Acta Sci. Math. (*Szeged*)
62 (1996), 145–151

Alternative approaches to alternative algebras

RICHARD ROSSMANITH and JONATHAN D. H. SMITH

Communicated by P. T. Nagy

Abstract. Alternative algebras known as comtrans algebras have been used in the coordinatization of web structures, and in the formulation of quantum mechanics. Time reversal in quantum mechanics corresponds to transposition of comtrans algebras. The current paper presents an alternative axiomatization of comtrans algebras, namely as tercom algebras, that simplifies the description of transposition. A new class of simple comtrans or tercom algebras, corresponding to simple alternative Akivis algebras, is then obtained.

1. Introduction

Analogues of the classical correspondence between Lie algebras and Lie groups have recently been developed in a number of different contexts. Following the work of the 1960's connecting Mal'cev algebras and Moufang loops, Akivis [1], Hofmann–Strambach [2] and others studied the relationship between (what are now called) Akivis algebras and smooth binary loops. Later [6], ternary multilinear algebras known as comtrans algebras, in company with Akivis algebras, were shown to correspond to smooth n-loops. The smooth loops and n-loops arise naturally in differential geometry as coordinatizations of web structures. The multilinear algebras in the tangent spaces display the common feature of alternativity to varying extents.

Beyond their differential-geometric roots, comtrans algebras have been applied to special relativity and quantum mechanics [7]. In particular, time reversal in quantum mechanics is described by the transposition relationship between certain comtrans algebras. Now although the axiomatization of comtrans algebras is very succinct, it does not lend itself nicely to the formulation of the transposition

Received December 28, 1994 and in revised form November 17, 1995.
AMS Subject Classification (1991): 17D10, 17D99, 53A60.

relationship. A primary aim of the current paper is to present an alternative axiomatization in terms of so called ternary commutator or “tercom” algebras, having a left alternative left commutator and a right alternative right commutator. In this axiomatization, transposition just becomes an interchange of left and right.

Comtrans and tercom algebras are examples of a general class of algebras defined in terms of a pair of permutations of the arguments of multilinear operations. The identities satisfied by these algebras – generalized alternativity and Jacobi identities – correspond to the cyclic groups generated by these permutations and their quotient. This group-theoretical approach to the specification of multilinear algebras is discussed in the first section. The second section addresses the transposition of tercom algebras, while the brief third section defines ideals, the abelian property and simplicity of tercom algebra. The final section investigates connections between Akivis algebras, Mal’cev algebras and tercom or comtrans algebras. The main theorem of that section shows how simple Mal’cev algebras, and more generally simple Akivis algebras, correspond to simple tercom or comtrans algebras. The theorem is part of the continuing programme (cf. [3], [4], [5]) of classifying simple comtrans algebras.

2. Comtrans algebras and tercom algebras

A *comtrans algebra* [6] is a (unital) module E over a commutative ring R (with identity element), together with two trilinear operations, the *commutator* $\lambda\colon E\times E\times E \to E$, and the *translator* $\eta\colon E\times E\times E \to E$, such that the commutator is *left alternative*:

$$\lambda(x,y,z)+\lambda(y,x,z)=0, \tag{1}$$

the translator satisfies the *Jacobi identity for comtrans algebras*:

$$\eta(x,y,z)+\eta(y,z,x)+\eta(z,x,y)=0, \tag{2}$$

and λ and η together satisfy the *comtrans identity*:

$$\lambda(x,y,z)+\lambda(z,y,x)=\eta(x,y,z)+\eta(z,y,x). \tag{3}$$

A different axiomatization leads to *ternary commutator algebras* or *tercom algebras*. Here we also have two trilinear operations, called the *left commutator* $\lambda : E \times E \times E \to E$, and the *right commutator* $\rho\colon E \times E \times E \to E$. Again, the left commutator is left alternative:

$$\lambda(x,y,z)+\lambda(y,x,z)=0, \tag{4}$$

the right commutator is right alternative:

$$\rho(x,y,z)+\rho(x,z,y)=0, \tag{5}$$

and the *Jacobi identity for tercom algebras* holds:

$$\lambda(x,y,z)+\lambda(y,z,x)+\lambda(z,x,y)=\rho(x,y,z)+\rho(y,z,x)+\rho(z,x,y). \tag{6}$$

