u, z) \in C_j$. Indeed, there are $|C_k| = nn_k$ elements (x, z) of C_k. For each such pair, there are c_{ij}^k elements u such that the triangle configuration (5.4) is realized. Thus the $|Q| \cdot |C_k| c_{ij}^k$ *triangle quadruples* $\langle y, u, x, z\rangle$ each index an arbitrary element y of Q together with a triangle (5.4).

Now consider a fragment

(5.5)

	$(y\backslash x, t\backslash z)$
(y, t)	(x, z)

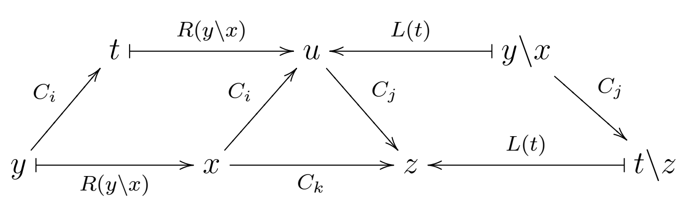
of the multiplication table of Q^2, with (y, t) as a row label in C_i, $(y\backslash x, t\backslash z)$ as a column label in C_j, and (x, z) as a table body entry in C_k. Such a *table fragment* is indexed by a *table quadruple* $[y, t, x, z]$. The transformations

$$[y, t, x, z] \mapsto \langle y, u, x, z\rangle \quad \text{with} \quad u = t(y\backslash x) \tag{5.6}$$

and

$$\langle y, u, x, z\rangle \mapsto [y, t, x, z] \quad \text{with} \quad t = u/(y\backslash x) \tag{5.7}$$

are mutually inverse. The diagram



confirms that

$$(x, u) = (y, t)R(y\backslash x) \in C_i G = C_i$$

and

$$(u,z) = (y\backslash x, t\backslash z)L(t) \in C_jG = C_j\,,$$

so (5.6) really does map a table quadruple to a triangle quadruple. Conversely, it also confirms that

$$(y,t) = (x,u)R(y\backslash x)^{-1} \in C_iG = C_i$$

and

$$(y\backslash x, t\backslash z) = (u,z)L(t)^{-1} \in C_jG = C_j\,,$$

so (5.7) really does map a triangle quadruple to a table quadruple. □

Remark 5.8. Let $(Q,\cdot,/,\backslash)$ be a nonempty finite quasigroup, with conjugacy class scheme (Q,Γ). The augmented quasigroup

$$(\mathbb{N}\Gamma,\mu,\rho,\lambda,\Delta,\varepsilon)$$

in the category $(\underline{\underline{\mathbb{N}}},\otimes,\mathbb{N})$ has the relational augmentation (3.16), with

$$\mu\colon C_i\otimes C_j \mapsto [C_k \mapsto c_{ij}^k|Q|\cdot|C_k|\,]$$

as the multiplication structure, with

$$\rho\colon C_k\otimes C_j \mapsto [C_i \mapsto c_{ij}^k|Q|\cdot|C_k|\,] \tag{5.8}$$

as the right division structure, and with

$$\lambda\colon C_i\otimes C_k \mapsto [C_j \mapsto c_{ij}^k|Q|\cdot|C_k|\,]$$

as the left division structure.

The right division structure (5.8) must not be confused with the multiplication structure in the augmented quasigroup $\mathbb{N}\Gamma(Q,/)$ which is determined by the conjugate quasigroup $(Q,/)$. For example, if $(Q,+)$ is an abelian group with conjugacy class scheme $\big((Q,+),\Gamma(Q,+)\big)$, then $|\Gamma(Q,+)|=|Q|$: the scheme is *thin.* But the right division conjugate is $(Q,-)$, with conjugacy class scheme $\big((Q,-),\Gamma(Q,-)\big)$ and one has $|\Gamma(Q,-)|<|Q|$ if $|Q|>2$ [28, Th. 6.9].

