ALGEBRAIC METHODS IN APPLIED LOGIC

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Preface

These are the notes for a six-lecture tutorial presented as part of the Summer School in Logic at the Jagiellonian University in Kraków, Poland, in connection with the twelfth conference (TACL 2026) in the Topology, Algebra, and Categories in Logic series. The tutorial focuses on finitary, algebraic methods, in deference to Marcus Tressl's parallel tutorial *Non-Hausdorff Topology in Algebra and Logic*. Likewise, no mention is made of co-algebras, in deference to Ana Sokolova's parallel tutorial *Coalgebra, Algebra, and Logics for Process Semantics*. The scheme of the six lectures is as follows.

1. **Categories:** A novel presentation treats categories as (directed) graphs with algebra structure. The basic notions are diagrams as graph maps from a graph to a category, and natural transformations as maps between diagrams.
2. **Limits and universal algebra:** Limits are presented as natural transformations. Created in word algebras over alphabets, universal algebra relies on two adjunctions: free algebras, and replications.
3. **Duality:** Duality is interpreted as dual equivalence between spaces and coordinate algebras. It is specified on finitary objects, and then extended by (co)limits. Diagrams are used to broaden the reach of any given duality.
4. **Logic action on sets:** Logical structures in commutative rings act on sets within modules. Over fields from the rationals to the reals, barycentric algebras synthesize convex sets and semilattices into hierarchical structures with multiple levels.
5. **Anharmonic action:** Fields are matched to universal algebra: Logical complementation and geometric inversion decompose the projective line over a field into three disjoint unions of a group with a Boolean algebra.
6. **The logic of cells:** Barycentric algebras map the gates of Boolean genetic networks to the continuous gates actually observed in biological experiments. Geometric inversion from Lecture 5 switches activation and inhibition.

As the default options, these notes follow the notational conventions of [43]. For example, if (X, S) denotes structure S on an object X, then a *reduct* of (X, S) is (X, R) for a subset R of S, or more generally for a subset of the full set of structure on X derived from S. In accord with the "algebraic" or "diagrammatic" convention which composes functions in the natural reading order from left to right, functions are placed to the right of their arguments, either on the line or as a superfix (as in $n!$ or x^2, for example).

Contents

1. Categories

1.1. **Quivers and their duals.**

Definition 1.1. A *quiver* or *directed graph* $C = (C_0, C_1, \partial_0, \partial_1)$ consists of two classes C_0, C_1 and two maps $\partial_0 \colon C_1 \to C_0$, $\partial_1 \colon C_1 \to C_0$.

(a) Elements of C_0 are called *vertices*, *points*, or *objects*.
(b) Elements of C_1 are called *edges*, *arrows*, or *morphisms*.
(c) The map ∂_0 is variously called the *tail* or *domain map*.
(d) The map ∂_1 is variously called the *head* or *codomain map*.

Edges f are depicted in the form $f \colon x \to y$ or $x \xrightarrow{f} y$ to indicate that $f^{\partial_0} = x$ and $f^{\partial_1} = y$. For a given pair (x, y) of vertices in a quiver C, set

$$C(x, y) = \{f \in C_1 \mid f^{\partial_0} = x \,, f^{\partial_1} = y\}. \tag{1.1}$$

A quiver $C = (C_0, C_1, \partial_0, \partial_1)$ is *small* if C_0 and C_1 are sets. The quiver is *locally small* if the class (1.1) is a set for each pair (x, y) of vertices. Usually, quivers are implicitly assumed to be locally small.

Definition 1.2. (a) A vertex or object $\top$ of C is said to be *terminal* if $C(x, \top)$ is a singleton for each object x of C.

(b) A vertex or object $\bot$ of C is said to be *initial* if $C(\bot, x)$ is a singleton for each object x of C.

Definition 1.3. The *opposite* or *dual* of a quiver $C = (C_0, C_1, \partial_0, \partial_1)$ is the quiver $C^{\mathsf{op}} = (C_0, C_1, \partial_1, \partial_0)$.

$$C : \quad \begin{tikzcd} & n & \\ & b \arrow[u] & \\ x \arrow[ur] & & y \arrow[ul] \\ & s \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad\qquad C^{\mathsf{op}} : \quad \begin{tikzcd} & n \arrow[d] & \\ & b \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & s & \end{tikzcd}$$

Figure 1. A quiver C and its dual. If the quiver C was considered as the Hasse diagram of a distributive lattice, then C^{op} would be the Hasse diagram of the dual lattice.

In Definition 1.3, note that each edge $x \xrightarrow{f} y$ of C corresponds to an edge $x \xleftarrow{f} y$ of C^{op}. Thus $C^{\mathsf{op}}(y, x) = C(x, y)$.

1.2. **Categories and their duals.**

Definition 1.4. A *category* $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1)$ is defined as a quiver $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1, \partial_0, \partial_1)$ with an *identity function*

$$\epsilon\colon \mathbf{C}_0 \to \mathbf{C}_1; X \mapsto 1_X$$

such that $\epsilon\partial_0 = 1_{\mathbf{C}_0} = \epsilon\partial_1$, and a *composition function*

$$\mathbf{C}(U,V) \times \mathbf{C}(V,W) \to \mathbf{C}(U,W); (f,g) \mapsto fg \tag{1.2}$$

for $\{U,V,W,X\} \subseteq \mathbf{C}_0$, such that

$$1_U f = f = f 1_V$$

for f in $\mathbf{C}(U,V)$, and the *associative law*

$$(fg)h = f(gh) \tag{1.3}$$

holds for f in $\mathbf{C}(U,V)$, g in $\mathbf{C}(V,W)$, and h in $\mathbf{C}(W,X)$.

Example 1.5 (Sets and concrete categories)**.** The *category* **Set** *of sets* has the class $\mathbf{Set}_0$ of sets as its object class and the class $\mathbf{Set}_1$ of functions between sets as its morphism class. For sets X, Y, the set $\mathbf{Set}(X,Y)$ is the set Y^X of functions from X to Y. Thus **Set** is locally small, but not small. The composition function (1.2) is given by the usual mixed associative law (or "law of exponents") $u^{fg} := (u^f)^g$ for $u \in U$. The empty set is initial, and any singleton $\{*\}$ is terminal.

In general, a category $\mathbf{C}$ is *concrete* if its objects are sets (possibly equipped with additional structure) and its morphisms are functions (say respecting the additional structure).

Example 1.6 (Pre-orders)**.** Given a reflexive, transitive relation R on a set P_0, have a category $P = (P_0, P_1)$ with

$$\forall\, x,y \in P_0\,, \quad |P(x,y)| \le 1 \quad \text{and} \quad (x,y) \in R \ \Leftrightarrow\ |P(x,y)| = 1\,. \tag{1.4}$$

In fact, (1.4) characterizes reflexive, transitive relations categorically. An upper bound is terminal; a lower bound is initial.

Example 1.7 (Monoids)**.** Given a monoid structure $(M_1, \bullet, 1)$ on a set M_1, have a category $M = (M_0, M_1)$ with $M_0 = \{*\}$. It has

$$\epsilon\colon M_0 \to M_1; * \mapsto 1 \quad \text{and} \quad \forall\, f,g \in M_1\,,\ fg = f \bullet g$$

for the identity and product structure. In fact, monoids are characterized categorically as small categories with singleton object class. For example, the *transformation monoid* on a set X is the category $\mathbf{Set}(X,X)$ induced from **Set**.

Remark 1.8. (a) Categories unify preorder and monoid structures.

(b) Note how the usual underlying set of the structure became the object set in Example 1.6 and the morphism set in Example 1.7.

A morphism $f\colon x \to y$ in some category is an *isomorphism* if there is a morphism $g\colon y \to x$ such that $fg = 1_x$ and $gf = 1_y$. In this case, the morphism g is unique, written as the *inverse* f^{-1} of the morphism f. Also,the objects x and y are said to be *isomorphic* in C, written $x \cong y$. Isomorphism in C is an equivalence relation.

Proposition 1.9. *Suppose that* $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1)$ *is a category. Then the dual* $\mathbf{C}^{\mathsf{op}}$ *is a category, where the composition function*

$$\mathbf{C}^{\mathsf{op}}(U,V) \times \mathbf{C}^{\mathsf{op}}(V,W) \to \mathbf{C}^{\mathsf{op}}(U,W); (f,g) \mapsto f \circ g$$

is the function

$$\mathbf{C}(V,U) \times \mathbf{C}(W,V) \to \mathbf{C}(W,U); (f,g) \mapsto gf$$

obtained from the composition function within $\mathbf{C}$ *by swapping the order of the factors in the direct product in the domain.*

Remark 1.10. If an object is initial in $\mathbf{C}$, then it is terminal in $\mathbf{C}^{\mathsf{op}}$. For example, in Figure 1, s is a lower bound in C, and an upper bound in C^{op}.

1.3. **Graph maps and diagrams.**

Definition 1.11. A *graph map* $F : D \to C$ from a quiver D to a quiver C consists of two functions, a *vertex map* or *object part* $F_0\colon D_0 \to C_0$ and an *edge map* or *morphism part* $F_1\colon D_1 \to C_1$, such that for each pair x, y of vertices of D, the map F_1 restricts to

$$F_1\colon D(x,y) \to C(x^{F_0}, y^{F_0}). \tag{1.5}$$

The respective suffices 0 and 1 on the object and morphism parts are usually suppressed.

Here are two examples of graph maps:

- The *identity map* $1_D : D \to D$ on a graph D comprises the respective identity maps 1_{D_0} and 1_{D_1} on the vertex and edge classes.
- If $\mathbf{C}$ is a category, and c is an object of $\mathbf{C}$, then the *constant map* $c^{\Delta}\colon D \to \mathbf{C}$ takes each vertex of the graph D to c and each arrow of the graph D to 1_c.

Definition 1.12. A *diagram* in a category $\mathbf{C}$ is a graph map $F\colon D \to \mathbf{C}$. The diagram is *proper* if its domain D is small. A *path* in a directed graph D is a sequence $e_1, \dots, e_l$ of edges such that each pair (e_1, e_2), $\dots$, (e_{l-1}, e_l) is composable. A diagram $F\colon D \to \mathbf{C}$ is said to *commute* if for each pair of paths $(e_1, \dots, e_l)$, $(f_1, \dots, f_m)$ in D with common starting point $e_1\partial_0 = f_1\partial_0$

and end point $e_l\partial_1 = f_m\partial_1$, the composite morphisms $e_1^F \dots e_l^F$ and $f_1^F \dots f_m^F$ in $\mathbf{C}$ agree.

Example 1.13 (Specification)**.** The commuting diagram
$$\begin{tikzcd} \top_1 \arrow[r] \arrow[d] \arrow[dr] & \top_2 \arrow[d] \\ \top_2 \arrow[r] & \top_1 \end{tikzcd}$$
shows that $\top_1 \cong \top_2$ for $\top_1$ and $\top_2$ terminal in some category. Thus objects are specified uniquely to within isomorphism when exhibited as terminal objects in a suitable category. By Remark 1.10, initial objects behave similarly, and may play a similar role in object specification.

1.4. **Natural transformations and functors.**

Definition 1.14. Suppose two diagrams $F\colon D \to \mathbf{C}$ and $G\colon D \to \mathbf{C}$ share a common domain quiver D and codomain category $\mathbf{C}$. Then a *natural transformation* $\tau\colon F \to G$ is a vector having a component $\tau_x\colon x^F \to x^G$ in $\mathbf{C}(x^F, x^G)$ for each vertex x of D, such that for each edge $f\colon x \to y$ of D, the *naturality property* $f^F\tau_y = \tau_x f^G$ is satisfied. The naturality corresponds to the commuting of the diagram in $\mathbf{C}$ on the right-hand side of the picture

$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} x^F \arrow[r, "\tau_x"] \arrow[d, "f^F"'] & x^G \arrow[d, "f^G"] \\ y^F \arrow[r, "\tau_y"'] & y^G \end{tikzcd}$$

$\dots$in D $\qquad\qquad$ $\dots$in $\mathbf{C}$

for every arrow $f\colon x \to y$ in D displayed on the left-hand side of the picture.

Example 1.15. Given a morphism $h\colon c \to d$ in $\mathbf{C}$, the *constant natural transformation* $h^\Delta\colon c^\Delta \to d^\Delta$ has component $h_x^\Delta = h$ at each object x of D.

Proposition 1.16. *For a quiver D and category $\mathbf{C}$, let $\left(\mathbf{C}^D\right)_0$ be the class of diagrams $F\colon D \to \mathbf{C}$. For given diagrams $F\colon D \to \mathbf{C}$ and $G\colon D \to \mathbf{C}$, let $\left(\mathbf{C}^D\right)(F,G)$ be the class of natural transformations $\tau\colon F \to G$. Define $(1_F)_x = 1_{x^F}$ at each vertex x of D. For H in $\left(\mathbf{C}^D\right)_0$ and $\upsilon\colon G \to H$, define $(\tau\upsilon)_x = \tau_x\upsilon_x$ at each vertex x of D. Then $\mathbf{C}^D$ forms a category.*

Definition 1.17. (a) For a quiver D and category $\mathbf{C}$, the category $\mathbf{C}^D$ described in Proposition 1.16 is known as a *diagram category.*

(b) Diagrams $F\colon D \to \mathbf{C}$ and $G\colon D \to \mathbf{C}$ are *naturally isomorphic* if they are isomorphic as objects of $\mathbf{C}^D$.

Definition 1.18. (a) A (*covariant*) *functor* $F\colon \mathbf{D} \to \mathbf{C}$ from a category $\mathbf{D}$ to a category $\mathbf{C}$ is a graph map satisfying the *functoriality properties* $1_x F = 1_{xF}$ for all objects x of $\mathbf{D}$ and

$$(fg)^F = f^F g^F \tag{1.6}$$

for all composable pairs (f, g) of $\mathbf{D}$.

(b) A *contravariant functor* $F\colon \mathbf{D} \to \mathbf{C}$ from a category $\mathbf{D}$ to a category $\mathbf{C}$ is a covariant functor from $\mathbf{D}$ to $\mathbf{C}^{\mathrm{op}}$.

(c) The *identity functor* $\mathbf{1_C}$ on a category $\mathbf{C}$ consists of the identity functions on the classes $\mathbf{C}_0$ and $\mathbf{C}_1$.

(d) For a directed graph D, the *diagonal functor* $\Delta\colon \mathbf{C} \to \mathbf{C}^D$ is defined by $\Delta\colon [h\colon c \to d] \mapsto [h^\Delta\colon c^\Delta \to d^\Delta]$. It encodes $\mathbf{C}$ into $\mathbf{C}^D$, repetitiously.

Example 1.19. If P and Q are pre-orders, as in Example 1.6, then functors $F\colon P \to Q$ are just the order-preserving functions from P_0 to Q_0. Indeed, simply by virtue of F being a graph map, F_1 takes order relationships in P, namely edges in P_1, to edges in Q_1. Equation (1.6) is trivially satisfied, since the arrow from $(fg)^{F\partial_0}$ to $(fg)^{F\partial_1}$ is unique.

Example 1.20. If M and N are monoid categories, as in Example 1.7, then by (1.6), functors $F\colon M \to N$ are just the monoid homomorphisms F_1 from M_1 to N_1.

Example 1.21. Let **BAlg** denote the category of Boolean algebras and homomorphisms between them.

(a) For a function $f\colon X \to Y$, the assignment

$$f \mapsto [\mathbf{2}^X \to \mathbf{2}^Y; A \mapsto Af = \{\, x^f \mid x \in A \,\}]$$

gives a covariant functor from **Set** to **BAlg**.

(b) For a function $f\colon X \to Y$, the assignment

$$f \mapsto [\mathbf{2}^Y \to \mathbf{2}^X; B \mapsto f^{-1}(B) = \{\, x \in A \mid x^f \in B \,\}]$$

gives a contravariant functor from **Set** to **BAlg**.

These are the respective *covariant* and *contravariant power set functors*.

1.5. Adjunctions and equivalence.

Definition 1.22. (a) An *adjunction* $(F, G, \eta, \varepsilon)$ consists of:

- A *left adjoint* functor $F\colon \mathbf{D} \to \mathbf{C}$;
- A *right adjoint* functor $G\colon \mathbf{C} \to \mathbf{D}$;
- A *unit* natural transformation $\eta\colon \mathbf{1_D} \to FG$; and
- A *counit* natural transformation $\varepsilon\colon GF \to \mathbf{1_C}$,

such that the following *triangular identities* are satisfied:

- $\eta_x^F \varepsilon_{xF} = 1_{xF}$ for all objects x of $\mathbf{D}$; and
- $\eta_{yG}\varepsilon_y^G = 1_{yG}$ for all objects y of $\mathbf{C}$.

(b) [14, §IV.1] The adjunction is summarized by the isomorphism

$$\mathbf{C}(xF, y) \cong \mathbf{D}(x, yG) \tag{1.7}$$

for objects x of $\mathbf{D}$ and y of $\mathbf{C}$, under which a morphism $g : xF \to y$ maps to $\eta_x g^G$, while a morphism $f : x \to yG$ maps to $f^F \varepsilon_y$. In particular, η_x corresponds to 1_{xF} and ε_y corresponds to 1_{yG}.

