ERRATA: "KEIMEL'S PROBLEM ON THE ALGEBRAIC AXIOMATIZATION OF CONVEXITY," A. KOMOROWSKI, A.B. ROMANOWSKA, AND J.D.H. SMITH

Replace p.17 of 20, l.−11 to p.18 of 20, l.−12 with the following text:

Definition 12.5. Set thresholds $0 \leq s < t \leq 1/2$. Set $\mathcal{B}^{s,t}$ to be the variety of idempotent, entropic, skew-commutative $\underline{\underline{I}}^{\circ}$-algebras defined by the following identities:

(1) $xy\,\underline{\underline{p}} = x$ for all $p < s$;
(2) $xy\,\underline{\underline{p}} = y$ for all $p > s'$;
(3) all identities true in the variety $\mathcal{B}^t_{\mathrm{mod}}$ of Definition 9.1.

Let $\mathcal{B}_{s,t}$ be the subvariety of $\mathcal{B}^{s,t}$ defined by all the identities $w = v$ true in $\mathcal{B}^s_{\mathrm{mod}}$ such that for all operation symbols $\underline{\underline{p}}$ appearing in $w = v$ with $s \leq p < t$ or $t' < p \leq s'$, the reduced form $w^t = v^t$ of $w = v$, described in Section 11, is an identity true in $\mathcal{B}^t_{\mathrm{mod}}$ (or both sides of it are equal to the same variable).

Theorem 12.6. *For thresholds $0 \leq s < t \leq 1/2$, the join $\mathcal{B}^s \vee \mathcal{B}^t$ of the varieties $\mathcal{B}^s$ and $\mathcal{B}^t$ is equal to the variety $\mathcal{B}_{s,t}$.*

Proof. First recall that for any $0 < r < 1/2$, each $\mathcal{B}^r$-algebra satisfies the identities $xy\,\underline{\underline{p}} = x$ for all small operations $\underline{\underline{p}}$, then $xy\,\underline{\underline{p}} = y$ for all large operations $\underline{\underline{p}}$ and all identities true in $\mathcal{B}^q_{\mathrm{mod}}$, for $q \geq r$, involving only moderate operations. Since $s < t$, it follows by Definition 12.5 that any identity true in $\mathcal{B}^{s,t}$ is satisfied in both the varieties $\mathcal{B}^s$ and $\mathcal{B}^t$. The same holds for the identities defining the subvariety $\mathcal{B}_{s,t}$, since in $\mathcal{B}^t$ any such identity reduce to an identity true in $\mathcal{B}^t_{\mathrm{mod}}$. Hence each identity true in $\mathcal{B}_{s,t}$ holds in $\mathcal{B}^s \vee \mathcal{B}^t$. Consequently, $\mathcal{B}^s \vee \mathcal{B}^t \leq \mathcal{B}_{s,t}$.

To verify the converse inequality, we will show that each identity true in both $\mathcal{B}^s$ and $\mathcal{B}^t$ (and hence in $\mathcal{B}^s \vee \mathcal{B}^t$) is also satisfied in $\mathcal{B}_{s,t}$. First note that all left-zero and all right-zero identities true in $\mathcal{B}^s$ also hold in $\mathcal{B}^s \vee \mathcal{B}^t$, and in $\mathcal{B}_{s,t}$.

Now let

$$w = v \tag{12.1}$$

be an identity satisfied in $\mathcal{B}^s \vee \mathcal{B}^t$ containing some operation symbols $\underline{\underline{p}}$ for $s \leq p \leq s'$.

Suppose that all the operation symbols appearing in (12.1) belong to $[t, t']$. Then the identity is satisfied by all $\mathcal{B}^t$-algebras, and hence by all $\mathcal{B}^t_{\mathrm{mod}}$-algebras. Consequently, it holds in all $\mathcal{B}^{s,t}$-algebras.

Now let (12.1) be an identity, true in the variety $\mathcal{B}^s$, containing both small and moderate operations. Then by results of Section 11, it is equivalent to the identity $w^s = v^s$ true in $\mathcal{B}^s_{\mathrm{mod}}$ containing only operation symbols from $[s, s']$.

So assume now that (12.1) contains operation symbols p only in the range $s \leq p \leq s'$. The identity holds also in the variety $\mathcal{B}^t$ precisely in two cases. Either all its operation symbols belong to $[s, t] \cup [t', s']$, and then both sides are equal to the same variable, or there are operation symbols in (12.1) belonging to $[t, t']$, and then the identity is equivalent to the identity $w^t = v^t$ true in $\mathcal{B}^t_{\mathrm{mod}}$. It follows that the identities true in both $\mathcal{B}^s$ and $\mathcal{B}^t$ satisfy the conditions of Definition 12.5. Hence they hold in $\mathcal{B}_{s,t}$, and $\mathcal{B}_{s,t} \leq \mathcal{B}^s \vee \mathcal{B}^t$. □

Note that the variety $\mathcal{B}_{s,t}$ is a proper subvariety of the variety $\mathcal{B}^{s,t}$. This is shown by the following example. First observe that the algebra $(I, \underline{I}^\circ)$ with appropriately defined operations may be considered as a member of each of the varieties $\mathcal{B}^t$ for $t \neq 0$ and $\mathcal{B}^{s,t}$. As a member I^t of $\mathcal{B}^t$ it satisfies the identities defining $\mathcal{B}^t$ and as a member $I^{s,t}$ of $\mathcal{B}^{s,t}$ it satisfies the identities defining $\mathcal{B}^{s,t}$.

Example. Let $0 < s < 1/5$ and $2/5 < t < 1/2$. Let $p = 1/4$ and $q = 1/5$. Then $p \circ q = 2/5$ and $q/(p \circ q) = 1/2$. Since $s < p, q, p \circ q < t$, it follows that the variety $\mathcal{B}^s$ satisfies skew-associativity for $p = 1/4$ and $q = 1/5$. On the other hand, the same identity holds in $\mathcal{B}^t$, since in this case both of its sides are equal to x. It follows that the identity holds in $\mathcal{B}_{s,t}$.

Now consider the algebra $I^{s,t}$ but satisfying additionally the following conditions: $xy1/5 = y$ and $xy2/5 = x$, and moreover $xy4/5 = x$ and $xy3/5 = y$. It is easy to check that this algebra is a member of $\mathcal{B}^{s,t}$. However, it does not belong to $\mathcal{B}_{s,t}$. The left-hand side of skew-associativity for $p = 1/4$ and $q = 1/5$ equals z, whereas the right-hand side equals x.