1. Directed graphs or quivers

What is category theory?

- Graph theory on steroids
- Comic book mathematics
- Abstract nonsense
- The secret dictionary

Sets and classes: For $S = \{X \mid X \notin X\}$, have $S \in S \Leftrightarrow S \notin S$

Directed graph or **quiver:** $C = (C_0, C_1, \partial_0 \colon C_1 \to C_0, \partial_1 \colon C_1 \to C_0)$
Class C_0 of objects, vertices, points, ...
Class C_1 of morphisms, (directed) edges, arrows, ...
For $x, y \in C_0$, write $C(x,y) := \{f \in C_1 \mid \partial_0 f = x, \partial_1 f = y\}$

tail, domain $\longrightarrow$ $\quad \partial_0 f \xrightarrow{f \in C_1} \partial_1 f \quad$ $\longleftarrow$ head, codomain

Opposite or **dual** graph of $C = (C_0, C_1, \partial_0, \partial_1)$ is $C^{\mathsf{op}} = (C_0, C_1, \partial_1, \partial_0)$

Graph homomorphism $F \colon D \to C$
has **object part** $F_0 \colon D_0 \to C_0$
and **morphism part** $F_1 \colon D_1 \to C_1$
with $\partial_i \circ F_1(f) = F_0 \circ \partial_i(f)$ for $i = 0, 1$.

Graph isomorphism has bijective object and morphism parts.

Poset $(X, \leq)$: set X with reflexive, antisymmetric, transitive **order** $\leq$

Hasse diagram of poset $(X, \leq)$: $x \to y$ if y **covers** x, i.e.,
$x \neq y$ and $[x, y] = \{x, y\}$, so $x \leq z \leq y \;\Rightarrow\; z = x$ or $z = y$.

Hasse diagram of $(\mathbb{N}, \leq)$ is $0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \ldots$

Hasse diagram of $(\{1,2,3,6\}, \mid\ )$ is

$$\begin{tikzcd} 3 \arrow[r] & 6 \\ 1 \arrow[u] \arrow[r] & 2 \arrow[u] \end{tikzcd}$$

2. Categories

Category: Quiver $C = (C_0, C_1, \partial_0 \colon C_1 \to C_0, \partial_1 \colon C_1 \to C_0)$ with:

- **composition:** $\forall\ x, y, z \in C_0$,
$C(x,y) \times C(y,z) \to C(x,z); (f,g) \mapsto g \circ f$
- satisfying **associativity:** $\forall\ x, y, z, t \in C_0$,
$\forall\ (f,g,h) \in C(x,y) \times C(y,z) \times C(z,t)\,,\ h \circ (g \circ f) = (h \circ g) \circ f$

$$\begin{tikzcd}
 & y \arrow[dl, "g"'] \arrow[dd, "h\circ g"] & \\
z \arrow[dr, "h"'] & & x \arrow[ul, "f"'] \arrow[ll, "g\circ f"] \arrow[dl, "h\circ(g\circ f)=(h\circ g)\circ f"] \\
 & t &
\end{tikzcd}$$

- **identities:** $\forall\ x, y, z \in C_0\,,\ \exists\ 1_y \in C(y,y)\,.$
$\forall\ f \in C(x,y)\,,\ 1_y \circ f = f$ and $\forall\ g \in C(y,z)\,,\ g \circ 1_y = g$

$$\begin{tikzcd}
y \arrow[d, "g"'] \arrow[dr, "1_y"] & x \arrow[l, "f"'] \arrow[d, "f"] \\
z & y \arrow[l, "g"]
\end{tikzcd}$$

Example: $\mathbb{N}_0 = \{x\}\,,\ \mathbb{N}_1 = \mathbb{N}\,,\ 1_x = 0\,,\ \forall\, m, n \in \mathbb{N}\,,\ n{\circ}m = m{+}n\,;$ —

one object, lots of arrows [**monoid** of natural numbers under addition]

Equation: $3 + 5 = 4 + 4$ **Commuting diagram:**
$$\begin{tikzcd}
x \arrow[r, "4"] \arrow[d, "3"'] & x \arrow[d, "4"] \\
x \arrow[r, "5"'] & x
\end{tikzcd}$$

Example: $\mathbb{N}_1 = \mathbb{N}\,,\ \forall\ m, n \in \mathbb{N}\,,\ |\mathbb{N}(m,n)| = \begin{cases} 1 & \text{if } m \leq n; \\ 0 & \text{otherwise} \end{cases}$

— lots of objects, lots of arrows [poset $(\mathbb{N}, \leq)$ as a category]

These two examples are **small categories:** have a set of morphisms.

Example: The category **Set** has the class of all sets as its object class, with $\mathbf{Set}(X,Y)$ as the set of all functions from X to Y, composition of functions: $g \circ f(x) = g\big(f(x)\big)$, usual identities $1_X \colon X \to X; x \mapsto x$.

This example is **large** (not small), but **locally small:**
just a set of arrows between each pair of objects.

3. Special morphisms and objects

Consider morphism $f\colon x \to y$.

- **Isom. or invertible:** $\exists\ f'\colon y \to x\ .\ f \circ f' = 1_y$ and $f' \circ f = 1_x$.
  Ex: Bijective function in **Set**.
- **Monomorphism:** $\forall\ g_i\colon z \to x\ ,\ f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$.
  Ex: Injective function in **Set**.
- **Epimorphism:** $\forall\ g_i\colon y \to z\ ,\ g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$.
  Ex: Surjective function in **Set**.
- **Retract** or **split epimorphism:** $r\colon y \to x$ with $r \circ f = 1_x$.
  Ex: $r\colon n \mapsto \max\{0, n-1\}$ retracts successor function on $\mathbb{N}$.
- **Section** or **split monomorphism:** $s\colon y \to x$ with $f \circ s = 1_y$.
  Ex: Successor function on $\mathbb{N}$ is a section of $r\colon \mathbb{N} \to \mathbb{N}$.
- **Idempotent:** $x = \partial_0 f = \partial_1 f$ and $f \circ f = f$.
  Ex: $\mathbb{R}^2 \to \mathbb{R}^2; (x_1, x_2) \mapsto (x_1, 0)$ in **Set**.

Lemma. For $\begin{tikzcd} x \arrow[r, bend left, "r"] & y \arrow[l, bend left, "s"] \end{tikzcd}$ with $r \circ s = 1_y$:

- r is an epimorphism.
- s is a monomorphism.
- $s \circ r$ is an idempotent (said to **split**).

Isomorphic objects $x \cong y$: Have isomorphism $f\colon x \to y$.

Terminal object $\top$ for C has $\forall\ x \in C_0\ ,\ |C(x, \top)| = 1$.

Examples: $\{0\}$ in **Set**, upper bound in a poset, ...

Initial object $\perp$ for C has $\forall\ x \in C_0\ ,\ |C(\perp, x)| = 1$.