(c) The adjunction provides an *equivalence* between the categories $\mathbf{C}$ and $\mathbf{D}$ if the components η_x and ε_y of the unit and counit are always (natural) isomorphisms. In this case (by the triangular identities), both F and G are mutual right and left adjoints.

Example 1.23 (Free monoids and underlying sets)**.** For a prototypical example of an adjunction, take $\mathbf{C}$ to be the category $\mathbf{Mon}$ of monoids and monoid homomorphisms, with $\mathbf{D}$ as the category $\mathbf{Set}$ of sets. The right adjoint is the underlying set functor $G : \mathbf{Mon} \to \mathbf{Set}$, while the left adjoint $F : \mathbf{Set} \to \mathbf{Mon}$ takes a set X, considered as an alphabet, to the free monoid X^* of words in the alphabet X (with the empty word as the identity element). Words are multiplied by concatenation.

The component $\eta_X : X \to X^*$ at a set X is the *insertion* [1] sending letters (elements) from X into X^* as one-letter words. For a monoid Y, the counit $\varepsilon_Y : Y^* \to Y$ takes a word in the alphabet Y to the product of its letters computed within the monoid Y. Morally, the counit acts like a *multiplication table*.

Under the isomorphism (1.7), a function $f : X \to Y$ from a set X to (the underlying set of) a monoid Y is mapped to its canonical extension to a monoid homomorphism from X^* to Y. In the converse direction, a monoid homomorphism $g : X^* \to Y$ is mapped to its restriction down to the set X of one-letter words.

Example 1.24 (Pre-orders are equivalent to posets)**.** For a prototypical example of an equivalence, take D to be a preorder, a reflexive, transitive relation. Take $C = \{\, [x] \mid x \in D_0 \,\}$, the set of isomorphism classes $[x]$ of objects x of D, as ordered by the well-defined reflexive, antisymmetric and transitive relation $[x] \le [y] \;\Leftrightarrow\; x \le y$ for x, y in D_0.

As in Example 1.19, the order-preserving map $F\colon D_0 \to C_0; x \mapsto [x]$ is a functor. If x_0 is a chosen representative of $[x]$ for any x in D_0, the *election map* $G\colon C_0 \to D_0; [x] \mapsto x_0$ selecting the representatives is also order-preserving, and therefore again gives a functor. These functors are adjoint,

[1] There are varieties of algebras where, unlike this case of monoids, the corresponding unit components are not injective. The identity $x_1 = x_2$ gives an extreme example.

the adjunction isomorphism (1.7) reducing to $|C([x],[y])| = |D(x,y_0)|$ for $x \in D_0$ and $[y] \in C_0$.

At each element $[x]$ of C_0, the component of the counit $\varepsilon\colon GF \to \mathbf{1}_C$ is $\varepsilon_{[x]}\colon [x_0] \mapsto [x]$. As the identity morphism $1_{[x]}$ in C_1, this is certainly an isomorphism, equal to its inverse.

At each element x of D_0, the component of the unit $\eta\colon \mathbf{1}_D \to FG$ is $\eta_x\colon x \to x_0$. This morphism in D_1 is inverted by the morphism $x_0 \to x$.

2. Limits and universal algebra

2.1. Limits.

2.1.1. *Products, coproducts, pullbacks.* Given objects x and y of a category C, their *product* is an object $x \times y$ of C, equipped with *projection* arrows $p : x \times y \to x$ and $q : x \times y \to y$, such that for each object z and arrows $f : z \to x$, $g : z \to y$, there is a unique arrow $f \times g$ or $(f, g) : z \to x \times y$ such that $(f, g)p = f$ and $(f, g)q = g$. This so-called *universality property* of the product may be expressed in the form of a diagram

(2.1)
$$\begin{tikzcd} x & x \times y \arrow[l, "p"'] \arrow[r, "q"] & y \\ x \arrow[u, equal] & z \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, dashed, "f\times g"] & y \arrow[u, equal] \end{tikzcd}$$

known as the *product diagram*. The *coproduct* or *sum* of x and y in C is an object $x + y$ of C, equipped with *insertion* arrows $i : x \to x + y$ and $j : y \to x + y$, such that for each object z and arrows $f : x \to z$, $g : y \to z$, there is a unique arrow $f + g : x + y \to z$ such that $i(f + g) = f$ and $j(f + g) = g$.

(2.2)
$$\begin{tikzcd} x \arrow[r, "i"] \arrow[d, equal] & x + y \arrow[d, dashed, "f+g"'] & y \arrow[l, "j"'] \arrow[d, equal] \\ x \arrow[r, "f"'] & z & y \arrow[l, "g"] \end{tikzcd}$$

Products and coproducts are dual: the coproduct of x and y in C is the product of x and y in C^{op}. Note that products and coproducts are only determined to within isomorphism. For example, in the category **Set** of sets and functions, the product of two sets X and Y may be realized by the "set-theoretical" construction

$$X \times Y = \{\{x, \{x, y\}\} \mid x \in X\,, y \in Y\} \tag{2.3}$$

or by the Cartesian construction

$$X \times Y = \{(x, y) \mid x \in X\,, y \in Y\} \tag{2.4}$$

of ordered pairs. The sum or coproduct $X + Y$ is the disjoint union. It may be realized as $(X \times \{0\}) \cup (Y \times \{1\})$ with insertions $i : X \to X+Y; x \mapsto (x, 0)$ and $j : Y \to X + Y; y \mapsto (y, 1)$.

Example 2.1 (Bounds in a poset)**.** Suppose that x and y are elements (objects) of a poset $(C_0, \leq)$. Then in the corresponding category C, their product $x \times y$ (if it exists) is their greatest lower bound (g.l.b.). Dually, their coproduct $x + y$ (if it exists) is their least upper bound (l.u.b.).

For a map $I \to C_0; i \mapsto x_i$ from a set I to the object class of a category C, the *product* $\prod_{i\in I} x_i$ is an object of C, coming equipped with *projections* $\pi_i\colon \prod_{j\in I} x_j \to x_i$ for each i in I, such that for each object z of C and arrow $f_i\colon z \to x_i$ for each i in I, there is a unique arrow $f = \prod_{i\in I} f_i\colon z \to \prod_{i\in I} x_i$ such that $f\pi_i = f_i$ for each i in I. If $I = \{0,1\}$, then $\prod_{i\in I} x_i$ is just $x_0 \times x_1$. If I is empty, then the product object $\prod_{i\in I} x_i$ is terminal in C. If the map $I \to C_0$ takes the constant value x, then $\prod_{i\in I} x_i$ is called the *power* x^I. The zeroth power X^0 of any object of the category of sets is the empty product, a terminal object or arbitrary singleton $\{*\}$.

Given arrows $f\colon x \to z$ and $g\colon y \to z$ in a category C, their *pullback* is an object $x \times_z y$ of C, equipped with *projection* arrows $p\colon x \times_z y \to x$ and $q\colon x \times_z y \to y$, such that for each object t of C and given arrows $h\colon t \to x$, $k : t \to y$ satisfying $hf = hg$, there is a unique arrow $h \times_z k$ such that $(h \times_z k)p = h$ and $(h \times_z k)q = k$. The pullback is usually displayed as

$$\begin{tikzcd} x \times_z y \arrow[r, "q"] \arrow[d, "p"'] & y \arrow[d, "g"] \\ x \arrow[r, "f"'] & z \end{tikzcd} \quad , \tag{2.5}$$

sometimes additionally with full arrows to its vertices from the object t of C labelled for h and k, together with a dotted arrow from t labelled for $h \times_z k$. Figure 2 provides a more faithful record of the limit process inherent to the pullback, leading to the general treatment of limits and colimits in §2.1.2 below.

Example 2.2. If C is taken as the category **Set**, with appropriate functions $f\colon X \to Z$ and $g\colon Y \to Z$, then the pullback $X \times_Z Y$ is realized as the subset $\{ (x,y) \in X \times Y \mid xf = yg \}$ of the product $X \times Y$. Then the functions $p\colon X \times_Z Y \to X$ and $q\colon X \times_Z Y \to Y$ are just the restrictions of the product projections displayed in (2.1).

Example 2.3. In a small category C, the set $C_1 \times_{C_0} C_1$ of composable pairs is the pullback of the head and tail maps of C in the category **Set**.

2.1.2. *Limits and colimits.* Let $K : J \to C$ be a graph map from a diagram J to a category C. Then the *limit* $\varprojlim K$ of K is an object of C, equipped with a natural transformation $\pi\colon (\varprojlim K)\Delta \to K$ from its diagonal encoding to K, such that the following *limit property* holds:

> Given $t \in C_0$ and a natural transformation $\kappa\colon t^\Delta \to K$, there is a unique morphism $\varprojlim \kappa\colon t \to \varprojlim K$ in C such that $(\varprojlim \kappa)\pi_v = \kappa_v$ for each vertex v of J.

$$\begin{tikzcd}
u \arrow[dd] \\
\\
w \\
\\
v \arrow[uu]
\end{tikzcd}$$

In J

$$\begin{tikzcd}
t \arrow[rr, "h"] \arrow[dr, dashed, "h\times_z k"'] \arrow[dd, equal] & & x \arrow[dd, "f"] \\
& x \times_z y \arrow[ur, "p"'] \arrow[d, equal] & \\
t \arrow[r, dashed, "h\times_z k"] \arrow[dd, equal] & x \times_z y \arrow[r, "pf=qg"] \arrow[d, equal] & z \\
& x \times_z y \arrow[dr, "q"] & \\
t \arrow[ur, dashed, "h\times_z k"] \arrow[rr, "k"'] & & y \arrow[uu, "g"']
\end{tikzcd}$$

In C

In C^J :

$$\begin{tikzcd}
t\Delta \arrow[rr, "\kappa"] \arrow[dr, dashed, "(\varprojlim \kappa)\Delta"'] & & K \\
& (\varprojlim K)\Delta \arrow[ur, "\pi"'] &
\end{tikzcd}$$

FIGURE 2. Commuting diagrams depicting the pullback in a category C as a typical limit of a diagram or graph map $K\colon (u \to w) \mapsto f, (v \to w) \mapsto g$ in the diagram category C^J. The columns in the upper right hand side of the picture (which is located in C) record the graph maps from J. The first two columns there give the diagonal encodings of the respective objects t and $h \times_z k$, while the third column gives the graph map K. The horizontal or up/down sloping arrows represent natural transformations between the graph maps. The respective components of the natural transformation κ are h at u, k at v, and $hf = kg$ at w. In the language of §2.1.2, the C-object $x \times_z y$ is identified as the *limit* $\varprojlim K$, while the C-morphism $h \times_z k$ is identified as the *limit* $\varprojlim \kappa$.

The natural transformation π, together with its components, are known as *projection*(s) from the limit $\varprojlim K$. Limits have also been described as *projective limits* or *inverse limits*. (This latter name is really confusing!) The pullback, as in Figure 2, provides a representative example of a limit.

Dually, the *colimit* $\varinjlim K$ of K is an object of C, equipped with a natural transformation $\iota\colon K \to (\varinjlim K)\Delta$ from K to its diagonal encoding, such that the following *colimit property* holds:

> Given $t \in C_0$ and a natural transformation $\kappa\colon K \to t^\Delta$, there is a unique morphism $\varinjlim \kappa\colon \varinjlim K \to t$ in C such that $\iota_v(\varinjlim \kappa) = \kappa_v$ for each vertex v of D.

The natural transformation ι, together with its components, are known as *insertion*(s) into the colimit $\varinjlim K$. Colimits have also been known as *inductive limits* or *direct limits* (sometimes under restrictions on the domain J of the functor K). Once again, the latter name may cause confusion, this time with the notions of the following definition.

Definition 2.4. (a) A preorder J is *directed* if the condition

$$\forall\ u, v \in J_0\,,\ \exists\ w \in J_0\,.\ \exists\ u \to w, v \to w \in J_1$$

is satisfied.

(b) A limit $\varprojlim K$ is *directed* if the domain J of K is a directed preorder.

(c) A colimit $\varinjlim K$ is *directed* if the domain J of K is a directed preorder.

2.1.3. *Limits and colimits as adjunctions.* A category C is *complete* if it has all limits, and *cocomplete* if it has all colimits. It is *bicomplete* if it is both complete and cocomplete.

Example 2.5 (Complete lattices)**.** A poset $(C_0, \leq)$ forms a complete lattice if and only if the corresponding category C is bicomplete.

If C is a complete category, then $\varprojlim\colon C^J \to C$ is a functor. Consider a diagram $K\colon J \to C$, an object of the diagram category C^J. Then the adjunction

$$C^J(t\Delta, K) \cong C(t, \varprojlim K) \tag{2.6}$$

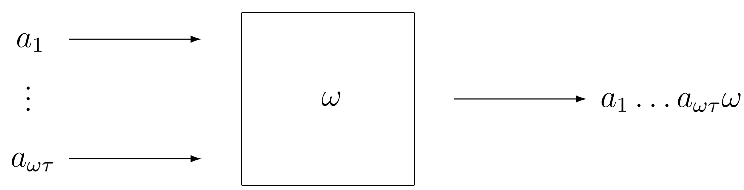
between the functors Δ and $\varprojlim$ captures the properties of the limit given in Section 2.1.2, i.e., $\varprojlim$ is a right adjoint of the diagonal functor Δ. Dually, $\varinjlim$ is a left adjoint of Δ, if C is cocomplete.

2.2. **Combinatorial universal algebra.** A *type* $\tau\colon \Omega \to \mathbb{N}$ is a function whose codomain is the set of natural numbers. (Note that $\mathbb{N}$, as the set of cardinalities of finite sets, includes the cardinality 0 of the empty set. The set of positive integers is denoted by $\mathbb{Z}^+$.) Elements of the domain of τ are called the *basic operators* of the type. An *algebra of type* τ or τ-*algebra* A or (A, Ω) is a set A equipped with *basic operations*

$$\omega : A^{\omega\tau} \to A; (a_1, \ldots, a_{\omega\tau}) \mapsto a_1 \ldots a_{\omega\tau}\omega \tag{2.7}$$

for each basic operator. The class of all τ-algebras is denoted by $\underline{\underline{\tau}}_0$.

Remark 2.6. The operation (2.7) may be interpreted as a circuit element



which, given inputs $a_1, \ldots, a_{\omega\tau}$ from A, returns the output $a_1 \ldots a_{\omega\tau}\omega$. This output may then become an input to another operation.

Definition 2.7. In a τ-algebra A, let B be a subset of A.

(a) The subset B of A is a *subalgebra* of (A, Ω) if

$$\forall\ \omega \in \Omega\,,\ (\forall\ 1 \le i \le \omega\tau\,,\ x_i \in B) \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B. \tag{2.8}$$

(b) The subset B of A is a *sink* of (A, Ω) if

$$\forall\ \omega \in \Omega,\ (\exists\ 1 \le i \le \omega\tau\,.\ x_i \in B) \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B. \tag{2.9}$$

(c) The subset B of A is a *wall* of (A, Ω) if

$$\forall\ \omega \in \Omega\,,\ (\forall\ 1 \le i \le \omega\tau\,,\ x_i \in B) \Leftrightarrow x_1 \ldots x_{\omega\tau}\omega \in B. \tag{2.10}$$

2.2.1. *Subalgebras, products, homomorphisms.* Intersections of subalgebras are subalgebras, so each subset X of A determines a smallest subalgebra $\langle X \rangle$ of A containing X, the *subalgebra generated by* X. Given a family of algebras (A_i, Ω), their product $\prod A_i$ becomes an algebra $\prod(A_i, \Omega)$ or $(\prod A_i, \Omega)$, the *product algebra*, when it is equipped with componentwise operations. A function $f\colon A \to B$ between the underlying sets of algebras $(A, \Omega), (B, \Omega)$ is a *homomorphism* (*of* τ-*algebras*) $f\colon (A, \Omega) \to (B, \Omega)$ if its *graph*

$$\{(a, b) \in A \times B \mid af = b\} \tag{2.11}$$

is a subalgebra of $(A \times B, \Omega)$. (Note that it is often convenient to identify a function with its graph.) The category $\underline{\underline{\tau}}$ has $\underline{\underline{\tau}}_0$ as its object class, and these homomorphisms as its morphisms.

2.2.2. *Congruences.* An equivalence relation V on an algebra A is said to be a *congruence* if it is a subalgebra of $(A \times A, \Omega)$. This implies that the *natural projection*

$$\operatorname{nat} V \colon A \to A^V; a \mapsto a^V, \tag{2.12}$$

mapping an element a of A to its equivalence class $a^V = \{b \in A \mid a\, V\, b\}$ in the *quotient* $A^V = \{a^V \mid a \in A\}$, is a homomorphism. Conversely, the *kernel*

$$\ker f = \{(a, a') \in A^2 \mid af = a'f\} \tag{2.13}$$

of a homomorphism $f \colon (A, \Omega) \to (B, \Omega)$ is a congruence on the domain of the homomorphism. An algebra (A, Ω) is defined to be *simple* if it is not the domain of a non-trivial, non-injective homomorphism. In other words, its only congruences are the *diagonal* $\widehat{A} = \{(a, a) \mid a \in A\}$ and the *universal* or improper congruence $A \times A$.