Both kinds of algebras are (term) equivalent in the sense of universal algebra: if (E,λ,η) is a comtrans algebra, then the definition

$$\rho(x,y,z)=(\eta-\lambda)(y,x,z) \tag{7}$$

yields a tercom algebra (E,λ,ρ). Conversely, given a tercom algebra (E,λ,ρ), one obtains a comtrans algebra (E,λ,η) by setting

$$\eta(x,y,z)=(\rho-\lambda)(y,x,z). \tag{8}$$

The sets of identities (1)–(3) and (4)–(6) can be generalized in the following way: let n be a positive integer, and let M be the set of all n-linear operations on E. Then the symmetric group $\mathrm{Sym}(n)$ (as the group of bijections of $\{1,\dots,n\}$) operates on M via

$$g\varphi\colon E^n\to E;\ x\mapsto\varphi(x\circ g) \tag{9}$$

for $\varphi\in M,\ g\in\mathrm{Sym}(n)$; here we interpret E^n to be the set of all maps from $\{1,\dots,n\}$ to E. For $x_1,\dots,x_n\in E$, we have in particular $(g\varphi)(x_1,\dots,x_n)=\varphi(x_{g(1)},\dots,x_{g(n)})$. The general form of the identities is then

$$\Sigma\langle g\rangle\alpha=0 \tag{10}$$

$$\Sigma\langle h\rangle\beta=0 \tag{11}$$

$$\Sigma\langle gh^{-1}\rangle\alpha=\Sigma\langle gh^{-1}\rangle\beta \tag{12}$$

with $\alpha,\beta\in M$ and $g,h\in\mathrm{Sym}(n)$. These identities describe a comtrans algebra with $\lambda=\alpha$ and $\eta=\beta$ if we set $n=3$, $g=(1\ 2)$, and $h=(1\ 2\ 3)$, which implies $gh^{-1}=(1\ 3)$. For a tercom algebra with $\lambda=\alpha$ and $\rho=\beta$ we set similarly $n=3$, $g=(1\ 2)$, and $h=(2\ 3)$ (hence $gh^{-1}=(1\ 2\ 3)$). Note that both $\{(1\ 2),(1\ 2\ 3)\}$ and $\{(1\ 2),(2\ 3)\}$ are minimal generating sets for $\mathrm{Sym}(3)$.

Proposition. *For every n-linear map $\mu : E^n \to E$, the maps $\alpha := \mu - g\mu$ and $\beta := \mu - h\mu$ satisfy* (10)–(12).

Proof.

$$\Sigma\langle g\rangle\alpha = \Sigma\langle g\rangle\mu - \Sigma \underbrace{\langle g\rangle g}_{=\langle g\rangle} \mu = 0,$$

$$\Sigma\langle gh^{-1}\rangle\beta = \Sigma\langle gh^{-1}\rangle\mu - \Sigma\langle gh^{-1}\rangle h\mu = \Sigma\langle gh^{-1}\rangle\mu - \Sigma\langle gh^{-1}\rangle gh^{-1}h\mu = \Sigma\langle gh^{-1}\rangle\alpha.$$

■

The proposition describes a "standard construction" of a comtrans or tercom algebra for any given, "genuine" trilinear multiplication $\mu\colon E \times E \times E \to E$. In [6] it is shown that in fact every comtrans or tercom algebra is the result of such a standard construction.

3. Transposed tercom algebras

The concept of the *transpose* $(E, \lambda, \eta)^\tau$ of a comtrans algebra (E, λ, η) was introduced in [4] and has important mathematical and physical applications [7]. The transpose is defined by

$$\lambda^\tau = (1\ 3)\lambda + (1\ 2\ 3)\eta, \tag{13}$$

$$\eta^\tau = -(2\ 3)\eta, \tag{14}$$

using the language of the group action (9). We have $\lambda^{\tau\tau} = \lambda$ and $\eta^{\tau\tau} = \eta$, hence the transpose is (term) equivalent to the genuine algebra.