Examples: $\emptyset$ in **Set**, lower bound in a poset, $\mathbb{Z}$ in **Ring**, ...

Zero object 0 for C is both initial and terminal.

Examples: $\{0\}$ in categories of groups, vector spaces, ...

Groupoid: Category where all morphisms are invertible.

Examples: For a set X:

- **Discrete category** $(X, \{1_x \mid x \in X\}, \partial_0\colon 1_x \mapsto x, \partial_1\colon 1_x \mapsto x)$.
- **Symmetric group** $X!$ of all bijections $X \to X$.
- The collection $\operatorname{Inv} X$ of all bijections between subsets of X.

4. Functors

Functor: A graph homomorphism $F\colon D \to C$, thus with restrictions

$$\forall\ x, y \in D_0\,,\ F_1\colon D(x,y) \to C(F_0x, F_0y)\colon f \mapsto F_1 f\,, \tag{4.1}$$

respecting identities, compositions: $F_1 1_x = 1_{F_0x}$, $F_1(g \circ f) = F_1 g \circ F_1 f$.

Global conditions: $\begin{cases} \textbf{Isomorphism: } F_0 \text{ and } F_1 \text{ are isomorphisms.} \\ \textbf{Essentially surjective: } \forall\ c \in C_0\,,\ \exists\ d \in D_0\,.\ c \cong F_0 d \end{cases}$

Local conditions: $\begin{cases} \textbf{Full: } \text{Each restriction (4.1) is surjective.} \\ \textbf{Faithful: } \text{Each restriction (4.1) is injective.} \end{cases}$

Example: While the **forgetful** or **underlying set functor**

$$U\colon \mathbf{Grp} \to \mathbf{Set}; [f\colon (G_1, \cdot, ^{-1}, 1) \to (G_2, \cdot, ^{-1}, 1)] \mapsto [f\colon G_1 \to G_2]$$

is faithful, $U_0\colon \mathbf{Grp}_0 \to \mathbf{Set}_0$ is not injective (same set, different groups). Also U not full: Some functions between groups are not homomorphic.

Example: For a monoid $(M, \cdot, 1_M)$, write M^* for the group of invertible elements or **units**. Then $\mathbf{Mon} \to \mathbf{Grp}$;

$$[f\colon (M_1, \cdot, 1) \to (M_2, \cdot, 1)] \mapsto [f|_{M_1^*}\colon (M_1^*, \cdot, ^{-1}, 1) \to (M_2^*, \cdot, ^{-1}, 1)]$$

is a functor between large categories, the **group of units** functor.

Moral: $\boxed{\text{Mathematical constructions are functors!}}$

Example: A monoid homomorphism $f\colon M_1 \to M_2$ yields a functor between the corresponding small one-object categories.
Note $(\mathbb{R}, \cdot, 1) \to ([0, \infty[, \cdot, 1); n \mapsto n^2$ is full, but not faithful.

Example: A functor $F\colon (P_1, \leq) \to (P_2, \leq)$ between poset catgeories corresponds to an **order-preserving** function:

$$x \leq y \ \text{ in } \ P_1 \quad \Rightarrow \quad F_0 x \leq F_0 y \ \text{ in } \ P_2\,.$$

Trivially faithful.

Example: Inclusion of a subcategory always gives a faithful functor.

Full subcategory: The inclusion functor is full.

Example: Category **FinSet** of finite sets is full in **Set**.

Example: Functor

$(\mathbb{N}, \leq) \to \mathbf{FinSet}; [n < n+1] \mapsto [\{0, 1, \ldots, n-1\} \hookrightarrow \{0, 1, \ldots, n-1, n\}]$

is essentially surjective.

5. Natural transformations

Given graph maps $F, G\colon D \to C$ from a graph D to a category C, a **natural transformation** $\tau\colon F \to G$ is a "vector" $(\tau_x \mid x \in D_0)$ of **components** $\tau_x\colon Fx \to Gx$ in C_1 such that,

for all $f\colon x \to y$ in D_1, the rectangle of the **naturality diagram**

$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} Fx \arrow[rr, "\tau_x"] \arrow[d, "Ff"'] & & Gx \arrow[d, "Gf"] \\ Fy \arrow[rr, "\tau_y"'] & & Gy \end{tikzcd}$$

$\ldots$ in D $\qquad$ $\ldots$ in C

commutes in the category C.

Natural isomorphism: Each component τ_x is an isomorphism in C.

Example: For a set A, have a functor
$L^A\colon \mathbf{Set} \to \mathbf{Set}; [f\colon X \to Y] \mapsto [A \times X \to A \times Y; (a, x) \mapsto (a, fx)]$.
Then a function $\alpha\colon A \to B$ gives a natural transformation
$L^\alpha\colon L^A \to L^B$ with components $L^\alpha_X\colon A \times X \to B \times X; (a, x) \mapsto (\alpha\, a, x)$
and naturality diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} (a,x) \arrow[rr, mapsto, "L^\alpha_X"] \arrow[d, mapsto, "L^A f"'] & & (\alpha\, a, x) \arrow[d, mapsto, "L^B f"] \\ (a, fx) \arrow[rr, mapsto, "L^\alpha_Y"'] & & (\alpha\, a, fx) \end{tikzcd}$$

Example:
Category $\mathcal{L}$ of (linear transformations between) real vector spaces.
Dual space $V^* = \mathcal{L}(V, \mathbb{R})$ of linear functionals on vector space V.
Double dual $V^{**} = \mathcal{L}(V^*, \mathbb{R}) = \mathcal{L}(\mathcal{L}(V, \mathbb{R}), \mathbb{R})$.
Identity functor $I\colon \mathcal{L} \to \mathcal{L}$.
Double dual functor $DD\colon \mathcal{L} \to \mathcal{L}; V \mapsto V^{**}$.
Natural transformation $\tau\colon I \to DD$ with "evaluation" components

$$\tau_V\colon V \to V^{**}; v \mapsto [\theta \mapsto \theta(v)].$$

Gives a natural isomorphism in finite dimensions.

Contrast: Given basis $\{e_1, \ldots e_n\}$ of V, define $\widehat{e}_i\colon V \to \mathbb{R}; e_j \mapsto \delta_{ij}$.

Then $V \to V^*; e_i \mapsto \widehat{e}_i$ does not set up a natural isomorphism.

6. Duality and contravariant functors

Dual or **opposite** C^{op} of a category C is built on the dual graph C^{op}: Same identity morphisms, but composition as shown:

$$\begin{tikzcd} y \arrow[d, "g"'] & x \arrow[l, "f"'] \arrow[dl, "g\circ f"] \\ z & \boxed{\text{In } C} \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "f"] & x \\ z \arrow[u, "g"] \arrow[ur, "g\circ f \text{ or } gf"'] & \boxed{\text{In } C^{\mathsf{op}}} \end{tikzcd}$$

For Eulerian notation in C, algebraic notation would be natural in C^{op}.