2.2.3. *The word algebra.* Given a set X, let $X + \Omega$ be the disjoint union of X with Ω. The free monoid $(X + \Omega)^*$ over $X + \Omega$ becomes a τ-algebra under

$$\omega : (w_1, \dots, w_{\omega\tau}) \mapsto w_1 \dots w_{\omega\tau}\omega \tag{2.14}$$

for each basic operator ω. Define $X\Omega$, the subalgebra of $((X + \Omega)^*, \Omega)$ generated by X, to be the *word algebra* or *algebra of τ-words in X*. It may also be described as the *set of words in the language τ*.

Proposition 2.8. *Each function $f \colon X \to A$ from a set X to the underlying set of a τ-algebra (A, Ω) extends to a unique homomorphism*

$$\overline{f} \colon (X\Omega, \Omega) \to (A, \Omega)$$

of τ-algebras.

Proof. The graph of $\overline{f}$ is the subalgebra of $(X\Omega \times A, \Omega)$ generated by the graph of f. □

2.2.4. *Derived operations.* Fix a set D of *variables* or *arguments* using a bijection

$$\beta : \mathbb{Z}^+ \to D; n \mapsto x_n. \tag{2.15}$$

Make the power set 2^D into a τ-algebra by

$$\omega : (A_1, \dots, A_{\omega\tau}) \mapsto A_1 \cup \dots \cup A_{\omega\tau} \tag{2.16}$$

for ω in Ω. By Proposition 2.8, there is a homomorphism

$$\arg \colon D\Omega \to 2^D; x_n \mapsto \{x_n\}, \tag{2.17}$$

called the *argument map*. Note $\arg(x_1 \dots x_{\omega\tau}\omega) = \{x_1, \dots, x_{\omega\tau}\}$. Since the argument map sends each element of $D\Omega$ to a finite subset of D, there is a well-defined function

$$\tau' \colon D\Omega \to \mathbb{N}; w \mapsto \max(\beta^{-1}(\arg w)) \tag{2.18}$$

called the *derived type* of τ. Elements of $D\Omega$ are called *derived operators* of τ. Given a τ-algebra (A, Ω), one obtains a τ'-algebra $(A, D\Omega)$ with *derived operations*

$$u \colon A^{u\tau'} \to A; (a_1, \dots, a_{u\tau'}) \mapsto \overline{u(x_i \mapsto a_i)}. \tag{2.19}$$

In other words, u acts on A by sending $(a_1, \dots, a_{u\tau'})$ to the image of u under the homomorphic extension $\overline{f} : D\Omega \to A$ of the function $f \colon D \to A; x_i \mapsto a_i$. It is convenient to write a derived operator u in the form $x_1 \dots x_{u\tau'}u$, so that (2.19) becomes

$$u \colon A^{u\tau'} \to A; (a_1, \dots, a_{u\tau'}) \mapsto a_1 \dots a_{u\tau'}u. \tag{2.20}$$

A basic operator ω is identified with the corresponding derived operator $x_1 \dots x_{\omega\tau}\omega$.

Finally, fix a τ-algebra (A, Ω). For a subset Δ of $D\Omega$, and restriction $\sigma \colon \Delta \to \mathbb{N}$ of $\tau' \colon D\Omega \to \mathbb{N}$, the σ-algebra (A, Δ) is called a *reduct* of the τ-algebra (A, Ω). Subalgebras of such reducts are called *subreducts* of the original algebra (A, Ω).

2.3. **Categorical universal algebra.** Given a class $\mathbf{V}$ of algebras of type τ, the category $\mathbf{V}$ (same symbol) will denote the category whose object class is the class $\mathbf{V}$, and such that, for $\mathbf{V}$-algebras A and B, the set $\mathbf{V}(A, B)$ of morphisms from A to B is the set of all homomorphisms from A to B. The category $\mathbf{V}$ is the domain of two forgetful functors, the *inclusion*

$$G \colon \mathbf{V} \hookrightarrow \underline{\underline{\tau}} \tag{2.21}$$

and the *underlying set functor*

$$U \colon \mathbf{V} \longrightarrow \mathbf{Set}; (f \colon (A, \Omega) \to (B, \Omega)) \mapsto (f \colon A \to B) \tag{2.22}$$

to the category $\mathbf{Set}$ of sets and functions. Proposition 2.8 shows that $U \colon \underline{\underline{\tau}} \to \mathbf{Set}$ has a left adjoint $\Omega \colon \mathbf{Set} \to \underline{\underline{\tau}}$. For a set X, the unit $\eta_X \colon X \to X\Omega U$ just construes an element x of X as a one-letter word. On the other hand, for a τ-algebra (A, Ω), the counit $\varepsilon_A \colon AU\Omega \to A$ is the homomorphic extension of the identity function $AU \to A; a \mapsto a$ given by Proposition 2.8. In particular, for each basic operator ω, the counit ε_A maps a word $a_1 \dots a_{\omega\tau}\omega$ in $AU\Omega$ to the corresponding element $a_1 \dots a_{\omega\tau}\omega$ of A given by the image of (2.7).

2.3.1. *Prevarieties.* A class $\mathbf{V}$ of algebras of type τ is said to be a *prevariety* if isomorphic copies, subalgebras, and products of $\mathbf{V}$-algebras are $\mathbf{V}$-algebras. When $\mathbf{V}$ is a prevariety, the forgetful functors (2.21) and (2.22) each have left adjoints.

2.3.2. *The replication functor.* The left adjoint of the inclusion $G\colon \mathbf{V} \hookrightarrow \underline{\underline{\tau}}$ is the *replication functor* R_V or $R\colon \underline{\underline{\tau}} \to \mathbf{V}$. The unit of the adjunction, for a τ-algebra A, is the surjective corestriction $\eta_A\colon A \to ARG$ of the product of the set of natural projections $\operatorname{nat}\alpha\colon A \to A^\alpha$ of congruences α on A whose quotient lies in $\mathbf{V}$. In other words, η_A is the natural projection of the smallest congruence ρ_V or ρ on A whose quotient A^ρ lies in $\mathbf{V}$. The congruence ρ is called the $\mathbf{V}$-*replica congruence* of A, and the corresponding quotient A^ρ in $\mathbf{V}$ is called the $\mathbf{V}$-*replica* of A.

2.3.3. *The free algebra functor.* The left adjoint of $U\colon \mathbf{V} \to \mathbf{Set}$ is the *free* $\mathbf{V}$-*algebra functor* $V\colon \mathbf{Set} \to \mathbf{V}$. The unit $\eta_X\colon X \to XVU$ of the adjunction, for a set X, is the composite of the unit $\eta_X\colon X \to X\Omega U$ of the adjunction

$$(\Omega\colon \mathbf{Set} \to \underline{\underline{\tau}}, U\colon \underline{\underline{\tau}} \to \mathbf{Set}, \eta, \varepsilon)$$

and (the image under $U\colon \mathbf{V} \to \mathbf{Set}$ of) the unit $\eta_{X\Omega}\colon X\Omega \to X\Omega RG$ of the adjunction

$$(R\colon \underline{\underline{\tau}} \to \mathbf{V}, G\colon \mathbf{V} \to \underline{\underline{\tau}}, \eta, \varepsilon).$$

In other words, the adjunction

$$(V\colon \mathbf{Set} \to \mathbf{V}, U\colon \mathbf{V} \to \mathbf{Set}, \eta, \varepsilon) \tag{2.23}$$

is the composite of these two given adjunctions. Two τ-words in X are said to be $\mathbf{V}$-*synonymous* if they are related by the replica congruence ρ_V on $X\Omega$. Thus, the free $\mathbf{V}$-algebra on a set X is the algebra of $\mathbf{V}$-synonymy classes.

2.3.4. *Identities and varieties.* Fix a type $\tau\colon \Omega \to \mathbb{N}$. An *identity* in the type τ is a pair (u, v) of derived operators of τ. It is often convenient to write the identity (u, v) in the form $x_1 \dots x_{u\tau'}u = x_1 \dots x_{v\tau'}v$. For example, in the type $\{(\mu, 2)\}$ of a binary "multiplication" μ, the associative law is $x_1x_2\mu x_3\mu = x_1x_2x_3\mu\mu$, or $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$ using infix notation for the multiplication. A τ-algebra (A, Ω) is said to *satisfy* the identity (u, v) if the derived operations u and v coincide on A. The identity (u, v) is *regular* if $\arg u = \arg v$. For example, associativity is regular, but $x_1 = x_2$ is not.

Theorem 2.9 (Birkhoff's Variety Theorem)**.** *A class $\mathbf{V}$ of τ-algebras is closed under homomorphic images, subalgebras and products if and only if it is the class of all τ-algebras satisfying a given set of identities.*

A *variety* is a class $\mathbf{V}$ of τ-algebras that satisfies the two equivalent conditions of Birkhoff's Theorem. Note that a variety is a prevariety. The full set of identities satisfied by each algebra from a variety $\mathbf{V}$ is just the $\mathbf{V}$-replica congruence ρ_V on the algebra $D\Omega$ of derived operators.

2.3.5. *Quasivarieties and quasi-identities.* A prevariety $\mathbf{V}$ of algebras of type τ is said to be a *quasivariety* if it is closed under directed colimits. A *quasi-identity* is an implication of the form

$$\bigwedge_{j=1}^{m} (u^j, v^j) \ \Rightarrow (u, v)$$

with identities $(u^1, v^1), \ldots, (u^m, v^m), (u, v)$.

Theorem 2.10 (Mal'cev's Quasivariety Theorem)**.** [15, Ch. 5] [37, Th. 3.7.12, Cor. 3.7.13] *A class* $\mathbf{V}$ *of* τ*-algebras is a quasivariety if and only if it is the class of all* τ*-algebras satisfying a given set of quasi-identities.*

3. Duality

3.1. **Dual equivalence.**

3.1.1. *Equivalence and dual equivalence.* Recall Definition 1.22(c) for an *equivalence* between two categories $\mathbf{B}$ and $\mathbf{C}$. It consists of two covariant functors $F\colon \mathbf{B} \to \mathbf{C}$ and $G\colon \mathbf{C} \to \mathbf{B}$ such that:

(a) the functors FG and $\mathbf{1_B}$ are naturally isomorphic, and
(b) the functors GF and $\mathbf{1_C}$ are naturally isomorphic.

Definition 3.1. For a *dual equivalence*, the same conditions (a), (b) are imposed on a pair of contravariant functors $F\colon \mathbf{B} \to \mathbf{C}$ and $G\colon \mathbf{C} \to \mathbf{B}$.

Note that a dual equivalence between categories $\mathbf{B}$ and $\mathbf{C}$ amounts to an equivalence between $\mathbf{B}$ and $\mathbf{C}^{\mathsf{op}}$, or between $\mathbf{B}^{\mathsf{op}}$ and $\mathbf{C}$. The notation

$$F : \mathbf{B} \rightleftarrows \mathbf{C} : G$$

will be used to denote a dual equivalence as in Definition 3.1.

3.1.2. *Duality.* A *duality* will denote a dual equivalence

$$D : \mathfrak{A} \rightleftarrows \mathfrak{X} : E \tag{3.1}$$

where $\mathfrak{A}$ is a category of *algebras* (in the sense of modern universal algebra) and homomorphisms, while $\mathfrak{X}$ is a concrete category of objects known as *spaces*. For an algebra A, the image AD is called the *representation space* of A. For a space X, the image XE is called the *algebra represented* by X. The functor D is called the *dual space functor*. The functor E is called the *represented algebra functor*.

Remark 3.2. In (3.1), the determination of which side has the algebras and which side has the spaces may be purely conventional. For example, in the Lindenbaum-Tarski duality (§3.4.3 below), sets are taken as the algebras (with an empty set Ω of operators). One might equally well take the sets as the spaces and the Boolean algebras as the algebras.

3.2. **What functors do with limits.**

3.2.1. *Preservation of limits.* An adjunction $\mathbf{C}(xF, y) \cong \mathbf{B}(x, yG)$ extends pointwise to an adjunction $\mathbf{C}^J(xF\Delta, K) \cong \mathbf{B}^J(x\Delta, KG)$ for a diagram $K\colon J \to \mathbf{B}$. If the diagram $K\colon J \to \mathbf{B}$ has a limit $\varprojlim K \in \mathbf{B}_0$, the natural isomorphisms

$$B^J(x\Delta, KG) \cong C^J(xF\Delta, K) \cong C(xF, \varprojlim K) \cong B\big(x, (\varprojlim K)G\big)$$

then show that $\varprojlim(KG) = (\varprojlim K)G$: **Right adjoints preserve limits**. Dually, **left adjoints preserve colimits**.

3.2.2. *Prevarieties.* Consider a prevariety $\mathbf{V}$ of algebras of type $\tau\colon \Omega \to \mathbb{N}$, with the adjunction (2.23) where the free algebra functor V is left adjoint to the underlying set functor U.

(a) The underlying set of a product $(A, \Omega) \times (B, \Omega)$ in $\mathbf{V}$ is the (Cartesian) product $A \times B$ in $\mathbf{Set}$.

(b) For sets X and Y, the free algebra $(X + Y)V$ over the disjoint union $X + Y$ is the coproduct $XV + YV$ of the respective free algebras XV and YV.

Prevarieties are actually bicomplete [43, §IV.2.2]. As illustrated by (a) here, limits are just constructed pointwise over the corresponding limits in $\mathbf{Set}$. On the other hand, colimits are harder to construct.

3.2.3. *Limits and colimits from dualities.* Consider a dual equivalence

$$F : \mathbf{B} \rightleftarrows \mathbf{C} : G$$

as in Definition 3.1. Here, the functors F and G are contravariant, and mutually adjoint. Preservation of colimits implies that F takes colimits in $\mathbf{B}$ to colimits in $\mathbf{C}^{\mathsf{op}}$, which are limits in $\mathbf{C}$. In general, the functors F and G swap limits to colimits, and *vice versa*.

Now, suppose that a category $\mathfrak{A}$ of algebras features in a duality (3.1). The coproduct $A_1 + A_2$ of two objects A_1, A_2 of $\mathfrak{A}$ will then appear as the algebra $(A_1^D \times A_2^D)E$ represented by the product of the spaces of the individual objects. Typically, these spaces are built pointwise.

3.3. **Currying and residuation.**

3.3.1. *Currying.* In the category $\mathbf{Set}$ of sets and functions, *Currying*

$$\begin{gathered} \mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, Z^Y)) \\ [(x, y) \mapsto z] \mapsto \big[x \mapsto [y \mapsto z]\big] \end{gathered} \tag{3.2}$$

equates a function of two variables with a parametrized family of functions of a single variable. Note that Currying is an adjunction between the functors $X \mapsto X \times Y$ and $Z \mapsto Z^Y$ from the category of sets to itself.

3.3.2. *Heyting algebras.* Let $(H_0, +, \times, \bot, \top)$ be a bounded lattice. For each element y of H_0, Curry the product (or meet) to

$$S_y\colon H_0 \to H_0; x \mapsto y \times x\,,$$

thereby defining a functor $S_z\colon H \to H$ from the poset category to itself. A *residuation* consists of a right adjoint $R_y\colon H \to H$ to S_y, for each $y \in H_0$, which amounts to

$$H(xS_y, z) = H(x, zR_y), \tag{3.3}$$

for $x, y, z \in H_0$, or to having a binary operation $y \multimap z = zR_y$ on H_0 such that

$$x \times y \leq z \ \Leftrightarrow \ x \leq y \multimap z \tag{3.4}$$

for $x, y, z \in H_0$. In this case, the distributive law

$$(x + y) \times z = (x + y)S_z = xS_z + yS_z = (x \times z) + (y \times z)$$

holds (for $x, y, z \in H_0$) since the left adjoints S_z preserves coproducts. The structure $(H_0, +, \times, \multimap, \bot, \top)$ forms a *Heyting algebra*.

3.3.3. *Quasigroups.* A *magma* is a τ-algebra for the type $\tau \colon \{\cdot\} \to \{2\}$. The binary operation is generically construed as a *multiplication.* A magma $(Q, \cdot)$ is a *quasigroup* if the analogue

$$y = x \backslash z \ \Leftrightarrow \ x \cdot y = z \ \Leftrightarrow \ x = z/y \tag{3.5}$$

of the residuation (3.4) holds.

3.4. **Examples of duality.**

3.4.1. *Finite-dimensional vector spaces.* For a field K, take both $\mathfrak{A}$ and $\mathfrak{X}$ to be the category $\underline{\underline{K}}^{<\omega}$ of finite-dimensional vector spaces and linear transformations over K. Then for a finite-dimensional vector space V, both V^D and V^E are defined as the space $\underline{\underline{K}}^{<\omega}(V, K)$ of linear functionals on K.