5.4. **Character quasigroups of finite groups.** If A is an abelian group of finite order n, then the character group $\tilde{A}$ is also an abelian group of finite order n.[3] There is a full duality, in the sense that A is recovered from $\tilde{A}$ as (isomorphic to) the double dual $\tilde{\tilde{A}}$.

Aspects of this duality are now extended to nonabelian groups. Let A be an arbitrary group of finite order n. According to Example 5.5, the nonabelian group A amalgamates to the weighted quasigroup of its group conjugacy class scheme. Then there are *character quasigroups* $\tilde{A}$ of finite order n which amalgamate to the weighted quasigroup of the

[3]The notation $\tilde{A}$ is used in preference to the more customary $\widehat{A}$ here, in order to avoid conflict with the diagonal or equality relation on A.

dual of the group conjugacy class scheme of A. As noted in Remark 5.6, a full duality, recovering A from $\tilde{A}$, cannot be expected here.

Proposition 5.9. *Let A be a group of finite order n. Let (A, Γ) be the group conjugacy class scheme of A, with dual scheme $(A, \widetilde{\Gamma})$. Then the augmented quasigroup $(\mathbb{R}^+\widetilde{\Gamma}, \mu, \rho, \lambda, \Delta, \varepsilon)$ of Corollary 4.20 restricts to an augmented quasigroup $(\mathbb{N}\widetilde{\Gamma}, \mu, \rho, \lambda, \Delta, \varepsilon)$ in the compact closed category $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$.*

Proof. As noted in [2, Exs. 1.1, 3.1], the Krein parameter in this case is

$$\hat{c}_{ij}^k = \frac{\chi_i(1)\chi_j(1)}{\chi_k(1)} N_{ij}^k \,,$$

where N_{ij}^k is the multiplicity of the irreducible character χ_k in the tensor product $\chi_i \otimes \chi_j$. Thus the multiplication structure

$$\mu \colon \mathbb{R}^+\widetilde{\Gamma} \otimes \mathbb{R}^+\widetilde{\Gamma} \to \mathbb{R}^+\widetilde{\Gamma}^*; x_i \otimes x_j \mapsto \left[x_k \mapsto \hat{c}_{ij}^k \kappa_k\right]$$

restricts to

$$\mu \colon \mathbb{N}\widetilde{\Gamma} \otimes \mathbb{N}\widetilde{\Gamma} \to \mathbb{N}\widetilde{\Gamma}^*; x_i \otimes x_j \mapsto \left[x_k \mapsto \chi_i(1)\chi_j(1)\chi_k(1)N_{ij}^k\right]$$

in $(\underline{\underline{\mathbb{N}}}, \otimes, \mathbb{N})$. □

In this situation, it is convenient to identify $\widetilde{\Gamma}$ directly with the set $\{\chi_1, \dots, \chi_s\}$ of irreducible characters of the group A. Then the Krein parameters may be written as

$$\hat{c}_{ij}^k = \frac{\chi_i(1)\chi_j(1)}{\chi_k(1)} (\chi_i\chi_j|\chi_k)_A$$

with the scalar product

$$(\varphi|\psi)_A = \frac{1}{n}\sum_{a\in A} \varphi(a)\overline{\psi(a)}$$

for functions $\varphi, \psi \colon A \to \mathbb{C}$ [27, §2.3].

Corollary 5.10. *There is a weighted quasigroup $(\widetilde{\Gamma}, w, \alpha, \beta, \gamma)$ with*

$$w \colon \chi_i \mapsto \chi_i(1)^2 \tag{5.9}$$

and

$$\alpha \colon \widetilde{\Gamma} \times \widetilde{\Gamma} \times \widetilde{\Gamma} \to \mathbb{N}; (\chi_i, \chi_j, \chi_k) \mapsto \chi_i(1)\chi_j(1)\chi_k(1) \cdot (\chi_i\chi_j|\chi_k)_A \tag{5.10}$$

as the multiplication function.