Example: The dual of a one-object monoid category $(M, \cdot, 1_M)$ is the one-object monoid category of the monoid $(M, \circ, 1_M)$ with $x \circ y = y \cdot x$.

Example: For a set X, the dual of the poset category of $\big(\mathcal{P}(X), \subseteq\big)$ is the poset category of $\big(\mathcal{P}(X), \supseteq\big)$.

Contravariant functor $F\colon D \to C$
is a ("covariant" or usual) functor $F\colon D \to C^{\mathsf{op}}$ or $F\colon D^{\mathsf{op}} \to C$.

Thus $F(1_x) = 1_{Fx}$ as usual, but $F(g \circ f) = Ff \circ Fg$.

Generic examples: Locally small C, e.g., **Set** or lin. trans. cat. $\mathcal{L}$.
Fix a **dualizing object** $T \in C_0$, e.g., $\mathbf{2} = \{0, 1\} \in \mathbf{Set}_0$ or $\mathbb{R} \in \mathcal{L}_0$.
Functor $*\colon C \to \mathbf{Set}^{\mathsf{op}}; Z \mapsto Z^* := C(Z, T)$ with $(g \circ f)^* = f^* \circ g^*$:

$$\begin{tikzcd} Y \arrow[dd, "g"'] \arrow[dr, bend left] & & X \arrow[ll, "f"] \\ & T & \\ Z \arrow[ur, bend right] & & \end{tikzcd} \qquad \begin{tikzcd} \theta \circ g \arrow[r, mapsto, "f^*"] & \theta \circ g \circ f \\ \theta \arrow[u, mapsto, "g^*"] \arrow[ur, mapsto, "(g\circ f)^*"'] & \end{tikzcd}$$

$$\theta \xmapsto{g^*} \theta \circ g$$

From **Set**, set Z^* is the **power set** 2^Z of characteristic functions θ.

From $\mathcal{L}$, vector space Z^* is the **dual space** of linear functionals θ.

7. Diagram categories and functor categories

Diagram category C^D for diagram D and category C
has graph maps $F, G\colon D \to C$ as objects
and natural transformations $\sigma\colon F \to G$ as morphisms. Composition:

$$\begin{tikzcd} x \arrow[ddd, "f"'] \\ \\ \\ y \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} Fx \arrow[rr, "(\tau\bullet\sigma)_x"] \arrow[dr, "\sigma_x"'] \arrow[ddd, "Ff"'] & & Hx \arrow[ddd, "Hf"] \\ & Gx \arrow[ur, "\tau_x"'] \arrow[d, "Gf"'] & \\ & Gy \arrow[dr, "\tau_y"] & \\ Fy \arrow[ur, "\sigma_y"] \arrow[rr, "(\tau\bullet\sigma)_y"'] & & Hy \end{tikzcd}$$

$\ldots$ in D $\qquad\qquad$ $\ldots$ in C

Constant objects and morphisms:
$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} c \arrow[r, "\theta"] \arrow[d, equal, "1_c"'] & c' \arrow[d, equal, "1_c"] \\ c \arrow[r, "\theta"'] & c' \end{tikzcd}$$

Functor category: Category D, functors $F, G, \ldots$

Example: Linear representations of a group G are objects $R\colon G \to \mathcal{L}$
of the functor category $\mathcal{L}^G$ for the one-object group category G,
so group homomorphisms $R\colon G \to \mathcal{L}(V,V)^* = \operatorname{Aut} V = \mathsf{GL}(V)$.

The morphisms are **intertwiners** or **equivariant maps** $\tau\colon R_1 \to R_2$,
so $\forall\ g \in G$,
$$\begin{tikzcd} V_1 \arrow[r, "\tau_x"] \arrow[d, "R_1(g)"'] & V_2 \arrow[d, "R_2(g)"] \\ V_1 \arrow[r, "\tau_x"'] & V_2 \end{tikzcd}$$

E.g., $G = S_3$, $\quad V_1 = \mathbb{R}^3 = \operatorname{Span}\{e_1, e_2, e_3\}$, $\quad R_1(\pi)\colon e_i \mapsto e_{\pi(i)}$.

$V_2 = \mathbb{R}^2 = \operatorname{Span}\{e_2 - e_1, e_3 - e_2\}$, $\quad R_2(\pi)\colon (e_{i+1} - e_i) \mapsto (e_{\pi(i+1)} - e_{\pi(i)})$

$$\overbrace{\frac{1}{3}\begin{bmatrix} -2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}}^{\tau_x} \cdot \overbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}^{R_1(1\ 2\ 3)} = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}}^{R_2(1\ 2\ 3)} \cdot \overbrace{\frac{1}{3}\begin{bmatrix} -2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}}^{\tau_x}$$

8. Products and coproducts

Product $X \times Y = \{(x,y) \mid x \in X\,,\ y \in Y\}$ of sets X, Y:

$$\begin{tikzcd} X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \\ & Z \arrow[ul, bend left, "f"] \arrow[u, dashed, "f \sqcap g"] \arrow[ur, bend right, "g"'] & \end{tikzcd}$$

Universality property: $\forall\, Z \in \mathbf{Set}_0$, "solid"$\downarrow$ implies $\downarrow$"dashed"
bijection $\mathbf{Set}(Z,X) \times \mathbf{Set}(Z,Y) \to \mathbf{Set}(Z, X \times Y); (f,g) \mapsto f \sqcap g$
with $f = \pi_X \circ (f \sqcap g)$ and $g = \pi_Y \circ (f \sqcap g)$. Thus $f \sqcap g \colon z \mapsto (fz, gz)$.

Picture in $\mathbf{Set}^2$ for discrete "two spot" diagram $2 = \boxed{\bullet \quad \bullet}$:

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "f \sqcap g"'] & & X \\ & X \times Y \arrow[ur, "\pi_X"'] & \end{tikzcd}$$

$$\begin{tikzcd} & X \times Y \arrow[dr, "\pi_Y"] & \\ Z \arrow[rr, "g"'] \arrow[ur, "f \sqcap g"] & & Y \end{tikzcd}$$

Examples: Product in **Set** carries products in **Grp**, **Ring**, **Mon**, etc.

Example: Product in a poset category is a **greatest lower bound**.

$$\begin{tikzcd} a & b \\ c \arrow[u] \arrow[ur] & d \arrow[u] \end{tikzcd}$$

$a \times b = c$ exists,

but $c \times d$ does not.

Coproduct in C is the product in C^{op}:
$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, bend right, "f"'] & X + Y \arrow[d, dashed, "f \sqcup g"] & Y \arrow[l, "\iota_y"'] \arrow[dl, bend left, "g"] \\ & Z & \end{tikzcd}$$

Example: Coproduct in **Set** is the disjont union.

Example: Coproduct in a poset category is a **least upper bound**.