3.4.2. *Pontryagin duality.* [6],[29, §37]

Take $\mathfrak{A}$ to be the category **Ab** of abelian groups. Suppose that $\mathfrak{X}$ is the category **CHAb** of compact Hausdorff topological abelian groups and continuous homomorphisms. Then, for a (discrete) abelian group A, the representation space is taken as the group $A^D := \mathbf{Ab}(A, \mathbb{R}/\mathbb{Z})$ of *characters* or homomorphisms into the circle group $\mathbb{R}/\mathbb{Z}$, topologized as a subspace of the product space $(\mathbb{R}/\mathbb{Z})^A$. On the other side, a compact Hausdorff abelian group X represents the discrete group $X^E := \mathbf{CHAb}(X, \mathbb{R}/\mathbb{Z})$ of *continuous characters*, continuous homomorphisms into the circle group. The duality between discrete and compact Hausdorff abelian groups is extended to a self-duality on the category of locally compact abelian groups.

3.4.3. *Lindenbaum-Tarski duality.* [11, p. xiv], [46]

Let $\mathfrak{A}$ be the category **Set** of sets. Let $\mathfrak{X}$ be the category **CABA** of complete atomic Boolean algebras and homomorphisms preserving all joins and meets. Consider the set $2 = \{0, 1\}$, possibly endowed with Boolean algebra structure. For a set A, define the representation space A^D as the power set 2^A or $\mathcal{P}(A)$ of (characteristic functions of) subsets of A, with the singletons as atoms. The functor D of (3.1) is the contravariant power set functor of Example 1.21(b). For a complete atomic Boolean algebra B, the represented algebra $B^E := \mathbf{CABA}(B, 2)$ is naturally isomorphic to the set of atoms of B.

3.4.4. *Priestley duality.* [11, pp. 66, 75], [31]
Take $\mathfrak{A}$ to be the category $\mathbf{DL}$ of distributive lattices. Take $\mathfrak{X}$ to be the category $\mathbf{O'Stone}$ of partially ordered Stone spaces (a.k.a Priestley spaces) and monotone continuous maps. Consider the set $2 = \{0 < 1\}$ as a distributive lattice or as a partially ordered Stone space. For a distributive lattice L, the representation space $L^D = \mathbf{DL}(L, 2)$ carries the induced order and subspace topology from the product 2^L. A partially ordered Stone space S represents $S^E = \mathbf{O'Stone}(S, 2)$, a (distributive) sublattice of 2^S.

3.4.5. *Gel'fand duality.* [13, §26E]
Take $\mathfrak{A}$ to be the category $\underline{\underline{C}}^*$ of commutative C^*-algebras and norm-decreasing, involution-preserving algebra homomorphisms. Take $\mathfrak{X}$ to be the category $\mathbf{CH}$ of compact Hausdorff topological spaces. For a C^*-algebra A, the representation space A^D is the space of maximal ideals. A compact Hausdorff space X represents the C^*-algebra $C^*(X)$ of continuous complex-valued functions on X. Compare [10, §38.4] to justify consideration of $\underline{\underline{C}}^*$ as a category of algebras.

3.4.6. *Affine schemes.* [22, II.2 Cor. 1]
Take $\mathfrak{A}$ to be the category $\mathbf{CRing}$ of commutative, unital rings. Take $\mathfrak{X}$ to be the category $\mathbf{Aff}$ of affine schemes. For a ring A, the space A^D is the affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. For an affine scheme X or $(X, \mathcal{O}_X)$, the algebra X^E is the ring $\Gamma(X, \mathcal{O}_X)$ of global sections.

3.4.7. *Dualizing objects.* Each of the examples 3.4.1–3.4.4 is hinged upon a *dualizing object* T which appears in an *algebraic* guise as an object $\underline{T}$ of $\mathfrak{A}$, and in a *spatial* guise as an object $\underset{\sim}{T}$ of $\mathfrak{X}$. The dual space functor D is then naturally isomorphic with $\mathfrak{A}(\ \ , \underline{T})$, while the represented algebra functor E is naturally isomorphic with $\mathfrak{X}(\ \ , \underset{\sim}{T})$:

- In 3.4.1, both $\underline{T}$ and $\underset{\sim}{T}$ are the one-dimensional vector space K.
- In 3.4.2, T is the one-dimensional Torus $\mathbb{R}/\mathbb{Z}$, discrete in its algebraic guise, but with the usual topology in its spatial guise.
- In 3.4.3, T is the Two-element set, with no additional structure in its algebraic guise, but taken as a Boolean algebra in its spatial guise.
- In 3.4.4, T is the Two-element set, as a distributive lattice in its algebraic guise, and as a partially ordered discrete space in its spatial guise.

Note that the duality 3.4.6 does not arise from a dualizing object.

3.4.8. *Finiteness.* In (3.1), each object of $\mathfrak{A}$ generally appears as a directed colimit of its finitely generated subalgebras. To specify the duality (3.1), it then suffices to specify its restriction to finitely generated algebras. The somewhat pathological topologies used in axiomatizing $\mathfrak{X}$ (for example, in the Priestley duality of Section 3.4.4) are often best understood quite simply as directed limits of the representation spaces of finitely generated algebras.

3.5. **Applications of duality.** One important application, construction of coproducts, was already discussed in Section 3.2.3. Some other applications are illustrated here by the examples from Section 3.4.

3.5.1. *Simplification.* It may happen that the space A^D is much simpler than the original algebra A, so the duality may be regarded as a useful way of constructing A from the simple object A^D as A^{DE}. Thus in the finite case, the Priestley duality of Section 3.4.4 reconstructs a distributive lattice as the lattice of down-sets of its poset of join-irreducibles.

3.5.2. *Logic.* Many dualities provide backgrounds for logic. A basic example is Lindenbaum-Tarski duality $D : \mathbf{Set} \rightleftarrows \mathbf{CABA} : E$, as in Section 3.4.3. Propositions describing properties of the elements of a set S live in the complete atomic Boolean algebra S^D.

3.5.3. *Coordinatization.* Dualities like those presented in Section 3.4.1 and Section 3.4.6 may be understood as coordinatizing geometries. For example, in Section 3.4.1, a non-zero linear functional on a vector space V (or element of V^D) measures coordinates along a certain axis.

3.5.4. *Conjugate variables.* In physics, economics, and quite generally in constrained optimization with Lagrange multipliers, the notion of a pair of *conjugate variables* is very important: position and momentum, time and energy, cost and profit, etc. The objects on each side of a duality then provide homes for each of the conjugate variables. Basic illustrations are furnished by the vector space duality of Section 3.4.1 and its extensions.

3.6. **Diagrammatic duality.**

3.6.1. *Diagrammatic algebras.*

Definition 3.3. Suppose that both $\mathfrak{C}$ and $\mathfrak{A}$ are categories of algebras and homomorphisms. Suppose that there is an equivalence $\mathfrak{C} \cong \mathfrak{C}_{\mathfrak{A}}$ between $\mathfrak{C}$ and a subcategory $\mathfrak{C}_{\mathfrak{A}}$ of a diagram category $\mathfrak{A}^J$ with given domain diagram J. In this situation, the objects of $\mathfrak{C}$ are known as *diagrammatic algebras* (relative to $\mathfrak{A}$).

In the context of Definition 3.3, it will often be convenient to abuse the notation and suppress the distinction between $\mathfrak{C}$ and $\mathfrak{C}_{\mathfrak{A}}$, merely stating that a particular $\mathfrak{C}$-algebra C is equivalent to a diagram $\gamma\colon J \to \mathfrak{A}$, or further that a $\mathfrak{C}$-homomorphism $f\colon C_1 \to C_2$ is equivalent to an $\mathfrak{A}^J$-morphism $\varphi\colon \gamma_1 \to \gamma_2$. For this reason, the functors producing the equivalence $\mathfrak{C} \cong \mathfrak{C}_{\mathfrak{A}}$ are not mentioned explicitly in the definition.

3.6.2. *Duality of diagram categories.* The theorem below promotes a dual eqivalence between categories to a dual equivalence between corresponding diagram categories.

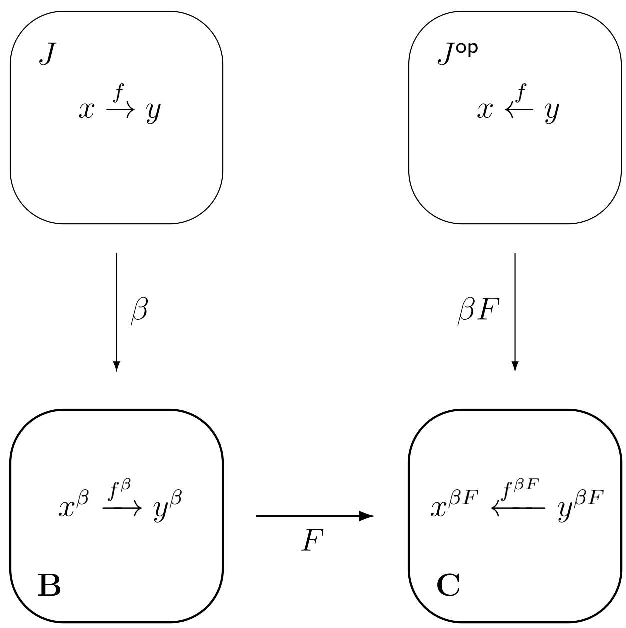


FIGURE 3. The object part of $F^J\colon \mathbf{B}^J \to \mathbf{C}^{(J^{\mathsf{op}})}$. Here, J is a quiver, and $F\colon \mathbf{B} \to \mathbf{C}$ is a functor. Then β is a diagram in $\mathbf{B}$, while βF is a diagram in $\mathbf{C}$.

Theorem 3.4. [40, Th. 31] *Let J be a quiver, and let $F : \mathbf{B} \rightleftarrows \mathbf{C} : G$ be a dual equivalence. Define a contravariant functor*

$$F^J\colon \mathbf{B}^J \to \mathbf{C}^{(J^{\mathsf{op}})}$$

by

$$(F^J)_0\colon \beta \mapsto \beta F$$

(compare Figure 3) and

$$(F^J)_1 \colon \varphi \mapsto \varphi^F$$

for $\varphi \in F^J(\beta_1, \beta_2)$*, where the component of* φ^F *at a vertex* x *of* J *is given by* $\left(\varphi^F\right)_x = (\varphi_x)^F$*. Define a contravariant functor*

$$G^J \colon \mathbf{C}^{(J^{\mathrm{op}})} \to \mathbf{B}^J$$

in similar fashion. Then

$$F^J : \mathbf{B}^J \rightleftarrows \mathbf{C}^{(J^{\mathrm{op}})} : G^J$$

is a dual equivalence.

3.6.3. *Diagrammatic duality.* Consider a duality

$$D : \mathfrak{A} \rightleftarrows \mathfrak{X} : E \tag{3.6}$$

as given by (3.1). In the current context, (3.6) is known as the *seed duality*. Let $\mathfrak{C}$ be a category of algebras that are diagrammatic relative to $\mathfrak{A}$, by an equivalence $\mathfrak{C} \cong \mathfrak{C}_{\mathfrak{A}}$ of $\mathfrak{C}$ with a subcategory $\mathfrak{C}_{\mathfrak{A}}$ of a diagram category $\mathfrak{A}^J$. Then the dual equivalence

$$D^J : \mathfrak{A}^J \rightleftarrows \mathfrak{X}^{(J^{\mathrm{op}})} : E^J$$

given by Theorem 3.4 restricts to a dual equivalence

$$D^J : \mathfrak{C}_{\mathfrak{A}} \rightleftarrows \mathfrak{X}_{\mathfrak{C}} : E^J \tag{3.7}$$

of $\mathfrak{C}_{\mathfrak{A}}$ with a subcategory $\mathfrak{X}_{\mathfrak{C}}$ of $\mathfrak{X}^{(J^{\mathrm{op}})}$. The dual equivalence (3.7) becomes

$$D^J : \mathfrak{C} \rightleftarrows \mathfrak{X}_{\mathfrak{C}} : E^J \tag{3.8}$$

when rewritten in terms of the category $\mathfrak{C}$ that is equivalent to $\mathfrak{C}_{\mathfrak{A}}$.

Definition 3.5. The duality (3.8) is known as *diagrammatic duality* of $\mathfrak{C}$, or between $\mathfrak{C}$ and $\mathfrak{X}_{\mathfrak{C}}$, relative to the seed duality (3.6).

3.7. Three examples of diagrammatic duality.

3.7.1. *Distributive and interlaced bilattices.* Among the simplest examples of diagrammatic algebras are interlaced or distributive bilattices (taken without negation). Define the *superproduct* $B = L_1 \bowtie L_2$ of lattices $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ as the algebra

$$B = (L_1 \times L_2, \wedge, \vee, \cdot, +)$$

with basic operations given by

$$\wedge = (\wedge_1, \vee_2)\,, \quad \vee = (\vee_1, \wedge_2)\,, \quad \cdot = (\wedge_1, \wedge_2)\,, \quad + = (\vee_1, \vee_2)$$

[21]. The reducts $B_1 = (B, \vee, \wedge)$ and $B_2 = (B, +, \cdot)$ of B are lattices. In fact $B_1 \cong L_1 \times L_2^d$, with L_2^d as the dual of L_2, while $B_2 \cong L_1 \times L_2$. Thus, taking the discrete diagram $J = \{\,1, 2\,\}$, superproducts of lattices, called

interlaced bilattices, are diagrammatic with respect to lattices. As a special case, superproducts of distributive lattices, called *distributive bilattices*, are diagrammatic with respect to distributive lattices.

Diagrammatic duality for interlaced bilattices is then provided on the basis of lattice duality (e.g., [8], [39, Th. 4.9]), while diagrammatic duality for distibutive bilattices is built on Priestley duality (§3.4.4). An alternative approach to duality for general distributive bilattices was given in [3], and for bounded distributive lattices in [18, Cor. 12]. The latter paper surveys applications of bilattices in computer science.

3.7.2. *Nelson algebras.* These algebras [41], also known as "N-lattices" [33], provide the algebraic semantics for Nelson's *constructive logic with strong negation* [23, 34].

Definition 3.6. [25, Defn. 2.1] Consider an algebra $(B, \vee, \wedge, \rightarrow, ', 0, 1)$ equipped with three binary operations $\vee, \wedge, \rightarrow$, and one unary operation $'$, the *strong negation*. Suppose that $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, with $\leq$ as the lattice ordering. Then the algebra $(B, \vee, \wedge, \rightarrow, ', 0, 1)$ is a *Nelson algebra* if the following conditions are satisfied:

(i) The De Morgan identities $(x \vee y)' = x' \wedge y'$ and $x' \wedge y' = x' \vee y'$ hold, along with $x'' = x$;
(ii) A reflexive, transitive relation $\preceq$ is defined on B by setting $x \preceq y$ iff $(x \rightarrow y) \rightarrow (x \rightarrow y) = x \rightarrow y$;
(iii) The lattice order relation $x \leq y$ on B is equivalent to $x \preceq y$ and $x' \preceq y'$;
(iv) The equivalence relation χ, defined on B by $x \ \chi \ y$ iff $x \preceq y$ and $y \preceq x$, is a congruence on the reduct $(B, \vee, \wedge, \rightarrow)$ such that the quotient $(B, \vee, \wedge, \rightarrow, 0, 1)^{\chi}$ is a Heyting algebra;
(v) For all $x, y \in B$, one has $(x \wedge x', 0) \in \chi$ and $((x \rightarrow y)', x \wedge y') \in \chi$.

Remark 3.7. Nelson algebras form a variety [25].

Definition 3.8. A congruence α on a Heyting algebra H is *Boolean* if the quotient H^{α} is a Boolean algebra.

Theorem 3.9. [41, Th. 4.1] *The category* **Nelson** *of Nelson algebras is equivalent to the category of pairs* (H, α)*, where* H *is a Heyting algebra and* α *is a Boolean congruence on* H*.*

Consider the quiver J given as $h \xrightarrow{a} b$. By virtue of Theorem 3.9, a Nelson algebra B is equivalent to a diagram $\beta \colon J \rightarrow \mathbf{Heyt}$ sending the arrow a to the natural projection of the Boolean congruence α_B from the Heyting algebra B^{χ}, so Nelson algebras are diagrammatic relative to (the category **Heyt** of) Heyting algebras. This gives a diagrammatic duality for Nelson algebras, based on Esakia duality [7] for Heyting algebras. An alternative approach to duality for Nelson algebras has been discussed in [5, 41].

3.7.3. *Cabalistic duality.* For a *type* $\tau\colon \Omega \to \mathbb{N}$, consider the category $\underline{\underline{\tau}}$ of τ-algebras and homomorphisms between them, as given in Section 2.2.1. The Ω-*cospan* is a quiver Ω_∞ whose edge set is taken to be the operator set $\Omega = \{\,\omega', \omega'', \dots, \omega, \dots\}$ of the type (see Figure 4). Its vertex set is the disjoint union $\Omega + \top$ of Ω with a singleton $\top = \{\infty\}$. The tail map is the identity function on Ω, while the head map is the unique function $\Omega \to \top$.