Theorem 5.11. *The weighted quasigroup $(\widetilde{\Gamma}, w, \alpha)$ of Corollary 5.10 lifts to a quasigroup $\tilde{A}$ of order n.*

Proof. The gross weight of the weighted quasigroup $(\widetilde{\Gamma}, w, \alpha, \beta, \gamma)$ is $\sum_{i=1}^{s} \chi_i(1)^2 = n$. Then the weighted quasigroup lifts to a quasigroup of order n, by Theorem 5.3. □

Definition 5.12. A quasigroup lift $\tilde{A}$ of $(\widetilde{\Gamma}, w, \alpha)$ is called a *character quasigroup* of the group A.

Example 5.13. If A is a finite abelian group, then the character group of A is the unique character quasigroup of A.

Example 5.14. Consider the symmetric group S_3 on three symbols, with character table

S_3	1	t	c
χ_1	1	1	1
χ_2	1	-1	1
θ	2	0	-1

in the notation of [27, §2.5]. Note $\theta^2 = \chi_1 + \chi_2 + \theta$, for example. Then one may choose $\tilde{S_3} \cong C_3 \times C_2$ as follows:

(5.11)

$\tilde{S_3}$	χ_1	χ_2	θ_1	θ_2	θ_3	θ_4
χ_1	χ_1	χ_2	θ_1	θ_2	θ_3	θ_4
χ_2	χ_2	χ_1	θ_2	θ_1	θ_4	θ_3
θ_1	θ_1	θ_2	θ_3	θ_4	χ_1	χ_2
θ_2	θ_2	θ_1	θ_4	θ_3	χ_2	χ_1
θ_3	θ_3	θ_4	χ_1	χ_2	θ_1	θ_2
θ_4	θ_4	θ_3	χ_2	χ_1	θ_2	θ_1

,

with the covering

$$f\colon \tilde{S_3} \to \tilde{\Gamma}; \chi_1 \mapsto \chi_1, \chi_2 \mapsto \chi_2, \theta_i \mapsto \theta \,. \tag{5.12}$$

By way of illustration, (5.1) yields

$$\begin{aligned}\alpha\colon (\theta,\theta,\theta) \mapsto & \left|\{(a,b) \in f^{-1}\{\theta\} \times f^{-1}\{\theta\} \mid (ab)f = \theta\}\right| \\ &= |\{\theta_1,\theta_2\}^2 \cup \{\theta_3,\theta_4\}^2| = 8 = \theta(1)^3 \cdot 1 \,,\end{aligned}$$

which combines with $\theta(1)^2 = |f^{-1}\{\theta\}| = 4$ from (3.9) and (5.9) to recover the equation $(\theta^2|\theta)_A = 1$ from (5.10).

Note that this particular choice of $\tilde{S_3}$ produces the character group $\tilde{C_6}$ of the cyclic group of order 6. However, to obtain the character scheme of C_6, the covering is bijective, unlike the covering (5.12) used for S_3 here.

Alternative choices for $\tilde{S_3}$ are obtained, for example, by intercalate changes (in the sense of [12, §3]), such as

$$\begin{bmatrix}\theta_3 & \theta_4 \\ \theta_4 & \theta_3\end{bmatrix} \mapsto \begin{bmatrix}\theta_4 & \theta_3 \\ \theta_3 & \theta_4\end{bmatrix} \tag{5.13}$$

in the middle of the body of the multiplication table (5.11).

6. Conclusions and future work

The paper has dealt with two main themes:

- Augmented structures in compact closed categories; and
- Covering quasigroups of weighted quasigroups.

These themes raise a number of questions that form a basis for future work.

6.1. **Augmented structures in compact closed categories.** In this paper, the augmented structures have been motivated primarily by diverse algebras (character algebras, fusion algebras, Bose-Mesner algebras, etc.) from algebraic combinatorics. These algebras typically incorporate preferred bases as an intrinsic part of the data, which are now encoded canonically by the augmentation and comultiplication. Moving outside the domain of algebraic combinatorics, there are other areas where algebras with selected bases make an appearance. For a representative example, take the *algebras with genetic realization* [25, 35] or *stochastic algebras* [22] of mathematical biology that are used in the analysis of population genetics.