Biproduct $\begin{tikzcd} U \arrow[r, bend left, "\iota_U"] & U \oplus V \arrow[l, bend left, "\pi_U"] \arrow[r, bend right, "\pi_V"'] & V \arrow[l, bend right, "\iota_U"'] \end{tikzcd}$ in $\mathcal{L}$ is product and coproduct.

9. More limits and colimits

$$\begin{tikzcd}
Z \arrow[r, dashed, "r"'] \arrow[rr, bend left, "k"] \arrow[dr, bend right, "h"'] & X \times_B Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & B
\end{tikzcd}$$

Pullback: or with poset diagram

$\boxed{\bullet \to \bullet \leftarrow \bullet}$ category picture:

$$\begin{tikzcd}
Z \arrow[rr, "h"] \arrow[dr, dashed, "r"'] \arrow[dd, equal] & & X \arrow[dd, "f"] \\
& X \times_B Y \arrow[ur, "\pi_X"'] \arrow[d, equal] & \\
Z \arrow[r, dashed, "r"] \arrow[rr, bend right, "f \circ h = g \circ k"'] \arrow[dd, equal] & X \times_B Y \arrow[r, "f \circ \pi_X = g \circ \pi_Y"] \arrow[d, equal] & B \\
& X \times_B Y \arrow[dr, "\pi_Y"] & \\
Z \arrow[rr, "k"'] \arrow[ur, dashed, "r"] & & Y \arrow[uu, "g"']
\end{tikzcd}$$

Ex: Domain of category composition is pullback of $C_1 \xrightarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1$.
Pushout is the dual of a pullback.

Equalizer: $\begin{tikzcd} Z \arrow[r, dashed, "r"'] \arrow[rr, bend left, "k"] & E \arrow[r, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$ so $f \circ k = g \circ k \Rightarrow \exists_! \, r \, . \, e \circ r = k$

In $\mathcal{L}$, $E = \mathrm{Ker}(f - g) \overset{e}{\hookrightarrow} X$. In **Set**, $E = \{x \in X \mid fx = gx\} \overset{e}{\hookrightarrow} X$.

Coequalizer: $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "u"'] \arrow[rr, bend left, "k"] & C \arrow[r, dashed, "r"'] & Z \end{tikzcd}$; $k \circ f = k \circ g \Rightarrow \exists_! \, r . \, r \circ u = k$

In **Set**, C is quotient of Y by equiv. rel'n. gen. by $\{(fx, gx) \mid x \in X\}$.

In $\mathcal{L}$, u projects from Y to $C = \mathrm{Coker}(f - g) := Y/\mathrm{Im}(f - g)$.

Extended First Isomorphism Theorem in $\mathcal{L}$ is the exact sequence

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker} f \longrightarrow 0$$

where **exact** means $\mathrm{Im}\, g_1 = \mathrm{Ker}\, g_2$ for each $\xrightarrow{g_1} \bullet \xrightarrow{g_2}$.

Similar in **Ab**, ${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$, $\mathbf{Mod}_K$ (commutative unital ring K), or any **abelian category** A where each $A(X, Y)$ is an abelian group.

10. General limits and colimits

Diagram D, category C, **constant** or **diagonal** for $\theta \in C(c, c')$ is nat. tr. $\Delta\theta \colon \Delta c \to \Delta c'$ with $\Delta \colon D \to C; [f \colon x \to y] \mapsto$

$$\begin{tikzcd} c \arrow[r, "\theta"'] \arrow[d, equal] & c' \arrow[d, equal] \\ c \arrow[r, "\theta"] & c' \end{tikzcd}.$$

Limit of graph map $F \colon D \to C$ is **projection** $\pi \colon \Delta \varprojlim F \to F$ such that $\forall\ \kappa \colon \Delta Z \to F\,,\ \exists_!\ r = \varprojlim \kappa \in C(Z, \varprojlim F)\,.\ \pi \circ \Delta \varprojlim \kappa = \kappa$.

$$\begin{tikzcd} x \arrow[ddd, "f"'] & & & Z \arrow[rr, "\kappa_x"] \arrow[dr, "r=\varprojlim \kappa"'] \arrow[ddd, equal] & & Fx \arrow[ddd, "Ff"] \\ & & & & \varprojlim F \arrow[ur, "\pi_x"'] \arrow[d, equal] & \\ & & & & \varprojlim F \arrow[dr, "\pi_y"] & \\ y & \boxed{\text{in } D} & \boxed{\text{in } C} & Z \arrow[rr, "\kappa_y"'] \arrow[ur, "r=\varprojlim \kappa"] & & Fy \end{tikzcd}$$

A.k.a "projective limit" or "inverse limit", written as lim.

Colimit of graph map $F \colon D \to C$ is **insertion** $\iota \colon F \to \Delta \varinjlim F$ such that $\forall\ \kappa \colon F \to \Delta Z\,,\ \exists_!\ r = \varinjlim \kappa \in C(\varinjlim F, Z)\,.\ \Delta \varinjlim \kappa \circ \iota = \kappa$.

A.k.a "inductive limit" or "direct limit", written as colim.

Example: Functor (order-preserving) between poset categories $x \colon (\mathbb{N}, \leq) \to (\mathbb{R} \cup \{\infty\}, \leq) \colon n \mapsto x_n$. Then $\varinjlim x = \lim_{n\to\infty} x_n$.

Example: $F \colon \mathbb{N} \smallsetminus \{0,1\} \to \mathbf{Ring}; n \mapsto \mathbb{Z}/n\mathbb{Z}$. Then $r = \Delta \varprojlim \kappa \colon \mathbb{Z} \hookrightarrow \varprojlim F = \prod_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$ for $\kappa_n \colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}; x \mapsto x + n\mathbb{Z}$.

Directed diagram D: $\forall\ x, y \in D_0\,,\ \exists\ z \in D_0\,.\ x \to z \leftarrow y$.

Then have **directed limits** and **directed colimits**.

Example: (Real) vector space V,
directed poset $\big(\mathcal{P}_{\mathsf{fin}}(V), \subseteq\big)$ of finite subsets.
Functor $F \colon \mathcal{P}_{\mathsf{fin}}(V) \to \mathcal{L}; X \mapsto \mathrm{Span}(X)$. Then $\varinjlim F = V$.

Theorem: Each algebra is the (directed) colimit of its finitely generated subalgebras.

11. Product categories and bifunctors

Product $B \times C$ of quivers B, C has $(B \times C)_0 = B_0 \times C_0$,
$(B \times C)_1 = B_1 \times C_1$, pointwise $\partial_i(f, g) = (\partial_i f, \partial_i g)$ for $i = 0, 1$.

