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Alternative Clifford-like algebras

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Abstract

The real, complex, and quaternion division algebras may be constructed as Clifford algebras. However, while the octonions and split octonions share many features with Clifford algebras, such as an involutary anti-automorphism, their nonassociativity prevents their realization as Clifford algebras. We now introduce *Kingdon algebras*: alternative Clifford-like algebras over vector spaces which are equipped with a symmetric bilinear form, such that the octonions and split octonions arise as Kingdon algebras. In the process, they acquire 2-graded (superalgebra) structure. A comparable construction yields 3-graded algebras over spaces equipped with an alternating trilinear form.

Keywords: Clifford algebra, Kingdon algebra, alternative algebra, graded algebra, trilinear form.

1 Introduction

The algebra in this paper rests on some geometric definitions.

Definition 1. Consider a field F (not of characteristic 2), and a vector space V over F.

(a) A (*bilinearly*) *formed space* (V, B) is a vector space over F with a symmetric bilinear form $B \colon V \times V \to F$.

(b) A trilinear form $T \colon V \times V \times V \to F$ is said to *alternate* whenever $T(x_{1\pi}, x_{2\pi}, x_{3\pi}) = (\operatorname{sgn} \pi) T(x_1, x_2, x_3)$ for each permutation π of $\{ 1, 2, 3 \}$.

(c) A *trilinearly formed space* (V, T) is a vector space over F with an alternating trilinear form $T\colon V \times V \times V \to F$.

(d) A *doubly formed space* (V, B, T) is a bilinearly formed space with an alternating trilinear form.

The *Clifford algebra* $\mathrm{Cl}(V, B)$ over a formed space (V, B) is the quotient of the free associative algebra or *tensor algebra* TV over V by the ideal

$$J_B = \langle uv + vu - B(u, v) | u, v \in V\rangle \tag{1}$$

of TV [1, p.102]. While $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$ all appear as Clifford algebras, the octonions $\mathbb{O}$ or split octonions $\widetilde{\mathbb{O}}$ do not, as they are not associative. Nevertheless, they have recently [2, 3] been given a Clifford-like construction (still with a $\mathbb{Z}/_2$-grading) over formed spaces (V, B), as shown in §2 below. In [2, Ch. 3], the Clifford-like construction was extended to ungraded algebras over doubly formed spaces, which actually yield $\mathbb{Z}/_3$-graded algebras over trilinearly formed spaces (§3).

2 Kingdon algebras

For elements x, y, z of a binary algebra A, the (*additive*) *associator* is the element $(x, y, z) = (xy)z - x(yz)$ of A. The algebra A is said to be *alternative* if the associator is alternating, in the sense that $(x_{1\pi}, x_{2\pi}, x_{3\pi}) = (\operatorname{sgn} \pi)(x_1, x_2, x_3)$ for all permutations π of $\{1, 2, 3\}$. The octonions $\mathbb{O}$ and split octonions $\widetilde{\mathbb{O}}$ are alternative.

Over a given field F, write $\mathrm{Alt}[V]$ for the free alternative algebra over V [4, §1.2]. The *Kingdon algebra* $K(V, B)$ over a formed space (V, B) is the quotient of the free alternative algebra $\mathrm{Alt}[V]$ over V by the ideal

$$I_B = \langle uv + vu - B(u, v), (uv)w - w(vu) | u, v, w \in V\rangle \tag{2}$$

of $\mathrm{Alt}[V]$ [3, Def'n 3.1]. The comparison between the ideals (2) and (1) is immediate. In particular, if the dimension of (V, B) is less than 3, then $\mathrm{Alt}[V] = TV$ and $I_B = J_B$, so $K(V, B) = \mathrm{Cl}(V, B)$ [3, Prop. 3.9]. Kingdon algebras inherit the $\mathbb{Z}/_2$-grading of $\mathrm{Alt}[V]$ [3, Prop. 3.2].

Theorem 2. *[3, Th. 3.14] Consider a three-dimensional vector space V spanned by $\{ i, j, k \}$ over a field F which is not of characteristic 2. Suppose that the set $\{ i, j, k \}$ is orthogonal with respect to a symmetric bilinear form B on V.*

(a) *If $B(i,i) = B(j,j) = B(k,k) = -2$, then $K(V,B) \cong \mathbb{O}$ over F.*

(b) *If $B(i,i) = B(j,j) = -2, B(k,k) = 2$, then $K(V,B) \cong \widetilde{\mathbb{O}}$ over F.*

3 Ternary forms and algebras

The *BT-Kingdon algebra* $K(V,B,T)$ over a doubly formed space (V,B,T) is the quotient of $\mathrm{Alt}[V]$ over V by the ideal

Table 1. Multiplication table of a T-Kingdon algebra $K(V,T)$ with basis $\{ 1, i, j, k, ij, jk, ki, \omega \}$ over a trilinearly formed space (V,T) with basis $\{ i, j, k \}$ (compare [2, Table 3.2]). The first header row gives the $\mathbb{Z}/_3$ degrees of the column label elements that appear in the second header row below them. Here 1 is the unit, $\omega = (ij)k$, $T = T(i,j,k)$.

	1	1	1	2	2	2	0
	i	j	k	ij	jk	ki	ω
i	0	ij	$-ki$	0	$T-\omega$	0	Ti
j	$-ij$	0	jk	0	0	$T-\omega$	Tj
k	ki	$-jk$	0	$T-\omega$	0	0	Tk
ij	0	0	ω	0	$-Tj$	Ti	0
jk	ω	0	0	Tj	0	$-Tk$	0
ki	0	ω	0	$-Ti$	Tk	0	0
ω	0	0	0	Tij	Tjk	Tki	$T\omega$

$$I_{B,T} = \langle uv + vu - B(u,v), (uv)w - w(vu) - T(u,v,w) | u,v,w \in V\rangle$$

of $\mathrm{Alt}[V]$ [2, Def'n 3.3.1]. The ideal $I_{B,T}$ is inhomogeneous with respect to the $\mathbb{Z}/_2$-grading of $\mathrm{Alt}[V]$ whenever the alternating trilinear form T is nonzero, so the algebra $K(V,B,T)$ does not carry a $\mathbb{Z}/_2$-grading in that case.

On the other hand, if the bilinear form vanishes, then the ideal $I_{B,T}$ is homogeneous with respect to the $\mathbb{Z}/_3$-grading of $\mathrm{Alt}[V]$. In this case, the BT-Kingdon algebra $K(V,B,T)$ is defined to be the *T-Kingdon algebra* $K(V,T)$ of the trilinearly formed space (V,T). Such algebras do inherit the $\mathbb{Z}/_3$-grading, as exhibited in the example of Table 1.

References

[1] C. Chevalley. *The Algebraic Theory of Spinors and Clifford Algebras: Collected Works Vol. 2*, Springer, Berlin, 1997.

[2] C.M. Depies. *New Approaches to the Octonions and Their Maximal Orders*, Ph.D. Thesis, Iowa State University, 2024. DOI: 10.31274/td-20240617-149

[3] C.M. Depies, J.D.H. Smith. M.D. Ashburn. *Octonions as Clifford-like algebras*, J. Algebra 644 (2024), pp. 761–795. arxiv:2310.09972; DOI: 10.1016/j.jalgebra.2024.01.020

[4] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov. *Rings that are Nearly Associative*, (tr. H.F. Smith), Academic Press, New York, NY, 1982. Russian original: Nauka, Moskva, 1978.

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