In the various application areas, relational structures may be used in place of linear algebras with selected bases, particularly when one wishes to adopt a qualitative or topological rather than quantitative or geometric approach. (Compare [5], say, in algebraic combinatorics, or [32] in population genetics.) The language of augmented structures in compact closed categories now provides a framework for the unification and comparative study of these various approaches.

6.2. **Covering quasigroups of weighted quasigroups.** Covering quasigroups are constructed for Bose-Mesner algebras of association schemes in §5.2, and for character algebras of finite groups in §5.4. The latter construction extends abelian group duality to nonabelian groups. These constructions open up new fields of study, designed to answer the how, the what, and the why of covering quasigroups.

6.2.1. *How are covering quasigroups obtained?* The proof of the main Theorem 5.3 given by Hilton and Wojciechowski in [8, Th. 1], [9, Th. 1] relied on an iterative algorithm (detailed on [8, p.228]) for expanding a weighted quasigroup to a covering quasigroup: row by row, column by column, and symbol by symbol. The expansion is driven by de Werra's regularity result [34]. The general construction is purely combinatorial, and completely disregards any further structure that might be carried by the weighted quasigroup in question.

On the other hand, the illustrative examples of covering quasigroups provided in this paper, namely (5.3) for the Petersen graph or Johnson scheme, and (5.11) for the character quasigroup of the symmetric group S_3, were both constructed manually, essentially by the kind of trial-and-error procedure that lay people use to solve sudoku puzzles. This procedure was guided to some extent by an awareness of the inner structure of the given weighted quasigroup. Thus one pressing research problem opened up by the current paper concerns the development of more specific algorithms for the expansion of weighted quasigroups into covering quasigroups that do take account of the structure inherent to the weighted quasigroups in question.

6.2.2. *What covering quasigroups are obtained?* It has been seen that a given weighted quasigroup may well have several distinct covering quasigroups. Certainly, the iterative construction algorithm of Hilton and Wojciechowski, discussed in the preceding paragraph, may branch at numerous steps and produce distinct covering quasigroups. A full classification of these quasigroups would require a detailed tracing of the algorithm. Emergence under the algorithm from a common weighted quasigroup X of gross weight W defines a new relation, say *cogeneration* relative to X, between certain quasigroups of the same cardinality W. An inverse problem is to examine all the weighted quasigroups of gross weight W that are covered by a given quasigroup of cardinality W. More completely, covering builds a bipartite graph between the set of weighted quasigroups of given gross weight W, and the set of quasigroups of cardinality W.

A direct way to obtain a new covering quasigroup A' from a given one A, both lifting from a specific weighted quasigroup X, is by the kind of intercalate change, compatible with the constraints of the lifting, that is exhibited in (5.13). Classification of all the compatible intercalate changes within a given lifting situation would be relatively tractable.

Remark 5.6 went beyond the kind of combinatorial coincidences that have been discussed so far in this paragraph, suggesting deeper reasons, at least in certain situations, that a given weighted quasigroup might lift to distinct covering quasigroups. It would be instructive to locate similar examples of such phenomena.

Now, rather than seeking to identify distinct lifts of a given weighted quasigroup, an alternative question to is ask for a specific covering quasigroup with particular desired properties. The covering quasigroup

(5.11) provides an example: It is actually a group. This example raises the following group-theoretical problem.

Problem 6.1. Let A be a finite, solvable group. Let $A_1, \dots, A_l$ be the set of cyclic groups appearing as the quotients of a composition series for A. Under what circumstances is $\bigoplus_{i=1}^{l} A_i$ a character group for A?

6.2.3. *What purposes may be served by covering quasigroups?* Weighted quasigroups from combinatorial contexts are not required or expected to carry any additional algebraic structure (although [9] does couch weighted quasigroups in an algebraic language of *simplex zeroids*). On the other hand, covering quasigroups certainly share the rich general algebraic structure of quasigroups, as outlined for example in [28], and supplemented by the subsequent Sylow theory of quasigroups [20, 29]. Whenever a weighted quasigroup X is lifted to a covering quasigroup Q, the lifting may then be used to transfer any selected part of the algebraic structure of Q down to X, thereby enriching the structure of the weighted quasigroup.