Product $B \times C$ of categories B, C : pointwise identities, composition:

$$(B \times C)\big((x, x'), (y, y')\big) \times (B \times C)\big((y, y'), (z, z')\big)$$
$$\to (B \times C)\big((x, x'), (z, z')\big) \colon \big((f, f'), (g, g')\big) \mapsto (f' \circ f, g' \circ g).$$

Universality:
$$\begin{tikzcd} B & B \times C \arrow[l, "\pi_B"'] \arrow[r, "\pi_C"] & C \\ & D \arrow[ul, bend left, "F"] \arrow[u, dashed, "F \sqcap G"] \arrow[ur, bend right, "G"'] & \end{tikzcd}$$
— graph maps or functors.

Example:
$$\begin{tikzcd} B' & B' \times C' \arrow[l, "\pi'_B"'] \arrow[r, "\pi'_C"] & C' \\ B \arrow[u, "F"] & B \times C \arrow[l, "\pi_B"'] \arrow[r, "\pi_C"] \arrow[u, dashed, "F \times G"] & C \arrow[u, "G"'] \end{tikzcd}$$

Bifunctor S to D on B and C is a functor $S \colon B \times C \to D$

— **graph, diagram** or **quiver bimap** if B, C are just quivers.

Proposition: Given bifunctor $S \colon B \times C \to D$: For $(b, c) \in (B \times C)_0$, define $R_b := S(b, \_) \colon C \to D$ and $L_c := S(\_, c) \colon B \to D$.

Then $\forall\, f \colon b \to b',\ g \colon c \to c'$:
$$\begin{tikzcd} S(b, c) \arrow[r, "R_b(g) = S(b, g)"] \arrow[d, "L_c(f) = S(f, c)"'] \arrow[dr, dashed, "S(f, g)"] & S(b, c') \arrow[d, "L_{c'}(f) = S(f, c')"] \\ S(b', c) \arrow[r, "R_{b'}(g) = S(b', g)"'] & S(b', c') \end{tikzcd}$$

Conversely, given $R_b \colon C \to D$ and $L_c \colon B \to D$
with $\forall b \in B_0, c \in C_0$, $L_c(b) = R_b(c)$
and commuting solid square,
the diagonal defines a bifunctor $S \colon B \times C \to D$. $\square$

Example: Locally small C, $B = C^{\mathsf{op}}$, $R_b \colon C \to \mathbf{Set}; b \mapsto C(b, c)$,
$L_c \colon C^{\mathsf{op}} \to \mathbf{Set}; b \mapsto C(b, c)$ (like dualizing), $R_b(c) = C(b, c) = L_c(b)$.

For $h \in C(b', c)$, so $b \xrightarrow{f} b' \xrightarrow{h} c \xrightarrow{g} c'$, have

$$\begin{tikzcd} h \circ f \arrow[r, mapsto, "R_b(g)"] & g \circ h \circ f \\ h \arrow[u, mapsto, "L_c(f)"] \arrow[r, mapsto, "R_{b'}(g)"'] & g \circ h \arrow[u, mapsto, "L_c(f)"'] \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} C(b, c) \arrow[r, "R_b(g)"] & C(b, c') \\ C(b', c) \arrow[u, "L_c(f)"] \arrow[r, "R_{b'}(g)"'] & C(b', c') \arrow[u, "L_{c'}(f)"'] \end{tikzcd}$$

12. Cartesian monoidal categories

Cartesian monoidal category: category C with all finite products.

Idea: Think of $(C, \times, \top)$ as like a monoid, say $(\mathbb{N}, +, 0)$ or $(\mathbb{R}, \cdot, 1)$.

Problem of **non-associativity:** e.g. in **Set**, $(x, (y, z)) \neq ((x, y), z)$.

Fix: Bifunctor $C \times C \to C; (X, Y) \mapsto X \times Y$,
trifunctors $C \times C \times C \to C; (X, Y, Z) \mapsto X \times (Y \times Z)$ or $(X \times Y) \times Z$,
nat. isom. α with components $\boxed{\alpha_{X,Y,Z} \colon X \times (Y \times Z) \to (X \times Y) \times Z}$
which commute with projections; both sides give a product of X, Y, Z.
[Typical two-stage projection $\pi_Y \colon X \times (Y \times Z) \xrightarrow{\pi_{Y \times Z}} Y \times Z \xrightarrow{\pi_Y} Y$.]

Problem of **non-unitality:** e.g. in **Set** with $\top = \{*\}$, have $(*, x) \neq x$.

Fix: Functors $C \to C; X \mapsto \top \times X$ or $X \times \top$ nat. isom. to identity,
so components $\boxed{\lambda_X \colon \top \times X \to X}$ and $\boxed{\rho_X \colon X \times \top \to X}$.

Potentially large "monoid" $(C, \times, \top)$ "up to natural isomorphisms".

Pentagon:

$$\begin{tikzcd}
 & (W \times X) \times (Y \times Z) \arrow[dl, "\alpha_{W \times X, Y, Z}"'] & \\
((W \times X) \times Y) \times Z & & W \times (X \times (Y \times Z)) \arrow[ul, "\alpha_{W, X, Y \times Z}"'] \arrow[d, "1_W \times \alpha_{X,Y,Z}"'] \\
(W \times (X \times Y)) \times Z \arrow[u, "\alpha_{W,X,Y} \times 1_Z"] & & W \times ((X \times Y) \times Z) \arrow[ll, "\alpha_{W, X \times Y, Z}"']
\end{tikzcd}$$

Triangle:
$$\begin{tikzcd}
X \times (\top \times Y) \arrow[rr, "\alpha_{X,\top,Y}"] \arrow[dr, "1_X \times \lambda_Y"'] & & (X \times \top) \times Y \arrow[dl, "\rho_X \times 1_Y"] \\
 & X \times Y &
\end{tikzcd}$$

Digon:
$$\begin{tikzcd}
\top \times \top \arrow[r, bend left, "\lambda_\top"] \arrow[r, bend right, "\rho_\top"'] & \top
\end{tikzcd}$$

Coherence [CWM §VII.2]: If these three diagrams commute, then w.l.o.g. have a **strict** monoidal category: the nat. isoms. are identities.

- Coherence holds for Cartesian monoidal categories.

13. Groups in categories

Group $(G, \nabla\colon G\times G \to G, S\colon G \to G, \eta\colon \top \to G)$ in **Set**, satisfying:

$$
\begin{tikzcd}
G\times G\times G \arrow[r, "1_G\times\nabla"] \arrow[d, "\nabla\times 1_G"'] & G\times G \arrow[d, "\nabla"] \\
G\times G \arrow[r, "\nabla"'] & G
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
G\times G \arrow[dr, "\nabla"] & G\times\top \arrow[l, "1_G\times\eta"'] \arrow[d, "\rho_G"] \\
\top\times G \arrow[u, "\eta\times 1_G"] \arrow[r, "\lambda_G"'] & G
\end{tikzcd}
\quad\text{(so a monoid)}
$$

and

$$
\begin{tikzcd}
 & G\times G \arrow[r, "1_G\times S"] & G\times G \arrow[dr, "\nabla"] & \\
G \arrow[ur, "\Delta"] \arrow[r] \arrow[dr, "\Delta"'] & \top \arrow[rr, "\eta"] & & G \\
 & G\times G \arrow[r, "S\times 1_G"'] & G\times G \arrow[ur, "\nabla"'] &
\end{tikzcd}
$$

with

$$
\begin{tikzcd}
G\times G \arrow[r, "\pi_G"] \arrow[d, "\pi_G"'] & G \\
G & G \arrow[l, "1_G"] \arrow[u, "1_G"'] \arrow[ul, dashed, "\Delta"']
\end{tikzcd}
$$

(so a group).