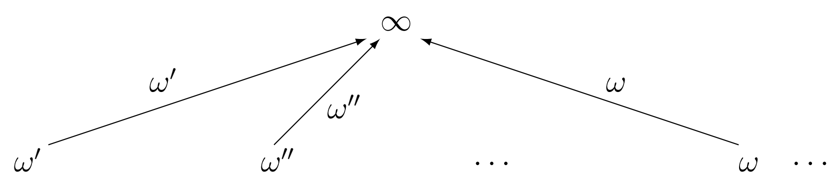


FIGURE 4. The Ω-cospan Ω_∞ for a type $\tau\colon \Omega \to \mathbb{N}$.

A τ-algebra A is equivalent to a diagram $\alpha\colon \Omega_\infty \to \mathbf{Set}$ with edge map $\alpha_1\colon \omega \mapsto (\omega\colon A^{\omega\tau} \to A)$, sending each operator to its operation on the set A. Likewise, each τ-homomorphism $f\colon (A, \Omega) \to (B, \Omega)$ is equivalent to a $\mathbf{Set}^{\Omega_\infty}$-morphism φ with component $\prod_{i=1}^{\omega\tau} f\colon A^{\omega\tau} \to B^{\omega\tau}$ at a vertex ω, and component $f\colon A \to B$ at the unique element ∞ of $\top$. Write $\underline{\underline{\tau}}_{\mathbf{Set}}$ for the subcategory of $\mathbf{Set}^{\Omega_\infty}$ consisting of all these morphisms φ (along with their domains and codomains). For the equivalences, consider the following.

Proposition 3.10. *Suppose that (A, Ω) and (B, Ω) are τ-algebras, with respective equivalent diagrams α and β in $\underline{\underline{\tau}}_{\mathbf{Set}}$. Then a $\mathbf{Set}^{\Omega_\infty}$-morphism $\varphi\colon \alpha \to \beta$ is a $\underline{\underline{\tau}}_{\mathbf{Set}}$-morphism iff $\varphi_\omega = \prod_{i=1}^{\omega\tau} \varphi_\infty$ for each operator ω.*

Proof. The naturality property of φ is the commuting of

(3.9)
$$\begin{tikzcd} A^{\omega\tau} \arrow[r, "\varphi_\omega"] \arrow[d, "\omega"'] & B^{\omega\tau} \arrow[d, "\omega"] \\ A \arrow[r, "\varphi_\infty"'] & B \end{tikzcd}$$

for each operator ω. If $\varphi_\omega = \prod_{i=1}^{\omega\tau} \varphi_\infty$, the commuting of (3.9) shows that $\varphi_\infty\colon A \to B$ is a τ-homomorphism. Conversely, if $\varphi_\infty = f$ and $\varphi_\omega = \prod_{i=1}^{\omega\tau} f$ for operators ω and a τ-homomorphism $f\colon A \to B$, the commuting of (3.9) is given, along with the condition $\varphi_\omega = \prod_{i=1}^{\omega\tau} \varphi_\infty$. $\square$

The equivalence specializes to any class of τ-algebras.

Theorem 3.11. [40, Th. 25] *For any type $\tau\colon \Omega \to \mathbb{N}$, each class* $\mathbf{V}$ *of τ-algebras is equivalent to a subcategory $\mathbf{V}_{\mathbf{Set}}$ of the diagram category $\mathbf{Set}^{\Omega_\infty}$ given by the Ω-cospan Ω_∞.*

Now consider Lindenbaum-Tarski duality

$$D : \mathbf{Set} \rightleftarrows \mathbf{CABA} : E$$

between sets and complete atomic Boolean algebras (§3.4.3). Then with Lindenbaum-Tarski duality as the seed duality (3.6), the corresponding diagrammatic duality

$$D^{\Omega_\infty} : \mathbf{V} \rightleftarrows \mathbf{CABA}_{\mathbf{V}} : E^{\Omega_\infty} \tag{3.10}$$

given by (3.8) is known as *CABAlistic* or *cabalistic duality* for $\mathbf{V}$.

The dual of the Ω-cospan Ω_∞ is known as the Ω-*span* Ω^∞. Thus in the cabalistic duality (3.10), the representation spaces in $\mathbf{CABA}_{\mathbf{V}}$ are objects in the diagram category $\mathbf{CABA}^{\Omega^\infty}$.

Example 3.12 (Quasigroups coordinatize 3-nets)**.** [40, §9] In Section 3.3.3, a quasigroup Q was presented as a residuated magma. To apply cabalistic duality, take left projection $p_1\colon (x_1, x_2) \mapsto x_1$, right projection p_2, and the magma multiplication p_3 as the basic quasigroup operations. Acting on the atoms $\{\, q \,\}$ of the power set $\mathcal{P}(Q)$, the respective cabalistic duals of these basic operations are

$$p_1^{-1}\{q\} = \{(x, y) \in Q^2 \mid x = q\}\,, \tag{3.11}$$

$$p_2^{-1}\{q\} = \{(x, y) \in Q^2 \mid y = q\}\,, \text{ and} \tag{3.12}$$

$$p_3^{-1}\{q\} = \{(x, y) \in Q^2 \mid x \cdot y = q\}\,. \tag{3.13}$$

Then (3.11)–(3.13) are the respective 1-, 2-, and 3-lines which constitute the 3-net as a geometric structure on the set $Q \times Q$. Note, for instance, that $p_1^{-1}\{z/y\}$ is the unique 1-line that passes through $p_2^{-1}\{y\} \cap p_3^{-1}\{z\}$.

4. Logic action on sets

4.1. **Clones.** Suppose that $\mathbf{V}$ is a variety of algebras, construed according to Birkhoff's Theorem 2.9 as a class closed under homomorphic images, subalgebras and products. The adjunction $(V\colon \mathbf{Set} \to \mathbf{V}, U\colon \mathbf{V} \to \mathbf{Set}, \eta, \varepsilon)$ of (2.23) alone suffices to assign a type

$$\kappa_{\mathbf{V}} = \{\, \{\, x_1, \dots, x_n \,\} V \times \{\, n \,\} \mid n \in \mathbb{N} \,\} \tag{4.1}$$

to $\mathbf{V}$-algebras, known as a *clone* (for "closed set of operations").

Remark 4.1. Typically, an abuse of notation that conflates operators and operations may arise in the specification (4.1) if the objects of $\mathbf{V}$ are already construed as τ-algebras for a given type τ. In this case, while $\{\, x_1, \dots, x_n \,\} V$ in (4.1) strictly consists of $\mathbf{V}$-synonymy classes, it can be interpreted as the union of these classes.

For instance, suppose that monoids are construed as having a basic binary multiplication and nullary identity element. Then inside the free monoid $\{\, x_1, x_2, x_3 \,\}^*$ (compare Example 1.23) occuring in (4.1), one may take both $(x_1 \cdot x_2) \cdot x_3$ and $x_1 \cdot (x_2 \cdot x_3)$ as separate clone operators, in place of the single word $x_1 x_2 x_3$. Of course, the corresponding clone operations $(x_1 \cdot x_2) \cdot x_3$ and $x_1 \cdot (x_2 \cdot x_3)$ on monoids coincide, namely with the clone operation $x_1 x_2 x_3$.

Example 4.2. (a) Given an original type τ, the type $\kappa_{\underline{\underline{\tau}}}$ is described as the *closure* $\overline{\tau}$ of τ [43, p.292]. A derived operation $x_1 \dots x_{u\tau'} u$ as in (2.20) is recovered in $\overline{\tau}$ as $(x_1 \dots x_{u\tau'} u, u\tau')$.

(b) For positive $n \in \mathbb{N}$ and $1 \le i \le n$, the clone operations

$$\pi_n^i \colon (x_1, \dots, x_n) \mapsto x_i \,, \tag{4.2}$$

coming from the clone operators (x_i, n), are known as *projections*.

(c) In particular, if U and V in (2.23) are just the identity functor on $\mathbf{Set}$, then the clone $\kappa_{\mathbf{Set}}$ consists entirely of projections.

4.2. **Idempotent and entropic algebras: Modes.**

Definition 4.3. Consider an algebra (A, Ω) of type $\tau\colon \Omega \to \mathbb{N}$.

(a) It is *idempotent* if each element is a subalgebra.

(b) It is *entropic* if each operation

$$\omega\colon (A, \Omega)^{\omega\tau} \to (A, \Omega); (a_1, \dots, a_{\omega\tau}) \mapsto a_1 \dots a_{\omega\tau}\omega$$

is a homomorphism.

(c) It is a *mode* if it is idempotent and entropic.

(d) Operations ω_1 and ω_2 are *mutually entropic*, or simply *commute*, if each operation

$$\omega_i\colon (A,\omega_j)^{\omega_i\tau} \to (A,\omega_j); (a_1,\dots,a_{\omega_i\tau}) \mapsto a_1\dots a_{\omega_i\tau}\omega_j$$

is a homomorphism for $\{\, i,j \,\} = \{\, 1,2 \,\}$.

Remark 4.4. (a) The conditions (a)–(d) on the algebra A are equivalent to identities on the algebra. Thus, a variety $\mathbf{V}$ is said to be *entropic* if each of its algebras is entropic, and a *mode variety* if each of its algebras is a mode.

(b) A magma $(A,\cdot)$ is idempotent if it satifies the identity $x \cdot x = x$, and entropic if the identity

$$(x\cdot u)\cdot(v\cdot z) = (x\cdot v)\cdot(u\cdot z) \tag{4.3}$$

is satisfied. The word "entropic" means "inner turning," referring to the switch of u and v in (4.3).

(d) Many synonyms of "entropic" appear in the literature. The "interchange law" of category theory means the mutual entropy of two binary operations. In semigroup theory, idempotent semigroups are called *bands*, and mode semigroups, entropic bands, are called *normal bands*.

Proposition 4.5. *A variety $\mathbf{V}$ is entropic if and only if $\mathbf{V}(A,B)$ is a subalgebra of the (pointwise) product algebra B^A for any algebras A,B in $\mathbf{V}_0$.*

4.3. **Constructing mode varieties.**

4.3.1. *Direct sums and tensor products.*

Definition 4.6. Suppose that $\mathbf{V}_i$ is a variety of modes of type $\tau_i\colon \Omega_i \to \mathbb{N}$, for $i \in \{\, 1,2 \,\}$, with respective free algebra functors V_i.

(a) [35, Th. 231] The *direct sum* $\mathbf{V}_1 \oplus \mathbf{V}_2$ is the variety that consists of products $A_1 \times A_2$ with $A_i \in \mathbf{V}_i$. Products $(f_1,f_2)\colon (A_1,A_2) \to (B_1,B_2)$ of homomorphisms $f_i\colon A_i \to B_i$ of $\mathbf{V}_i$-algebras are the homomorphisms. The free algebra functor $V_1 \oplus V_2\colon \mathbf{Set} \to \mathbf{V}_1 \oplus \mathbf{V}_2$ is given by $X(V_1 \oplus V_2) = XV_1 \times XV_2$.

(b) [35, Prop. 233] The *tensor product* $\mathbf{V}_1 \otimes \mathbf{V}_2$ is the variety of algebras $(A, \Omega_1 + \Omega_2)$ of type $\tau_1 + \tau_2$ whose τ_i-reducts lie in V_i, and where each τ_1-operation is mutually entropic with each τ_2-operation.

4.3.2. *Sets and zero bands.* As was observed in Example 4.2(c), the clone $\kappa_{\mathbf{Set}}$ consists entirely of the projections (4.2), which are all idempotent and mutually entropic. Thus, $\mathbf{Set}$ is a variety of modes. It acts as an identity element for the mode variety tensor product of Definition 4.6(b).

The basic magma type $\tau\colon \{\cdot\} \to \{2\}$ closes to $\kappa_{\mathbf{Set}}$ if a set A is taken with the *left projection* $x \triangleleft y = x$, creating a *left zero band* $(A, \triangleleft)$, or with the *right projection* $x \triangleright y = x$, creating a *right zero band* $(A, \triangleright)$. The respective magma varieties are written as **Lz** and **Rz**.

4.3.3. *Affine combinations and affine spaces.* Throughout this section, take S as a commutative, unital ring. Suppose that A is an S-module. An (S-)*affine combination* of arguments $x_1, \dots, x_n$ from A is defined as a linear combination $x_1 s_1 + \dots + x_n s_n$ with $s_1 + \dots + s_n = 1$. Let Ω be the set of all S-affine combinations. Then (A, Ω) and its subalgebras (including $\emptyset$) are *affine spaces* (over S). The affine space is *faithful* if the S-module A is faithful.

The class $\underline{\underline{S}}$ of all affine spaces over S constitutes a variety of modes, being closed under homomorphic images, subalgebras and products. Affine combinations in the free S-module $\sum_{x \in X} S$ on a set X create the free affine space on $\{0\} + X$, while $\emptyset$ is the free affine space over $\emptyset$. Furthermore, $\underline{\underline{R \oplus S}} = \underline{\underline{R}} \oplus \underline{\underline{S}}$ and $\underline{\underline{R \otimes S}} = \underline{\underline{R}} \otimes \underline{\underline{S}}$ for commutative, unital rings R and S.

Remark 4.7. Modules over a commutative, unital ring S are diagrammatic relative to $\underline{\underline{S}}$, in the sense of Section 3.6.1. The terminal object $\top$ of $\underline{\underline{S}}$ is the singleton affine space $\{0\}$. An S-module A then corresponds to a diagram $\top \to A$ in $\underline{\underline{S}}$. Since a ring is specified (up to Morita equivalence) by its category of modules (as the endomorphism ring of the free module over a singleton), it is similarly specified by its category of affine spaces.

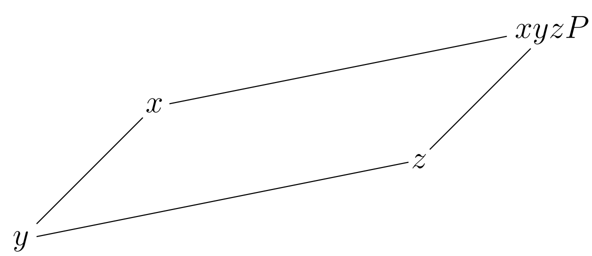


FIGURE 5. The Mal'cev operation $xyzP$, corresponding to the affine combination $x - y + z$, completes the parallelogram with initial sides xy and yz. It satisfies identities $xxzP = z$ and $xyyP = x$. In a Heyting algebra $(H_0, +, \times, \multimap, \bot, \top)$, it is implemented by $((x \multimap y) \multimap z) \times ((z \multimap y) \multimap x)$.

For each element p of the ring S, it is notationally convenient to consider the binary affine combination

$$xy\underline{p} = x(1-p) + yp \tag{4.4}$$

as a basic binary operation on each affine space A over S, creating the algebraic structure $(A, \underline{S})$. For any subset U of S, write

$$\underline{U} = \{\, \underline{p} \mid p \in U \,\}\ . \tag{4.5}$$

The *Mal'cev parallelogram* $(x, y, z)P$ or

$$xyzP = x - y + z \tag{4.6}$$

(Figure 5) is often needed as a basic ternary operation. The basic set $\underline{S} + \{\, P \,\}$ closes to the full set Ω of all S-affine combinations [37, Th. 6.3.4].

If 2 is invertible in S, the equation

$$xyzP = y\,xz\,\underline{1/2}\,\underline{2} \tag{4.7}$$

shows that $\underline{S}$ alone closes to Ω in this case. Then, the variety $\underline{\underline{S}}$ of affine spaces over S is the variety of modes $(A, \underline{S})$ satisfying the identities

$$xy\,\underline{0} = x = yx\,\underline{1} \quad \text{and} \quad xy\,\underline{p}\ xy\,\underline{q}\ \underline{r} = xy\,\underline{pq\,\underline{r}} \tag{4.8}$$

[27],[37, Cor. 6.3.5]. Note that the parsing of the second identity is unique, even without the insertion of any brackets. This is one of the advantages of the algebraic notation (4.4) for binary operations.

The specification (4.8) of the mode variety $\underline{\underline{S}}$ holds true if S has no ideal, even trivial, of index 2 [37, Th. 6.3.6]. More generally, the variety $\underline{\underline{S}}$ of affine spaces over any commutative unital ring S is the variety of modes $(A, \underline{S} + \{\, P \,\})$ where the identities (4.8) and

$$yxyP = xy\,\underline{2}\,, \qquad (xy\,\underline{p}, xy\,\underline{q}, xy\,\underline{r})P = xy\,\underline{pqrP} \tag{4.9}$$

are satisfied [37, Th. 6.3.4].

4.3.4. *Eliminating projections.* Suppose that S is a commutative, unital ring, providing operators for the operations (4.4). Since

$$xy\,\underline{0} = x = x \triangleleft y \quad \text{and} \quad xy\,\underline{1} = y = x \triangleright y \tag{4.10}$$

are just the projections π_1^2 and π_2^2, the following important result holds.