A natural way to extend transposition to tercom algebras is as follows:

- Transform any given tercom algebra to its corresponding comtrans algebra using equation (8).
- Transpose the obtained comtrans algebra with equations (13) and (14).
- Go back to tercom algebras via (7).

After a short computation, one finds the transpose $(E, \lambda, \rho)^\tau$ of a tercom algebra (E, λ, ρ) to be given by

$$\lambda^\tau = (1\ 3)\rho, \tag{15}$$

$$\rho^\tau = (1\ 3)\lambda. \tag{16}$$

Since this operation merely interchanges the left and right commutators and reverses the order of their arguments, it is easier to handle than equations (13) and (14).

4. Ideals

There are several equivalent ways of defining an *ideal* of a tercom algebra. The most basic one describes an ideal J of a tercom algebra E as an R-submodule of E such that for all $x, y, z \in E$ the implication

$$\{x, y, z\} \cap J \neq \emptyset \Rightarrow \lambda(x, y, z), \rho(x, y, z) \in J \tag{17}$$

holds. Ideals can also be characterized as the kernels of tercom algebra homomorphisms [4]. A tercom algebra is said to be *abelian* if both the left and the right commutator are identically zero. A nonabelian tercom algebra E is *simple* if it has no proper nontrivial ideals.

5. Tercom algebras and Akivis algebras

An *Akivis algebra* [2] (formerly called a W-*algebra* [1]) is an R-module E equipped with an anticommutative bilinear product $[\cdot, \cdot]\colon E \times E \to E$ (called the *commutator*) and a trilinear map $\alpha\colon E \times E \times E \to E$ (the *associator* of the algebra), such that the *Jacobi identity for Akivis algebras* or *Akivis identity*

$$\begin{aligned}[][[x, y], z] + [[y, z], x] + [[z, x], y] = \alpha(x, y, z) - \alpha(x, z, y) + \\ + \alpha(y, z, x) - \alpha(y, x, z) + \\ + \alpha(z, x, y) - \alpha(z, y, x)\end{aligned} \tag{18}$$

is satisfied for all $x, y, z \in E$. This is a more general equation than the one for Lie algebras (where we have $\alpha = 0$).

If $E \times E \to E; (x, y) \mapsto xy$ is a bilinear (not necessarily associative) product on E, then the standard construction of a Lie algebra may be extended to Akivis algebras by setting

$$[x, y] = xy - yx, \tag{19}$$

$$\alpha(x, y, z) = (xy)z - x(yz). \tag{20}$$

Ideals of Akivis algebras are defined much as in Section 4. An Akivis algebra is *abelian* if both its commutator and associator are identically zero, and a nonabelian Akivis algebra with no proper nontrivial ideals is said to be *simple*.

An Akivis algebra is called *alternative* (in [2]) if $t\alpha = -\alpha$ for all transpositions $t \in \mathrm{Sym}(3)$. In this case the Jacobi identity collapses to

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 6\alpha(x,y,z). \tag{21}$$

Thus the associator is uniquely determined by the commutator if 6 is invertible in the underlying ring R. Note that every Lie algebra is also an alternative Akivis algebra, since then we have $\alpha = 0$. A *Mal'cev algebra* is an R-module E equipped with an anticommutative bilinear product $[\cdot,\cdot]$ satisfying the identity

$$[[x,y],[x,z]] = [[[x,y],z],x] + [[[y,z],x],x] + [[[z,x],x],y]. \tag{22}$$

If 6 is invertible in R, then one may use (21) to define a trilinear map $\alpha\colon E\times E\times E \to E$ such that $(E,[\cdot,\cdot],\alpha)$ is an alternative Akivis algebra (cf. [1]).