Group in a Cartesian monoidal category: interprets the diagrams.

Example: A **topological group** is a group in the category **Top** of continuous maps between topological spaces.

Example: The additive group functor
$G_a\colon K \mapsto (K, +, -, 0)$ is a group in $\mathbf{Grp}^{\mathbf{CRing}}$.

Example: The multiplicative group or group-of-units functor
$G_m\colon K \mapsto (K^*, \cdot, {}^{-1}, 1)$ is a group in $\mathbf{Grp}^{\mathbf{CRing}}$.

Example: The p-th roots of unity functor
$\mu_p\colon K \mapsto (\{k \in K \mid k^p = 1\}, \cdot, {}^{p-1}, 1)$ is a group in $\mathbf{Grp}^{\mathbf{CRing}}$.

Example: $(\mathsf{SL}_2, \nabla, S, \eta)$ as a group in $\mathbf{Grp}^{\mathbf{CRing}}$:

$$\mathsf{SL}_2\colon \mathbf{CRing} \to \mathbf{Grp}; K \mapsto \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in K\ ,\ ad - bc = 1 \right\}.$$

$$\nabla_K\colon \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) \mapsto \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix},$$

$$S_K\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ and } \eta_K\colon 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

14. Spaces, bases, adjunctions

Category $\mathcal{L}$ of linear transformations of vector spaces over a field K.

Forgetful or underlying set functor $U\colon \mathcal{L} \to \mathbf{Set}$.

For set X, v. sp. with basis X is $FX = \left\{ \sum_{i=1}^{r} \lambda_i \cdot x_i \mid \lambda_i \in K\,,\ x_i \in X \right\}$,

the space of formal linear combinations $\sum_{i=1}^{r} \lambda_i \cdot x_i$ of elements of X.

At X, have **unit** $\eta_X\colon X \to UFX; x \mapsto 1 \cdot x$ which **inserts** the basis.

At V, have **counit** $\varepsilon_V\colon FUV \to V; \sum_{i=1}^{r} \lambda_i \cdot v_i \mapsto v$, where
$v = \lambda_1 v_1 + \ldots + \lambda_r v_r$, the formal combination **worked out** in V.

For $f\colon X \to Y$, lin. transf. $Ff\colon FX \to FY; \sum_{i=1}^{r} \lambda_i \cdot x_i \mapsto \sum_{i=1}^{r} \lambda_i \cdot f(x_i)$.

So have functors $F\colon \mathbf{Set} \to \mathcal{L}$ and $U\colon \mathcal{L} \to \mathbf{Set}$.

Nat. isom. with components $\varphi_{X,V}\colon \boxed{\mathcal{L}(FX, V) \cong \mathbf{Set}(X, UV)}$ (*)

Mutually inverse $\varphi_{X,V}\colon [FX \xrightarrow{\theta} V] \mapsto [X \xrightarrow{\eta_X} UFX \xrightarrow{U\theta} UV]$
(informally, restricting θ to X); note unit $\eta_X = \varphi_{X,FX}(1_{FX})$;

and dually, $\varphi^{-1}_{X,V}\colon [X \xrightarrow{f} UV] \mapsto [FX \xrightarrow{Ff} FUV \xrightarrow{\varepsilon_V} V]$
(informally, extending f to FX); note counit $\varepsilon_V = \varphi^{-1}_{UV,V}(1_{UV})$.

Adjunction $\boxed{(F, U, \eta, \varepsilon)}$ with **left adjoint** F and **right adjoint** U.

Thus $\varphi_{X,V}\colon \theta \mapsto U\theta \circ \eta_X$ and $\varphi^{-1}_{X,V}\colon f \mapsto \varepsilon_V \circ Ff$.

Triangular identities: $\forall\, X \in \mathbf{Set}_0\,,\ 1_{FX} = \varepsilon_{FX} \circ F\eta_X\ [= \varphi^{-1}_{X,FX}(\eta_X)]$
and $\forall\, V \in \mathcal{L}_0\,,\ 1_{UV} = U\varepsilon_V \circ \eta_{UV}\ [= \varphi_{UV,V}(\varepsilon_V)]$.

The triangular identities are necessary and sufficient for an adjumction.

Other notations: $F \dashv U$ or
$$\begin{tikzcd} \mathcal{L} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Mnemonic: In the box (*), put the functors at the extreme edges.
The left adjoint (F) is on the left; the right adjoint (U) is on the right.

15. Three adjunctions with monoids

- **Free module functor** $F\colon \mathbf{Set} \to \mathbf{Mon}$ is left adjoint to the forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$.

"Tensor" notation $x_1 \otimes \ldots \otimes x_n$ for n-tuple $(x_1, \ldots, x_n)$.

For set or **alphabet** X, coproduct $FX := \sum_{n\in\mathbb{N}} X^n$, with $X^0 = \{1\}$ and **word concatenation** associative product

$$(x_1\otimes\ldots\otimes x_m, y_1\otimes\ldots\otimes y_n) \mapsto x_1\otimes\ldots\otimes x_m\otimes y_1\ldots\otimes y_n.$$

Note $\lambda_X = 1_X$ gives $1 \otimes x = x$ and similarly $x \otimes 1 = x$ by $\rho_X = 1_X$.

Unit $\eta_X\colon x \mapsto x$ ("alphabet letter makes a one-letter word") and

counit $\varepsilon_M\colon FUM \to M; m_1 \otimes m_2 \mapsto m_1 \cdot m_2$ ("multiplication table").

- **Group of Units functor** $U\colon \mathbf{Mon} \to \mathbf{Grp}; (M, \cdot, 1) \mapsto (M^*, \cdot, ^{-1}, 1)$ is right adjoint to Forgetful $F\colon \mathbf{Grp} \to \mathbf{Mon}; (G, \cdot, ^{-1}, 1) \mapsto (G, \cdot, 1)$.

Natural isomorphism $\varphi_{G,M}\colon \mathbf{Mon}(FG, M) \cong \mathbf{Grp}(G, UM); \theta \mapsto \theta$,

since $g \cdot g^{-1} = 1 \ \Rightarrow \theta(g) \cdot \theta(g)^{-1} = 1$ and dually, so $\theta(g) \in M^*$.