Proposition 4.8. (a) *For any subset U of S, the closure of $\underline{U}$ is the closure of $\underline{U \smallsetminus \{\, 0, 1 \,\}}$.*

(b) *In particular, the clone of $\underline{\underline{\mathsf{GF}(2)}}$ is the set $\kappa_{\mathbf{Set}}$ of projections.*

4.3.5. *Set products and rectangular bands.* As an illustration of the mode variety direct sum of Definition 4.6(a), it is instructive to examine the clone $\kappa_{\mathbf{Set}\oplus\mathbf{Set}}$. For sets A_i with $i \in \{\, 1, 2 \,\}$, implement the product $R = A_1 \times A_2$

as the set of 1×2 matrices, with elements of A_i in the i-th slot. Consider the binary operation

$$\mathbf{diag}\colon R \times R \to R; \begin{bmatrix} \begin{bmatrix} x_{11} & x_{12} \end{bmatrix} \\ \begin{bmatrix} x_{21} & x_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} x_{11} & x_{22} \end{bmatrix} \tag{4.11}$$

that sets up the leading diagonal of a matrix as a row vector.

A semigroup is a *rectangular band* if it satisfies the identity $xyx = x$ (which implies idempotence and entropicity). Write $\mathbf{Re}$ for the variety of rectangular bands. The magma $(R, \cdot)$, whose multiplication denotes infix notation for the binary operation (4.11), is a rectangular band. Conversely, if $(R, \cdot)$ is a rectangular band, the underlying set R of the band decomposes as the product $eR \times Re$, given any element e of R. If R is empty, it has the form $\emptyset \times A$ for any set A.

4.4. **Barycentric algebras.**

4.4.1. *Operations on a ring.* Let S be a commutative, unital ring. Define the *complementation*

$$K\colon S \to S; p \mapsto 1 - p\,, \tag{4.12}$$

which specializes to the Boolean $\neg p$ or NOT p on the set $\{\,0, 1\,\}$ of binary digits. Note that the complementation is *involutive*: K^2 is the identity map 1_S on S. When x and y are elements of an affine space A over a commutative, unital ring S, binary affine combinations (4.4) with $p \in S$ may be rewritten as

$$xy\,\underline{p} = x(1-p) + yp = xp^K + yp\,. \tag{4.13}$$

For $p, q \in S$, define the *product* or *conjunction*

$$p \cdot q = pq \tag{4.14}$$

specializing to the Boolean $\wedge$ or AND on $\{0, 1\}$. Define the *dual product* or *disjunction*

$$p \circ q = p + q - pq \tag{4.15}$$

specializing to the Boolean $\vee$ or (non-exclusive) OR on $\{\,0, 1\,\}$. Note that the dual product may be defined in terms of the product and complementation using *De Morgan's law* $p \circ q = (p^K q^K)^K$ or $(p \circ q)^K = p^K \cdot q^K$.

4.4.2. *Rectangular bands revisited.* Consider the set

$$B = \left\{ \mathrm{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2 \right\}$$

of idempotents of the ring $\mathsf{GF}(2)^2_2$ of 2×2-matrices over the field of order 2. It forms a Boolean algebra $(B, \circ, \cdot, K, \mathrm{O}_2, \mathrm{I}_2)$ under the operations of Section 4.4.1. The diagonal operation (4.11) may be realized as the instance

$$\begin{bmatrix} x_{11} & x_{12} \end{bmatrix} \begin{bmatrix} x_{21} & x_{22} \end{bmatrix} \underline{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} x_{11} & x_{22} \end{bmatrix}$$

of the binary operation (4.13) for the primitive idempotent $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ of $\mathsf{GF}(2)^2_2$ that corresponds to the projection π^2_2 in the clone of the rectangular band. Compare, say, [44, 45].

4.4.3. *If-then-else algebras.* If p is a binary digit 0 or 1, the operation (4.13) makes sense when the inputs x and y are elements of some arbitrary set X, with

$$xy\,\underline{p} = \textbf{if } (p = 1) \textbf{ then } y \textbf{ else } x\,.$$

by virtue of (4.10). This may be written as

$$xy\,\underline{[\![P]\!]} = \textbf{if } P \textbf{ then } y \textbf{ else } x\,, \tag{4.16}$$

recalling that the (Boolean) *truth value* $[\![P]\!]$ of a proposition P is 0 if P is false, and 1 if P is true. Compare, say, [16, 17].

4.4.4. *Convex sets.* If p is any real number, then the operation (4.13) makes sense when the inputs x and y lie in a real vector space. In particular, if p is an element of the closed real unit interval

$$I = [0, 1] = \{p \in \mathbb{R} \mid 0 \le p \le 1\}\,, \tag{4.17}$$

then the operation (4.13) makes sense when the inputs x and y lie in some convex set C, for example some interval on the real line. Indeed, a subset C of a real vector space is closed under all the binary operations $\underline{p}$ for $p \in I$ if and only if it is a convex subset.

Given an arbitrary set X, take the convex set XB of all finite probability distributions on X, identifying each element x of X with the distribution δ_{xy} that puts weight 1 on x and 0 on each of the elements of $X \smallsetminus \{\,x\,\}$. For elements x and y of X, and p in I, the operation (4.13) produces the distribution selecting y with probability p and x with probability p^K. Each element of XB has a unique expression of the form

$$\delta_{x_0}\delta_{x_1}\ldots\delta_{x_r}\,\underline{q}_1\cdots\underline{q}_r \tag{4.18}$$

for some $r \in \mathbb{N}$, ordered subset $\{ x_0 < x_1 \cdots < x_r \}$ of X, and operators $q_1, \dots, q_r$ taken from the interior

$$I^\circ =]0,1[= \{ p \in \mathbb{R} \mid 0 < p < 1 \} \tag{4.19}$$

of I (cf. [37, Lemma 5.8.1]). If $0 < |X| = n \in \mathbb{N}$, then geometrically, XB is realized as a simplex of dimension $n-1$. Also, $\emptyset B = \emptyset$.

A variety $\mathcal{B}$ of so-called $\mathcal{B}$-*algebras* arises from an adjunction

$$(B\colon \mathbf{Set} \to \mathcal{B}, U\colon \mathcal{B} \to \mathbf{Set}, \eta, \varepsilon)$$

of the form (2.23), with $\eta\colon X \to XBU; x \mapsto \delta_{xy}$. In the following section, these algebras are identified as algebras of type $I^\circ \times \{ 2 \}$.

4.4.5. *Barycentric algebras.* The respective ring operations of disjunction, conjunction, and implication introduced in Section 4.4.1 were taken on the matrix ring $\mathsf{GF}(2)^2_2$ in Section 4.4.2 to create the 4-element Boolean algebra $(B, \circ, \cdot, K)$ of idempotents. Now, note that the real open unit interval I° of (4.19) is also closed under the set $\{ \circ, \cdot, K \}$, endowing I° with what may be regarded as a logic structure. The action of this logic structure is encoded in the following specification of the variety $\mathcal{B}$.

Definition 4.9. An algebra $(A, \underline{I}^\circ)$ of type $I^\circ \times \{ 2 \}$ is a *barycentric algebra* if the identities of *idempotence*

$$xx\,\underline{p} = x \tag{4.20}$$

for x in A and p in I°, *skew-commutativity*

$$xy\,\underline{p} = yx\,\underline{p^K} \tag{4.21}$$

for x, y in A and p in I°, and *skew-associativity*

$$xy\,\underline{p}\,z\,\underline{q} = x\,yz\,\underline{q/(p \circ q)}\,\underline{p \circ q} \tag{4.22}$$

for x, y, z in A and p, q in I°.

Remark 4.10. Note that the identities (4.20)–(4.22)that are used to specify barycentric algebras as algebras of type $I^\circ \times \{ 2 \}$ are all regular, in the sense of Section 2.3.4. In the literature, there are various specifications of barycentric algebras (under various names) as algebras of type $I \times \{ 2 \}$, including the left and right projections as basic operations in the type. This is unnecessary, as noted in Proposition 4.8. Worse, it is undesirable, as it complicates the interpretation of semilattices as barycentric algebras presented in Example 4.14.This interpretation underlies various structure theorems for barycentric algebras — compare [36], say.

Certain inequalities establish the soundness of the expression on the right hand side of the skew-associativity identity.

Lemma 4.11. *Suppose* $0 < p, q < 1$ *in* $\mathbb{R}$*. Then* $q < p \circ q$ *and* $q/(p \circ q) < 1$.

Proof. Note $p^K < 1$ gives $p^K q^K < q^K$. Then, since the complementation involution on $\mathbb{R}$ reverses the order, $q = q^{KK} < (p^K q^K)^K = p \circ q$. The second inequality follows. $\square$

Proposition 4.12. *Barycentric algebras, as presented in Definition 4.9, constitute the full class of elements of the variety $\mathcal{B}$.*

Proof. Since the identities (4.20)–(4.22) listed in Definition 4.9 suffice to reduce each word in the language $I^\circ \times \{ 2 \}$ to the form (4.18), it will be enough to show that the non-trivial binary convex combinations $\underline{I}^\circ$ of points of a simplex of dimension 2 satisfy those identities. The idempotence is immediate, and skew-commutativity holds since the complementation K is an involution.

Now, note that

$$xy\,\underline{p}\,z\,\underline{q} = (xp^K + yp)z\underline{q} = xp^K q^K + ypq^K + zq\,, \tag{4.23}$$

while

$$x\,yz\,\underline{r}\ \underline{s} = x(yr^K + zr)\underline{s} = xs^K + yr^K s + zrs \tag{4.24}$$

for $p, q, r, s \in I^\circ$. Equating coefficients of x across the final expressions of (4.23) and (4.24) yields $s = (p^K q^K)^K = p \circ q$. Equating coefficients of z then yields $rs = q$, whence $r = q/s = q/(p \circ q)$. This establishes the skew-associativity. $\square$

4.4.6. *Examples of barycentric algebras.*

Example 4.13 (Convex sets)**.** From §§4.4.4–4.4.5, it is apparent that the class $\mathcal{C}$ of convex sets forms a subclass of the variety $\mathcal{B}$. In fact, amongst all barycentric algebras, the quasi-identities

$$xy\,\underline{p} = xz\,\underline{p} \ \Rightarrow\ y = z$$

of *cancellativity*, for $p \in I^\circ$, characterize convex sets [37, Th. 5.8.6]. Thus, the class $\mathcal{C}$ forms a quasivariety of algebras of type $I^\circ \times \{ 2 \}$. Note that this specification of convex sets is completely intrinsic, and does not rely on any ambient space.

Example 4.14 (Semilattices)**.** In algebraic terms, a *semilattice* $(H, \times)$ is defined as a commutative band: an idempotent, commutative semigroup. Thus, setting $xy\,\underline{p} = x \times y$ for all $p \in I^\circ$, the algebra $(H, \underline{I}^\circ)$ becomes a barycentric algebra, in which skew-commutativity and skew-associativity reduce respectively to actual commutativity and associativity. Amongst all barycentric algebras, these (*iterated*) *semilattices* are characterized by the identities $xy\,\underline{p} = xy\,\underline{q}$ for all p, q in I°. They form the only proper, non-trivial subvariety $\mathcal{S}$ of the variety $\mathcal{B}$.

As a barycentric algebra, a finitely-generated convex set is a *polytope*. In algebraic terms, its vertices are its singleton walls (cf. Definition 2.7(c)), and (the meet reduct of) its *face lattice* is its semilattice replica.

Example 4.15 (The extended reals)**.** As a convex set, the set $\mathbb{R}$ forms a barycentric algebra. Then, the set $\mathbb{R}^\infty := \mathbb{R} + \{\infty\}$ of *extended reals* forms a barycentric algebra, with $\{\infty\}$ as a sink (cf. Definition 2.7(b)). Its semilatttice replica is the two-element semilattice $\{\{\infty\}, \mathbb{R}\}$, with $\{\infty\}$ as a zero element.

4.5. **Implications and residuations.** To complete the specification of barycentric algebras as a logic action on sets, the expression $q/(p \circ q)$ that appears on the right hand side of the skew-associative identity (4.22) will be interpreted, in two natural ways (4.25) and (4.26), as an implication on the interior I° of the closed real unit interval I that extends from the usual Boolean implication on the boundary $\partial I = \{0, 1\}$ of I. Since $0 < q < p \circ q$ in the skew-associativity by Lemma 4.11, it is clear that $q/(p \circ q)$ appears as $(p \circ q) \to q$ from (4.25), and as $(p \circ q) \to_\pi q$ from (4.26)

4.5.1. *Implication as total division.* The motivation for this approach, which will be set in a much broader context later (in Section 5.5), is the desire to place the field structure on $\mathbb{R}$ inside an algebra structure defined purely by total operations. Here, division is replaced by the (*total*) *implication*

$$x \to y = \textbf{if } (x = 0) \textbf{ then } 1 \textbf{ else } y/x \tag{4.25}$$

with *premise* x and *conclusion* y. In particular, a nonzero premise x is the denominator of a quotient. Recall that the quotient appears in the residuation (3.5) of the (quasi)group product $\cdot$ on the set of non-zero reals.

4.5.2. *Łukasiewicz-product logic.* An alternative to the total implication (4.25) is provide by the *product implication*

$$x \to_\pi y = \textbf{if } (x \leq y) \textbf{ then } 1 \textbf{ else } y/x \tag{4.26}$$

forming part of the (ŁΠ-)algebra structure on I arising from the combination of Łukasiewicz and product logic that is discussed in [20, p.110], compare [19, 26]. Here, the product implication residuates the product $\cdot$ on I°, i.e.

$$z \cdot x \leq y \ \Leftrightarrow \ z \leq x \to_\pi y$$

as an analogue of (3.4) [26, p.262].

5. Anharmonic action

5.1. **Projective lines.** Let $\mathbb{F}$ be a field. Consider

$$\widetilde{\mathbb{F}} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \,\middle|\, (a,b) \in \mathbb{F}^2 \smallsetminus \{ (0,0) \} \right\},$$

where (the top row $\begin{bmatrix} a & b \end{bmatrix}$ of) each matrix specifies a point in the punctured plane $\mathbb{F}^2 \smallsetminus \{ \begin{bmatrix} 0 & 0 \end{bmatrix} \}$. Define an equivalence relation

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} R \begin{bmatrix} a' & b' \\ 0 & a' \end{bmatrix} \quad \Leftrightarrow \quad \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = 0$$

that recognizes when $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} a' & b' \end{bmatrix}$ share a line through the origin $\begin{bmatrix} 0 & 0 \end{bmatrix}$ in the plane $\mathbb{F}^2$. Write

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} := \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}_R \tag{5.1}$$

for the equivalence class of a given matrix.

Definition 5.1. (a) The set

$$\widehat{\mathbb{F}} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, (a,b) \in \mathbb{F}^2 \smallsetminus \{ (0,0) \} \right\}$$

is the *projective line* over the field $\mathbb{F}$.

(b) Define the *puncture points*

$$0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \infty := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.2}$$

as the respective *zero*, *unity*, and *infinity* points on the projective line.

(c) The puncture points delimit three open intervals in $\mathbb{R}$:

- The *open unit* interval $I^\circ =]0,1[$;
- The *open large* interval $L^\circ =]1,\infty[$, and
- The *open minus* interval $M^\circ =]\infty, 0[:= \{ x \in \mathbb{R} \mid x < 0 \}$.

Their respective closures I, L, M partition $\widehat{\mathbb{R}}$, overlapping at the puncture points. In any field $\mathbb{F}$, set $\partial I = \{ 0, 1 \}$, $\partial L = \{ 1, \infty \}$, and $\partial M = \{ \infty, 0 \}$.

Remark 5.2. (a) The three points (5.2) are the only points of $\widehat{\mathsf{GF}(2)}$.

(b) Figure 6 shows the *stereographic projection* to the non-compact real line $\mathbb{R}$ from the compact circle S^1 in its role as the projective line $\widehat{\mathbb{R}}$. It may also be interpreted as showing the real part of the stereographic projection to the non-compact complex plane $\mathbb{C}$ from the compact sphere S^2 in its role as the projective line $\widehat{\mathbb{C}}$. Mathematicians refer to S^2 in this context as the *Riemann sphere*; quantum information theorists speak of the *Bloch sphere*.

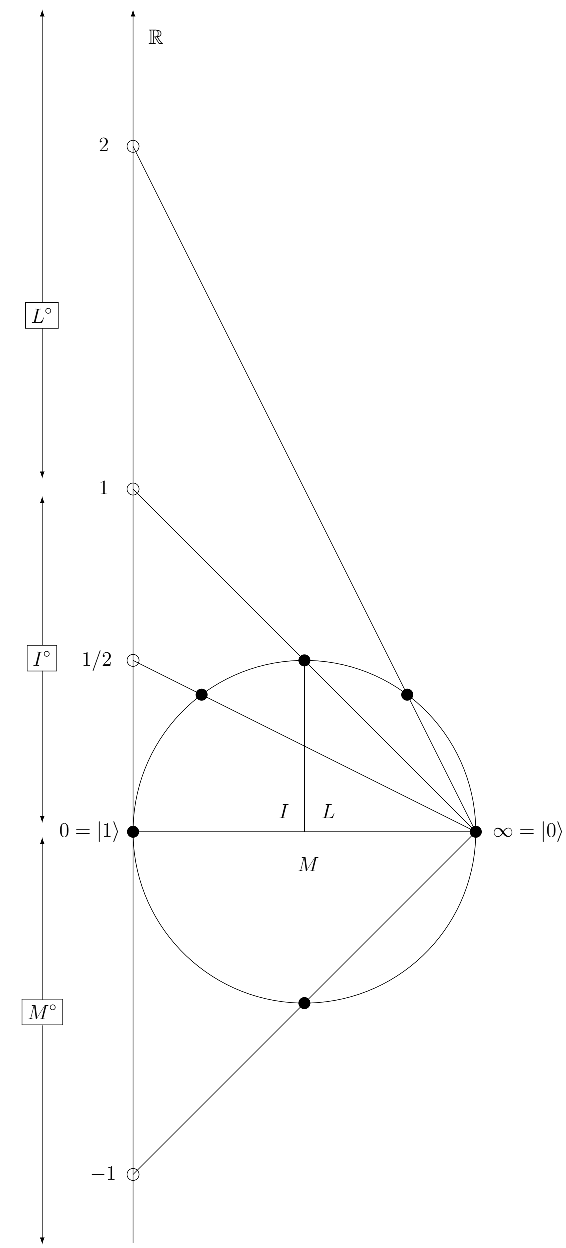


FIGURE 6. Stereographic projection from the circle S^1, as the real projective line $\widehat{\mathbb{R}}$, to the real line $\mathbb{R}$. The three intervals $I = [0, 1]$, $L = [1, \infty]$, $M = [\infty, 0]$ partition the projective line $\widehat{\mathbb{R}}$. The indicated midpoints $1/2$, 2, and -1 of the intervals are respective fixed points of the involutions K, JKJ, and J.