Every Akivis algebra furnishes a tercom algebra whose left and right commutator are given by

$$\lambda(x,y,z) = [[x,y],z], \tag{23}$$

$$\rho(x,y,z) = \alpha(x,y,z) - \alpha(x,z,y). \tag{24}$$

The verification of equations (4)–(6) is straightforward.

Shen and Smith proved [4] that simplicity of a Lie algebra is equivalent to simplicity of its associated comtrans or tercom algebras. Using equation (21), the proof may be extended to Mal'cev algebras and more general alternative Akivis algebras over rings where 6 is a unit:

Theorem.

(i) *Each ideal of an Akivis algebra is also an ideal of its associated tercom algebra.*

(ii) *An alternative Akivis algebra $(E,[\cdot,\cdot],\alpha)$ over a ring R with $6 \in R^\times$ is simple if and only if its associated tercom algebra is simple.*

Proof. (i) trivial.

(ii) "if": Implied by (i).

"only if": Let $(E,[\cdot,\cdot],\alpha)$ be a simple alternative Akivis algebra, and let (E,λ,ρ) be the associated tercom algebra. Then by (21),

$$E' = \left\{ \sum_{i=1}^{k} [x_i,y_i] : k \in \mathbb{N}_0, x_1,\ldots,x_k,y_1,\ldots,y_k \in E \right\}$$

is an ideal of the Akivis algebra. Since the Akivis algebra is nonabelian, we have $E' \neq 0$, hence $E' = E$. Let $x, y \in E$ such that $[x, y] \neq 0$, and let $z_1, \ldots, z_k,\ w_1, \ldots, w_k \in E$ such that $x = \sum_{i=1}^{k} [z_i, w_i]$. Then $\sum_{i=1}^{k} \lambda(z_i, w_i, y) = \sum_{i=1}^{k} [[z_i, w_i], y] = [x, y] \neq 0$. Therefore $\lambda \neq 0$, and the tercom algebra is also nonabelian.

Now let $J \subseteq E$ be a tercom ideal of E, and let $j \in J$, $x \in E$. Again let $z_1, \ldots, z_k, w_1, \ldots, w_k \in E$ such that $x = \sum_{i=1}^{k} [z_i, w_i]$. Then

$$[x, j] = \sum_{i=1}^{k} [[z_i, w_i], j] = \sum_{i=1}^{k} \lambda(z_i, w_i, j) \in J.$$

By (21), the associator α is a linear combination of commutator terms, hence $\alpha(x, y, j), \alpha(x, j, y), \alpha(j, x, y) \in J$ for all $x, y \in E$, $j \in J$. This shows that J is an Akivis ideal of E, whence $J = 0$ or $J = E$. Therefore, E is also simple as a tercom algebra.

■

References

[1] M. A. Akivis, Local algebras of a multidimensional three-web, *Sibirsk. Mat. Ž.*, **17** (1976), 5–11 (Russian); Translated in: Siberian Math. J., **17** (1976), 3–8.

[2] K. H. Hofmann and K. Strambach, Lie's fundamental theorems for local analytical loops, *Pacific J. Math.*, **123** (1986), 301–327.

[3] X. R. Shen and J. D. H. Smith, Comtrans algebras and bilinear forms, *Arch. Math.* (*Basel*), **59** (1992), 327–333.

[4] X. R. Shen and J. D. H. Smith, Simple multilinear algebras, rectangular matrices and Lie algebras, *J. Algebra*, **160** (1993), 424–433.

[5] X. R. Shen and J. D. H. Smith, Simple algebras of hermitian operators, *Arch. Math.* (*Basel*), **65** (1995), 534–539.

[6] J. D. H. Smith, Multilinear algebras and Lie's theorem for formal n-loops, *Arch. Math.* (*Basel*), **51** (1988), 169–177.

[7] J. D. H. Smith, Comtrans algebras and their physical applications, *Banach Center Publications* (*Warszawa*), **28** (1993), 319–326.

R. Rossmanith, Mathematisches Institut, Universität Jena, D-07740 Jena, Germany

J. D. H. Smith, Department of Mathematics, Iowa State University, Ames, Iowa 50011, U.S.A.