This process may be illustrated in the context of §5.4, where the character structure of the symmetric group S_3 was lifted to its character group, the cyclic group C_6. For notational convenience, implement C_6 as the additive group $(\mathbb{Z}/_6, +)$ of residues modulo 6. Using this notation, scalar multiplication by 3 (cubing in the character group) has multiset action

$$\langle 0\rangle \mapsto \langle 0\rangle\,, \quad \langle 3\rangle \mapsto \langle 3\rangle\,, \quad \langle 1,2,4,5\rangle \mapsto \langle 3,0,0,3\rangle\,,$$

which maps down by the covering function (5.12) to

$$\chi_1 \mapsto \chi_1(1)\chi_1\,, \quad \chi_2 \mapsto \chi_2(1)\chi_2\,, \quad \theta \mapsto \theta(1)(\chi_1+\chi_2)\,,$$

and thus reproduces the *Adams operation*

$$\Psi^3(\chi)\colon S_3 \to \mathbb{C}; g \mapsto \chi(g^3)$$

in the character ring of S_3 (compare [27, §9.1, Exercise 3] [31, (4.1.2)]) with

$$\Psi^3\colon \chi_1 \mapsto \chi_1\,, \quad \chi_2 \mapsto \chi_2\,, \quad \theta \mapsto \chi_1+\chi_2$$

as an extra structure on the weighted quasigroup.

In the context of association schemes, a natural first question about the application of the covering quasigroups would be to investigate the connections between their Sylow theory (as in [20, 29]) and the Sylow theory of association schemes studied in [10], say.

Appendix A. Quasigroups as quotients of groups

This appendix exhibits a nonempty quasigroup Q as a quotient of a group G by a subgroup H which is not necessarily normal, based on ideas going back to [1]. The notation of §2.3.1 is used.

Let $(Q, \cdot, /, \backslash)$ be a quasigroup with element e. Write $e^2 = e \cdot e$. If Q is a group, an appropriate choice for e would be the identity element. Define a quasigroup multiplication $+$ on the set Q by

$$x + y = (x/e) \cdot (e\backslash y) .$$

Note that $x + y = x \cdot y$ if e is the identity element in a group $(Q, \cdot)$. In general, $R_+(q) = R_{\cdot}(e)^{-1} R_{\cdot}(e\backslash q)$ for e, q in any quasigroup $(Q, \cdot, /, \backslash)$.

The respective quasigroup identities (IR) and (IL) imply that

$$e^2 + q = q \tag{A.1}$$

for $q \in Q$. Let G be the subgroup of the multiplication group of the quasigroup $(Q, \cdot, /, \backslash)$ generated by

$$T = \{R_+(q) \mid q \in Q\} . \tag{A.2}$$

Let H be the stabilizer of e^2 in the defining right action of G on Q.

Consider an element g of G. Suppose $q = e^2 g$ in Q. Now by (A.1), $q = e^2 + q = e^2 R_+(q)$, whence $g R_+(q)^{-1} \in H$ and $g \in H R_+(q)$. Thus (A.2) is a transversal to H in G, i.e., there is a disjoint union decomposition

$$G = \sum_{q \in Q} H R_+(q) \tag{A.3}$$

of the set G into cosets of H with representatives from T.

Theorem A.1. *For $x, y \in Q$, the membership*

$$R_+(xe) R_+(ey) \in H R_+(xy) \tag{A.4}$$

recovers the quasigroup product xy from the quotient structure (A.3) *of the group G by its subgroup H.*

Proof. Using (A.1), the actions

$$e^2 \xmapsto{R_+(xe)} e^2 + (xe) = xe \xmapsto{R_+(ey)} xe + ey = xy$$

witness the membership (A.4). □

Remark A.2. Note that Theorem A.1 specializes to Cayley's Theorem if e is the identity element of a group Q. In this case, the associativity of Q renders H trivial.

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