Unit $\eta_G\colon G \to G^*; g \mapsto g$ (note $G^* = G$) and

counit $\varepsilon_M\colon M^* \hookrightarrow M; u \mapsto u$ (embedding group of units into monoid).

- M**-sets for a monoid** M — **categorification** of a monoid M — e.g., permutation representations for M a group.

Functor $L\colon M \to \mathbf{Set}; * \mapsto X,\ m \mapsto [L_m\colon X \to X; x \mapsto mx]$,
can also be written as (X, M), a set X with "scalars" from M,
or as the monoid homomorphism $L\colon M \to \mathbf{Set}(X, X); m \mapsto L_m$.

Category $\mathbf{Set}^M$ of M-**sets**.

Forgetful functor $U\colon \mathbf{Set}^M \to \mathbf{Set}; L \mapsto L(*)$ or $(X, M) \mapsto X$.

Free M**-set functor** $F\colon \mathbf{Set} \to \mathbf{Set}^M; X \mapsto (M \times X, M)$
with $m(n, x) = (mn, x)$.

<u>Free algebra functor is left adjoint to the underlying set functor:</u>

Unit $\eta_X\colon X \to M \times X; x \mapsto (1, x)$
(embedding generators into the free algebra) and

counit $\varepsilon_{(X,M)}\colon (M \times X, M) \to (X, M); (m, x) \mapsto mx$
(action in the M-set).

16. Poset adjunctions and Galois correspondences

Poset categories $(\mathbf{A}, \leq)$, $(\mathbf{B}, \leq)$, functors $R\colon \mathbf{A} \to \mathbf{B}$, $S\colon \mathbf{B} \to \mathbf{A}$.

Galois connection: adjunction $\mathbf{A}(Sb, a) \cong \mathbf{B}(b, Ra)$,

$$\text{so } Sb \leq a \;\Leftrightarrow\; b \leq Ra.$$

Unit: $\forall\, b \in B\,,\ b \leq RSb$. **Counit:** $\forall\, a \in A\,,\ SRa \leq a$.

Thus $\forall\, a \in \mathbf{A}\,,\ Ra \leq RSRa$ and $\forall\, b \in \mathbf{B}\,,\ SRSb \leq Sb$ (plug in).

Also $\forall\, b \in \mathbf{B}\,,\ Sb \leq SRSb$ and $\forall\, a \in \mathbf{A}\,,\ RSRa \leq Ra$ (use S, R).

Closed elements: In $S(\mathbf{B}) \subseteq \mathbf{A}$ or $R(\mathbf{A}) \subseteq \mathbf{B}$.

Closure of $a \in \mathbf{A}$ is $SRa = \operatorname{dom} \varepsilon_a$, and of $b \in \mathbf{B}$ is $RSb = \operatorname{cod} \eta_b$.

Galois correspondence: Mut. inverse $\big(S(\mathbf{B}), \leq\big) \underset{S}{\overset{R}{\rightleftarrows}} \big(R(\mathbf{A}), \leq\big)$.

Polarity is a relation $\alpha \subseteq I \times J$. Gives Galois connection

$$S\colon (2^I, \subseteq) \to (2^J, \supseteq); X \mapsto \{y \in J \mid \forall x \in X\,,\ x\,\alpha\,y\}$$

$$R\colon (2^J, \supseteq) \to (2^I, \subseteq); Y \mapsto \{x \in I \mid \forall y \in Y\,,\ x\,\alpha\,y\}$$

Note $\forall\, X \subseteq I\,,\ \forall\, Y \subseteq J\,,$

$$SX \supseteq Y \quad \Leftrightarrow \quad \forall\, x \in X\,,\ \forall\, y \in Y\,,\ x\,\alpha\,y \quad \Leftrightarrow \quad X \subseteq RY.$$

Galois theory: Group permutation representation or G-set (X, G).
Fixed point relation $\{(x, g) \in X \times G \mid gx = x\}$.
Right adjoint $R\colon 2^G \to 2^X$ is the **fixed point functor**.
Left adjoint $S\colon 2^X \to 2^G$ is the (pointwise) **stabilizer functor**.

Polar geometry: Vector space V with quadratic form $\langle \mathbf{u}, \mathbf{v} \rangle$.
Polarity $\{(\mathbf{u}, \mathbf{v}) \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\} \subseteq V \times V$.
Closure of a subset is its **orthogonal complement**.

Alg. geometry: On $\mathbf{C}^n \times \mathbf{C}[X_1, \ldots, X_n]$, polarity $\{(\mathbf{x}, f) \mid f(\mathbf{x}) = 0\}$.
Closed subsets of $\mathbf{C}^n$ are **algebraic sets** or **varieties**.
Closed subsets of $\mathbf{C}[X_1, \ldots, X_n]$ are **radical ideals**.
Hilbert's Nullstellensatz: The closure of an ideal $\mathfrak{I} \lhd \mathbf{C}[X_1, \ldots, X_n]$ is its **radical** $\sqrt{\mathfrak{I}} = \{f \mid \exists\ 0 < n \in \mathbb{N}\,.\ f^n \in \mathfrak{I}\}$.

Example: Radical of $\langle X_1^2 \rangle$ in $\mathbf{C}[X_1, \ldots, X_n]$ is $\langle X_1 \rangle$.

17. Slice categories and comma categories

For $b \in C_0$, **slice category** $(C \downarrow b)$ or $(1_C \downarrow b)$ of C-**objects over** b has object class $\partial_1^{-1}(b)$, morphisms
$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "p"'] & & c' \arrow[dl, "p'"] \\ & b & \end{tikzcd}$$
(commuting),

composition
$$\begin{tikzcd} c \arrow[r, "f"'] \arrow[rr, bend left, "f'\circ f"] \arrow[dr, "p"'] & c' \arrow[r, "f'"'] \arrow[d, "p'"] & c'' \arrow[dl, "p''"] \\ & b & \end{tikzcd}$$
, terminal object $1_b \colon b \to b$.

Dually, **slice category** $(b \downarrow C)$ or $(b \downarrow 1_C)$ of C-**objects under** b.

Examples: Down-sets, and up-sets (or principal filters), in posets.

Example: For a group G and G-module A in $\mathbf{Ab}^G$,
the split extension $p \colon A \times G \to G; (a, g) \mapsto g$ in $(\mathbf{Grp} \downarrow G)$.
Here $(a, g)(a', g') = (a + ga', gg')$.

For $b \in C_0$ and $T \colon E \to C$,
comma category $(T \downarrow b)$ of **objects** T**-over** b
has morphisms
$$\begin{tikzcd} Te \arrow[rr, "Tf"] \arrow[dr, "p"'] & & Te' \arrow[dl, "p'"] \\ & b & \end{tikzcd}$$
.

Dually, for $b \in C_0$ and $S \colon D \to C$,
comma category $(b \downarrow S)$ of **objects** S**-under** b.