Proposition 5.3. [42, §7.8] (a) *With the exception of the undefined case* $\infty+\infty$*, a commutative and associative sum operation*

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} := \left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ 0 & a' \end{bmatrix} \right)_R \tag{5.3}$$

on $\widehat{\mathbb{F}}$ *is well-defined by the usual matrix product. Note* $x+0=x$ *for* $x\in\mathbb{F}$.

(b) *A commutative, associative product operation*

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} := \begin{bmatrix} aa' & bb' \\ 0 & aa' \end{bmatrix}_R \tag{5.4}$$

on $\widehat{\mathbb{F}}$*, distributing over the sum, is well-defined by the elementwise (or Hadamard) matrix product, with the unity as its neutral element.*

(c) *The function*

$$\mathbb{F} \to \widehat{\mathbb{F}} \smallsetminus \{\infty\}\,;x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{5.5}$$

is a unital ring isomorphism, which may be used to identify $\mathbb{F}$ *with its image.*

(d) *Note that* $x+\infty=\infty$ *for* $x\in\mathbb{F}$*, while* $y\cdot\infty=\infty$ *for* $y\in\widehat{\mathbb{F}}$.

5.2. **Anharmonic action.** The involutive *complementation*

$$K\colon \widehat{\mathbb{F}} \to \widehat{\mathbb{F}}; \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & a-b \\ 0 & a \end{pmatrix}, \tag{5.6}$$

well-defined on the projective line $\widehat{\mathbb{F}}$, provides an extension of the involutive complementation $K\colon x\mapsto 1-x$ of (4.12) defined on the commutative ring $\mathbb{F}$. The involutive *inversion*

$$J\colon \widehat{\mathbb{F}} \to \widehat{\mathbb{F}}; \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} b & a \\ 0 & b \end{pmatrix}, \tag{5.7}$$

well-defined on the projective line $\widehat{\mathbb{F}}$, provides an extension of the involutive inversion $J\colon x\mapsto 1/x$ defined on $\mathbb{F}\smallsetminus\{0\}$. Note

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^J \cdot \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}^J = \begin{pmatrix} bb' & aa' \\ 0 & bb' \end{pmatrix} = \begin{pmatrix} aa' & bb' \\ 0 & aa' \end{pmatrix}^J, \tag{5.8}$$

so that J is an automorphism of the product (5.4).

On the set $\{0,1,\infty\}$ of puncture points, the complementation K acts as the transposition $(0\ 1)$ interchanging the zero with the unity, and fixing infinity. Likewise, the inversion J acts as the transposition $(0\ \infty)$ that interchanges the zero with infinity, and fixes the unity. The equal composites $JKJ=KJK$ act as $(1,\infty)$, fixing the zero. Abstractly, all three of K,J, and JKJ are involutions.

$$\begin{tikzcd}
\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \arrow[r, dash, "J"] \arrow[d, dash, "K"'] & \begin{pmatrix} b & a \\ 0 & b \end{pmatrix} \arrow[r, dash, "K"] & \begin{pmatrix} b & b-a \\ 0 & b \end{pmatrix} \arrow[d, dash, "J"] \\
\begin{pmatrix} a & a-b \\ 0 & a \end{pmatrix} \arrow[r, dash, "J"'] & \begin{pmatrix} a-b & a \\ 0 & a-b \end{pmatrix} \arrow[r, dash, "K"'] & \begin{pmatrix} b-a & b \\ 0 & b-a \end{pmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\frac{x}{1}} \arrow[r, dash, "J"] \arrow[d, dash, "K"'] & \boxed{\frac{1}{x}} \arrow[r, dash, "K"] & \boxed{\frac{x-1}{x}} \arrow[d, dash, "J"] \\
\boxed{\frac{1-x}{1}} \arrow[r, dash, "J"'] & \boxed{\frac{1}{1-x}} \arrow[r, dash, "K"'] & \boxed{\frac{x}{x-1}}
\end{tikzcd}$$

FIGURE 7. The action of the anharmonic group on points of the projective line $\widehat{\mathbb{F}}$ over a field $\mathbb{F}$. The upper diagram shows the action in terms of the matrices (5.1). The lower diagram gives the values x' of the respective transformations $x \mapsto x'$.

Definition 5.4. [4, pp.110,120], [9, §2] Over a field $\mathbb{F}$, the *anharmonic group* is defined as the group of transformations of the projective line $\widehat{\mathbb{F}}$ that is generated by the complementation (5.6) and inversion (5.7).

The action of the anharmonic group on a general point of the projective line $\widehat{\mathbb{F}}$ is displayed in Figure 7. From the closure of the diagrams, it is clear that the anharmonic group over any field has at most 6 elements. Thus, the permutation representation of the anharmonic group on the three-element set $\{0, 1, \infty\}$ of puncture points is faithful. Indeed, as an abstract group, the anharmonic group is isomorphic to the symmetric group Σ_3 of degree 3. On the puncture points, JK is the 3-cycle $(0\ \infty\ 1)$, with inverse KJ.

The automaton diagram, action diagram, or Cayley diagram

$$\begin{tikzcd}
\infty \arrow[loop left, dash, "K"] \arrow[r, dash, "J"] & 0 \arrow[r, dash, "K"] & 1 \arrow[loop right, dash, "J"]
\end{tikzcd} \tag{5.9}$$

gives a convenient full description of the action of the anharmonic group on the puncture points; indicating, for example, that $0JK = \infty K = \infty$. It follows that the real closed intervals depicted in Figure 6 are permuted according to

$$ {}_K\big(\, I \overset{J}{\text{———}} L \overset{K}{\text{———}} M \,\big)_J \tag{5.10}$$

under the action of the anharmonic group, with a corresponding anharmonic action on their interiors as introduced in Definition 5.1(c). From (5.10), it is apparent that the involutions K, J, and JKJ respectively act on the intervals I, M, and L. Under the first of these actions, the mid-point $1/2$ of I is fixed by K. The anharmonic group acts on this mid-point as depicted in the diagram

$$ {}_K\big(\, 1/2 \overset{J}{\text{———}} 2 \overset{K}{\text{———}} -1 \,\big)_J \tag{5.11}$$

which shows that -1 is fixed by J and 2 is fixed by JKJ. Thus, the points -1 and 2 are identified as respective *mid-points* of M and L (Figure 6).

5.3. **De Morgan laws and boundary Booleans.**

5.3.1. *De Morgan laws.* The De Morgan laws relating conjunction $\cdot$ and disjunction $\circ$ on a field $\mathbb{F}$, as they appear in Section 4.4.1, make use of the complementation involution. Recalling $\infty\cdot\infty = \infty$ from Proposition 5.3(d), and the fact that K fixes ∞, it is clear that these classical De Morgan laws extend to the full projective line $\widehat{\mathbb{F}}$. In particular, the equation $\infty \circ \infty = \infty$ holds.

Lemma 5.5 (Classical De Morgan laws)**.** *For $x, y \in \widehat{\mathbb{F}}$, the identities*

$$(x \circ y)^K = x^K \cdot y^K \quad \textit{and} \quad (x \cdot y)^K = x^K \circ y^K \tag{5.12}$$

hold: The involution $K \colon \big(\widehat{\mathbb{F}}, \cdot, 1\big) \to \big(\widehat{\mathbb{F}}, \circ, 0\big)$ is a monoid isomorphism.

Definition 5.6. (a) The *extended De Morgan law*

$$x \times y = (x^{JKJ} \cdot y^{JKJ})^{JKJ} \tag{5.13}$$

defines the *injunction* operation $\times$ on the projective line $\widehat{\mathbb{F}}$. Indeed, the involution $JKJ \colon (\widehat{\mathbb{F}}, \cdot, 1) \to (\widehat{\mathbb{F}}, \times, \infty)$ is a monoid isomorphism.

(b) Collectively, it is convenient to refer to the disjunction, conjunction and injunction simply as the *junctions*. Note that they are all commutative.

Lemma 5.7 (More extended De Morgan laws)**.** *The identities*

$$(x \times y)^J = x^J \circ y^J \quad \textit{and} \quad (x \circ y)^J = x^J \times y^J \tag{5.14}$$

hold for $x, y \in \widehat{\mathbb{F}}$. In other words, the involution $J \colon \big(\widehat{\mathbb{F}}, \circ, 0\big) \to \big(\widehat{\mathbb{F}}, \times, \infty\big)$ is a monoid isomorphism.

Proof. Note that $x^J \times y^J \overset{(5.13)}{=} (x^{JJKJ} \cdot y^{JJKJ})^{JKJ} = (x^{KJ} \cdot y^{KJ})^{JKJ} \overset{(5.8)}{=} (x^K \cdot y^K)^{KJ} \overset{(5.12)}{=} (x \circ y)^J$. $\square$

$$\begin{tikzcd} \big(\widehat{\mathbb{F}},\times,\infty\big) \arrow[loop left, "K"] \arrow[r, dash, "J"] & \big(\widehat{\mathbb{F}},\circ,0\big) \arrow[r, dash, "K"] & \big(\widehat{\mathbb{F}},\cdot,1\big) \arrow[loop right, "J"] \end{tikzcd}$$

FIGURE 8. The anharmonic action (5.9) on the puncture points enriches to a diagram of monoid isomorphisms, where the puncture points are the neutral elements of the respective monoids. By (5.8), J is an automorphism of $\big(\widehat{\mathbb{F}},\cdot,1\big)$. By the classical De Morgan laws, K is a monoid isomorphism, and KJK is an automorphism of $\big(\widehat{\mathbb{F}},\circ,0\big)$. Finally, by the extended De Morgan laws, J is a monoid isomorphism, and $K = J(JKJ)J = J(KJK)J$ is an automorphism of $\big(\widehat{\mathbb{F}},\times,\infty\big)$.

As shown in Figure 8, the De Morgan laws imply that the anharmonic group forms a set of monoid isomorphisms.

5.3.2. *Boundary Booleans.* Over a field $\mathbb{F}$, the junctions of Definition 5.6(b) feature in Table 1, presenting Boolean structures on the interval boundaries $\partial I = \{\,0,1\,\}$, $\partial L = \{\,1,\infty\,\}$, and $\partial M = \{\,\infty,0\,\}$. To avoid confusion, verbal names are given in the header for the respective Boolean operations (binary, unary, and nullary). For example, the disjunction $\circ$, which as usual provides the OR connective on ∂I, implements the AND connective on ∂M. It is of interest to note that the boundaries appearing in Table 1 are the respective sets of non-invertible elements in the monoids of Figure 8, complementing the groups of units of these monoids. In other words, anharmonic action permutes three ways of exhibiting the projective line as a disjoint union of an associative quasigroup (i.e., a group or the empty set) with a Boolean algebra.

set	or	and	not	implication	false	true
$\partial I = \{\,0,1\,\}$	$\circ$	$\cdot$	K	$x^K \circ y$	0	1
$\partial L = \{\,1,\infty\,\}$	$\cdot$	$\times$	JKJ	$x^{JKJ} \cdot y$	1	∞
$\partial M = \{\,\infty,0\,\}$	$\times$	$\circ$	J	$x^J \times y$	∞	0

TABLE 1. Boolean structures on the respective interval boundaries $\partial I = \{\,0,1\,\}$, $\partial L = \{\,1,\infty\,\}$, and $\partial M = \{\,\infty,0\,\}$.

Natively, Figure 6 displayed the stereographic projection from the circle S^1, as the real projective line $\widehat{\mathbb{R}}$, to the real line $\mathbb{R}$. However, as observed in Remark 5.2(b), it also shows the real cross-section of the stereographic projection to the complex plane $\mathbb{C}$ from the Bloch sphere S^2 as the complex projective plane $\widehat{\mathbb{C}}$.

Recall that a *bit* or *binary digit* is a classical system with two states 0 and 1, logically interpreted as “false” and “true” in accord with the **top** row of the body of Table 1. A *qu(antum)bit* is a quantum system whose Hilbert space is $\mathbb{C}^2$, spanned by a classical bit with states written in Dirac's “ket” format as $|0\rangle$ and $|1\rangle$. The Bloch sphere represents the set of pure states of a qubit, including the two classical states $|0\rangle$ and $|1\rangle$. In stereographic projection, these respective states of “false” and “true” are aligned with the elements ∞ and 0 of $\widehat{\mathbb{C}}$, this time matching the **bottom** row of Table 1.

5.4. **The Three Ring Circus.** A field $\mathbb{F}$ is normally regarded as a special kind of commutative ring, where the set of non-zero elements forms a group. Anharmonic action builds on this traditional understanding, identifying the projective line $\widehat{\mathbb{F}}$ over $\mathbb{F}$ as a structure comprising three rings, with respective underlying sets that are complementary to the singletons of the puncture points. This structure is known (light-heartedly) as the *Three Ring Circus*, presented in Figure 9.

$$
\begin{tikzcd}
& \big(\overline{\{\infty\}},+,\cdot,0,1\big) \arrow[dl, "JK"'] & \\
\big(\overline{\{1\}},\circledast,\circ,\infty,0\big) \arrow[rr, "JK"'] & & \big(\overline{\{0\}},\boxplus,\times,1,\infty\big) \arrow[ul, "JK"']
\end{tikzcd}
$$

FIGURE 9. The *Three Ring Circus.* Here, $\big(\overline{\{\infty\}},+,\cdot,0,1\big)$ gives the usual ring structure on $\mathbb{F}$, the complement $\widehat{\mathbb{F}}\smallsetminus\{\infty\}$ of $\{\infty\}$ on the projective line $\widehat{\mathbb{F}}$. The 3-cycle JK in the anharmonic action isomorphically transfers this ring structure to copies on the complements of $\{1\}$ and $\{0\}$.

The junctions form the multiplications of the three rings. The addition of the ring $\overline{\{1\}}$ is given by $x \circledast y = (x^{KJ} + y^{KJ})^{JK}$, or by

$$
\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \circledast \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} = \begin{pmatrix} \operatorname{perm}\begin{bmatrix} a & a-b \\ a' & a'-b' \end{bmatrix} & \operatorname{perm}\begin{bmatrix} a & a'-b \\ a' & a-b' \end{bmatrix} \\ \\ 0 & \operatorname{perm}\begin{bmatrix} a & a-b \\ a' & a'-b' \end{bmatrix} \end{pmatrix}
$$

in terms of the matrix representation of $\widehat{\mathbb{F}}$ and matrix permanents. On the other hand, the rational form for the addition

$$x \boxplus y = (x^{JK} + y^{JK})^{KJ} = \frac{x \cdot y}{x \circ y}$$

of the ring $\overline{\{\,0\,\}}$, as the quotient of the conjunction by the disjunction (valid directly for $x \circ y \neq 0$ — mnemonically "x and y not both false"), is notable. Alternatively, the perhaps more elegant form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \boxplus \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} = \begin{pmatrix} ab' - bb' + ba' & bb' \\ 0 & ab' - bb' + ba' \end{pmatrix}$$

is given by the matrix representation.

5.5. **Residuations.** Working with the field $\mathbb{R}$, Section 4.5.1 introduced the implication

$$x \to y = \textbf{if } (x = 0) \textbf{ then } 1 \textbf{ else } y/x$$

of (4.25), which restricts to the usual implication $x^K \circ y$ on $\partial I = \{\,0, 1\,\}$ within any field $\mathbb{F}$, as displayed in the first row of Table 1.

On the other hand, consider the associative quasigroup $\widehat{\mathbb{F}} \smallsetminus \{\,\infty, 0\,\}$, which is empty if $|\mathbb{F}| = 2$, and otherwise is the group of units of the monoid $\big(\widehat{\mathbb{F}}, \cdot, 1\big)$ presented in Figure 8. There, (4.25) restricts to the usual residuation $x \backslash y = x^J \cdot y$ of the associative quasigroup, as discussed in Section 3.3.3.

Anharmonic action extends these specific implications and residuations to a complete set on the boundary Booleans and their complementary groups, as displayed in Table 2.

boundary	implication	complement	residuation
$\partial I = \{\,0, 1\,\}$	$x^K \circ y$	$\widehat{\mathbb{F}} \smallsetminus \{\,0, 1\,\}$	$x^K \times y$
$\partial L = \{\,1, \infty\,\}$	$x^{JKJ} \cdot y$	$\widehat{\mathbb{F}} \smallsetminus \{\,1, \infty\,\}$	$x^{JKJ} \circ y$
$\partial M = \{\,\infty, 0\,\}$	$x^J \times y$	$\widehat{\mathbb{F}} \smallsetminus \{\,\infty, 0\,\}$	$x^J \cdot y$

TABLE 2. Implications on boundary Booleans paired with residuations on their (associative quasi)group complements.

6. The logic of cells

6.1. **Beyond Boolean networks.** In the nineteenth century, George Boole described what we now call *Boolean algebra* as providing “the fundamental laws of those operations of the **mind** by which reasoning is performed” [2, p.1]. In more recent times, Boolean networks have extended the reach of Boolean algebra to “operations of the **body**,” describing biomolecular networks of gene expression [47]. Now, we will see how the calculi (4.16) of if-then-else algebras and of barycentric algebras (Definition 4.9) may be used in systems biology, to provide a virtually automatic translation from the basic Boolean models of gene expression to more realistic continuous or *fuzzy* models [38]. Experimentally observed logic gates in cells do not follow the pattern directly suggested by standard Boolean models, but their features match closely with models obtained using the barycentric algebra approach (Figures 14, 15).

6.2. **Systems biology: transcription factors.** Cells thrive by producing proteins in response to various signals that they receive. Here, a simplified model of the process will be adequate, but for fuller details, see [1], [12], [28, §19.2]. A protein is produced by the expression of a corresponding part of the cell’s DNA, namely the gene that encodes for that protein. The gene is first *transcribed* to messenger RNA (mRNA). The mRNA is then *translated* into the required protein. The transcription process, synthesis of the mRNA, is facilitated by the RNA polymerase (RNAp) enzyme. The enzyme binds itself to a regulatory region of the DNA, adjacent to the gene, known as the *promoter site* (Figures 10, 11).

Signals that are of importance to a cell may be physical, such as a change in temperature, or chemical, such as the presence of a nutrient like glucose. Received signals switch proteins that are known as *transcription factors* from a dormant to an active state. The active transcription factors attach themselves to the promoter site, where they change the binding probability of the RNAp. When a transcription factor is an *activator* or *promoter*, its adhesive interaction with the RNAp will increase the binding probability of the RNAp at the promoter site, thereby increasing the rate of transcription and protein production (Figure 10). Other transcription factors that are known as *repressors* or *inhibitors* have the opposite effect of inhibiting the expression of certain genes. In actual practice, there are various mechanisms for this, such as deformation of the DNA [28, Fig. 19.6], but in the schematic representation of Figure 11, a simple blocking of the access of the RNAp to its binding site is suggested.

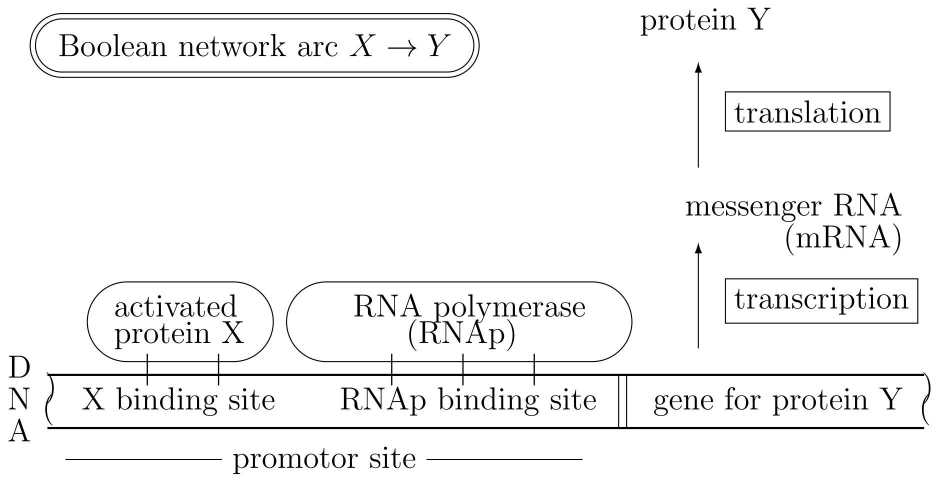


FIGURE 10. A simplified model of a cellular process, where the activated protein X (as a transcription factor) **promotes** the production of protein Y. The inset at the top left shows a typical Boolean network symbolism for "X promotes Y.

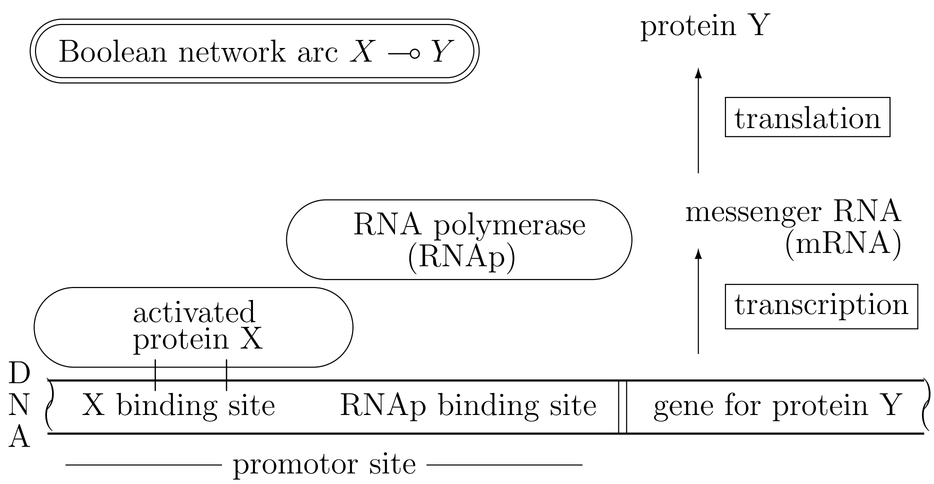


FIGURE 11. A simplified model of a cellular process, where the activated protein X (as a transcription factor) **represses** the production of protein Y. The inset at the top left shows a typical Boolean network symbolism for "X represses Y.

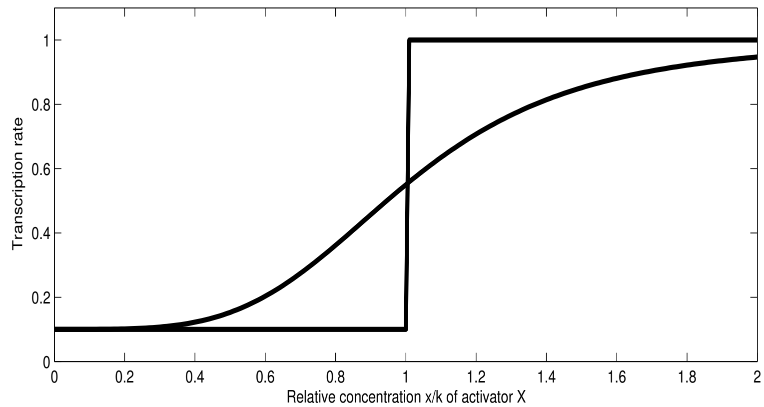


FIGURE 12. Crisp (Boolean) and fuzzy dependencies of the transcription rate on the concentration x of an activator X, relative to a fixed critical threshold concentration k.

6.3. **Boolean and fuzzy logic.**

6.3.1. *Boolean or crisp logic.* Fig. 12 displays sample dependencies of the transcription rate for production of a protein on the dimensionless relative concentration x/k of an activator X. Here, x is the actual concentration variable, while k is a fixed critical threshold concentration.

In the absence of the activator, the transcription rate assumes a *residual base level* v_0, in this case 0.1. (Often, a value of $v_0 = 0$ is appropriate.) If the activator is present in high concentrations, the transcription rate assumes a *maximal expression level* v_1, in this case 1.0.

The step function displays a crisp logical dependence of the transcription rate on the relative concentration x/k of the activator X. The transcription rate may be written as

$$v_0 v_1 \underline{\left[\!\!\left[\ 1 > \frac{k}{x}\ \right]\!\!\right]} \tag{6.1}$$

in the Boolean (if-then-else algebra) notation of (4.16).

6.3.2. *Complementation between activation and repression.* Now suppose that the transcription factor X is a repressor rather than an activator. The corresponding transcription rate would then appear in any of the forms

$$v_1 v_0 \underline{\left[\!\!\left[\ 1 > \frac{k}{x}\ \right]\!\!\right]} = v_0 v_1 \underline{\left[\!\!\left[\ 1 > \frac{k}{x}\ \right]\!\!\right]}^{K} = v_0 v_1 \underline{\left[\!\!\left[\ 1 > \left(\frac{k}{x}\right)^{J}\ \right]\!\!\right]}\,. \tag{6.2}$$

The first two forms are equivalent by the skew-commutativity (4.21). Note that use of the strict inequality in the last form neglects the improbable equality $x = k$. In terms of the boundary Boolean algebras in Table 1, it is interesting to observe the interchangeable use in (6.2) of the respective negations K and J.

6.3.3. *Fuzzy logic.* Because of its convenience, Boolean or crisp logic has been used widely for the construction of network models in systems biology [47]. The curved graph of Fig. 12 displays a more realistic description of the dependence of the transcription rates on the relative concentrations of the transcription factors. The formalism of abstract barycentric algebras allows one to convert easily from the crisp functions of §6.3.1 to more realistic fuzzy functions. The crisp activator dependence (6.1) is replaced by the classic barycentric-algebraic expression

$$v_0 v_1 \underline{\left[\ 1+\left(\frac{k}{x}\right)^n\ \right]^{-1}} \tag{6.3}$$

interpreted in the closed interval $[v_0, v_1]$. Fig. 12 illustrates the case $n = 4$. The barycentric operator

$$\frac{1}{1+(k/x)^n} = \frac{(x/k)^n}{1+(x/k)^n}$$

in (6.3) is known as a *Hill function* in the terminology of [1], [28, §6.4.3], with *Hill coefficient* n. For $n = 1$ (and $v_0 = 0$), the Hill function implements Michaelis-Menten kinetics [1, A.7], [28, §15.2.7]. The case $n > 1$ corresponds to *cooperative* reactions. The crisp repressor dependencies (6.2) are replaced by either of the equivalent forms

$$v_1 v_0 \underline{\left[\ 1+\left(\frac{k}{x}\right)^n\ \right]^{-1}} = v_0 v_1 \underline{\left[\ 1+\left(\frac{x}{k}\right)^n\ \right]^{-1}} .$$

From these expressions, it is clear that the passage from crisp or Boolean to fuzzy logic is formally achieved by the replacement

$$[\![1 > \lambda]\!] \qquad \longrightarrow \qquad [1+\lambda^n]^{-1} . \tag{6.4}$$

Here, the dimensionless quantity λ is taken as k/x for activators and x/k for repressors. The conversion process is illustrated in the following paragraph.

6.4. **Logic gates.** Transcription rates may depend on logical combinations of different transcription factors. For example, the dependence

$$v_0 v_1 \underline{[\![\, 1 > \frac{k}{x} \,]\!] \cdot [\![\, 1 > \frac{l}{y} \,]\!]} \tag{6.5}$$

requires high concentrations of each of two transcription factors X and Y. The concentration x of X must exceed the critical threshold k; the concentration y of Y must exceed the critical threshold l.

Lemma 6.1. *The identity*

$$x\, xy\, \underline{r}\ \underline{s} = xy\, \underline{r \cdot s} \tag{6.6}$$

holds in barycentric algebras.

Proof. When the left hand side of (6.6) is interpreted as a binary convex combination using (4.24), the coefficient of y is $r \cdot s$. ☐

Using Lemma 6.1, the crisp logical expression (6.5) may be rewritten as the concatenation

$$v_0\, v_0 v_1 \underline{[\![\, 1 > \frac{k}{x} \,]\!]}\ \underline{[\![\, 1 > \frac{l}{y} \,]\!]}$$

which then translates to

$$v_0\, v_0 v_1 \underline{\left[1 + \left(\frac{k}{x}\right)^n\right]^{-1}}\ \underline{\left[1 + \left(\frac{l}{y}\right)^n\right]^{-1}} \tag{6.7}$$

under the replacement (6.4). With the previously used parameter values $v_0 = 0.1$, $v_1 = 1$, $k = 1$, $n = 4$, along with $l = 1$, this fuzzy AND gate is displayed in Fig. 13.

FIGURE 13. The fuzzy AND gate (6.7), which replaces the crisp or Boolean AND gate (6.5).

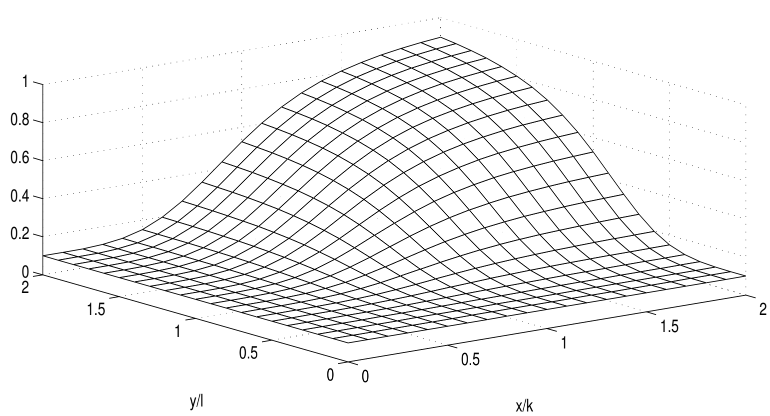
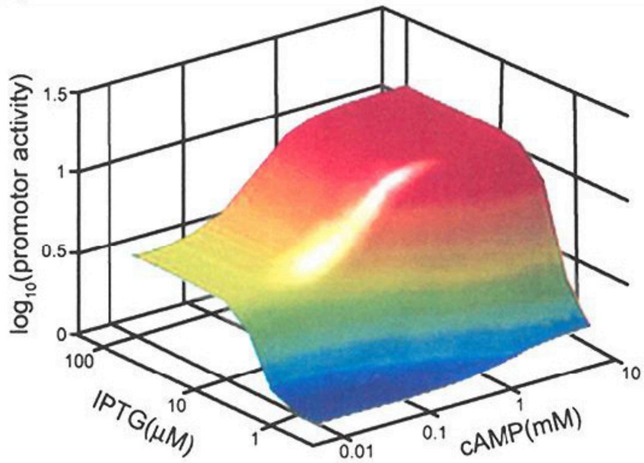


FIGURE 14. Experimentally observed AND gate.



Reproduced with permission from Setty et al., *Proc. Nat. Acad. Sci. USA* 100:7702, 2003.

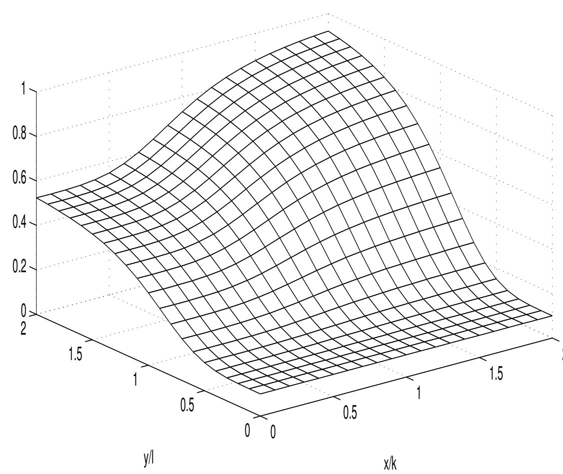


FIGURE 15. Modified fuzzy AND gate with the format $v_0\, v_{01} v_1\, \underline{p}\ \underline{q}$ of (4.24). Here, the distinct transcription rates $v_0 < v_{01} < v_1$ take $v_0 = 0.1, v_1 = 1, k = 1$ and $l = 1$ as in (6.7), but using an intermediate expression level $v_{01} = 0.55$.

6.4.1. *Experimental observations.* The fuzzy AND gate presented in (6.7) has the format $v_0\, v_0 v_1\, \underline{p}\ \underline{q}$ of Lemma 6.1. Here, the concatenated barycentric algebra operations have arguments (corresponding to transcription rates) that are repeated exactly. Exact repeats of this kind are improbable in biology, so it might appear at first glance that this would argue against the barycentric algebra approach.

However, real fuzzy AND gates as observed experimentally (see Figure 14) actually have the format $v_0\, v_{01} v_1\, \underline{p}\ \underline{q}$ of (4.24) with distinct transcription rates $v_0 < v_{01} < v_1$, as illustrated in Figure 15 with intermediate expression level $v_{01} = 0.55$. Thus:

Conclusion 6.2. The barycentric algebra formulation provides a natural framework for the dependence of expression levels on transcription factor concentrations.

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