Proposition: For adjunction $(F \colon \mathbf{X} \to \mathbf{A}, U \colon \mathbf{A} \to \mathbf{X}, \eta, \varepsilon)$,
unit $\eta_X \colon X \to UFX$ is an initial object of $(X \downarrow U)$ and
counit $\varepsilon_A \colon FUA \to A$ is a terminal object of $(F \downarrow A)$.

Proof.
$$\begin{tikzcd} & X \arrow[dl, "\eta_X"'] \arrow[dr, "p"] & \\ UFX \arrow[rr, dashed, "U\varphi^{-1}_{X,A}p"'] & & UA \end{tikzcd}$$
and
$$\begin{tikzcd} FX \arrow[rr, dashed, "F\varphi_{X,A}p"] \arrow[dr, "p"'] & & FUA \arrow[dl, "\varepsilon_A"] \\ & A & \end{tikzcd}$$
□

Cor: Given
$$\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$$
, unit and counit uniquely determined.

18. The Yoneda Lemma

Yoneda Lemma: Let $\mathbf{A}$ be locally small.
For object A_1 of $\mathbf{A}$, and $f\colon A_2 \to A_3$ in $\mathbf{A}_1$, remember
$\mathbf{A}(A_1, f)\colon \mathbf{A}(A_1, A_2) \to \mathbf{A}(A_1, A_3); h \mapsto f \circ h$ post-composes with f.

Then for $K\colon \mathbf{A} \to \mathbf{Set}$, have $\boxed{\mathbf{Set}^{\mathbf{A}}\big(\mathbf{A}(A_1, \_\,), K\big) \cong KA_1; \tau \mapsto \tau_{A_1}(1_{A_1})}$

Proof. • Injectivity:

$$
\begin{tikzcd}
A_1 \arrow[d, "h"'] \\
A_2
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
\mathbf{A}(A_1, A_1) \arrow[r, "\tau_{A_1}"] \arrow[d, "{\mathbf{A}(A_1,h) \atop = L_\circ(h)}"'] & KA_1 \arrow[d, "Kh"] \\
\mathbf{A}(A_1, A_2) \arrow[r, "\tau_{A_2}"'] & KA_2
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{A_1} \arrow[rr, mapsto] \arrow[d, mapsto] & & \tau_{A_1}(1_{A_1}) \arrow[d, mapsto] \\
h \arrow[r, mapsto] & \tau_{A_2}(h) = Kh\Big(\tau_{A_1}(1_{A_1})\Big) &
\end{tikzcd}
$$

$\boxed{\text{In } \mathbf{A}}$ $\boxed{\text{In } \mathbf{Set}}$

• Surjectivity, $\rho\colon \mathbf{A}(A_1, A_2) \to K$, $\rho_{A_2}\colon h \mapsto Kh(x)$ for $x \in KA_1$ nat:

$$
\begin{tikzcd}
A_2 \arrow[d, "f"'] \\
A_3
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
\mathbf{A}(A_1, A_2) \arrow[r, "\rho_{A_2}"] \arrow[d, "{\mathbf{A}(A_1,f) \atop = L_\circ(f)}"'] & KA_2 \arrow[d, "Kf"] \\
\mathbf{A}(A_1, A_3) \arrow[r, "\rho_{A_3}"'] & KA_3
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & Kh(x) \arrow[d, mapsto] \\
f \circ h \arrow[r, mapsto] & Kf\Big(Kh(x)\Big) = K(f \circ h)(x)
\end{tikzcd}
$$

$\boxed{\text{In } \mathbf{A}}$ $\boxed{\text{In } \mathbf{Set}}$ $\square$

Corollary: Full, faithful (covariant) **Yoneda embedding**

$$\boxed{ヨ\colon D \hookrightarrow \widehat{D} = \mathbf{Set}^{D^{\mathsf{op}}}; [f\colon x \to y] \mapsto \big[D(\_\,, f)\colon D(\_\,, x) \to D(\_\,, y)\big]}$$

Category $\widehat{D}$ of (set-valued) **pre-sheaves** over D.

Note: "ヨ" is Katakana for "Yo".

Example: Poset category $(P, \leq)$.
For element x, slice category $D(\_\,, x)$ is (ess.) the down-set $\downarrow x$ of x.
Then for $f\colon x \leq y$,
natural transformation $D(\_\,, f)$ is the inclusion $\downarrow x \hookrightarrow \downarrow y$.

19. Reflective subcategories and counit properties

Reflective subcategory $\mathbf{A}$ of $\mathbf{B}$ means
the inclusion $K\colon \mathbf{A} \hookrightarrow \mathbf{B}$ is full (not required in CWM), and has a left adjoint $L\colon \mathbf{B} \to \mathbf{A}$, called the **localization** or **reflector**.

Example: $K\colon \mathbf{Ab} \hookrightarrow \mathbf{Grp}$
Then $L\colon G \mapsto G/[G,G]$, the largest abelian quotient of G.

Reflective adjunction: $\mathbf{A}(LB, A) \cong \mathbf{B}(B, A)$

Unit: $\eta_B\colon B \to LB$; **counit:** $\varepsilon_A\colon LA \to A$ is an isomorphism.

So, when are counits of adjunctions isomorphisms? Need lemmata:

Lemma 1: $\tau\colon S \to T$ is $\begin{Bmatrix}\text{epi}\\ \text{mono}\end{Bmatrix}$ in $\mathbf{Set}^{\mathbf{A}}$ iff each $\tau_{A''}$ $\begin{Bmatrix}\text{epi}\\ \text{mono}\end{Bmatrix}$ in $\mathbf{A}$.

Proof.
$$\begin{tikzcd} S \arrow[r, "\tau"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\text{p-o}"] & T \arrow[d, "1_T"] \\ T \arrow[r, "1_T"'] & T \end{tikzcd} \quad \Leftrightarrow \quad \forall\ A'' \in \mathbf{A}_0, \quad \begin{tikzcd} SA'' \arrow[r, "\tau_{A''}"] \arrow[d, "\tau_{A''}"'] \arrow[dr, phantom, "\text{p-o}"] & TA'' \arrow[d, equal] \\ TA'' \arrow[r, equal] & TA'' \end{tikzcd} \qquad \square$$

Lemma 2: For $f\colon A' \to A$, natural transformation $R_\circ(f)$ or
$\mathbf{A}(A, f)\colon \mathbf{A}(A, \_) \to \mathbf{A}(A', \_)$ is $\begin{Bmatrix}\text{mono}\\ \text{epi}\end{Bmatrix}$ iff f is $\begin{Bmatrix}\text{epi}\\ \text{split mono}\end{Bmatrix}$.

[Note $R_\circ(f) \mapsto R_\circ(f)(1_A) = f$ under the Yoneda Lemma.]

Proof. $\mathbf{A}(A, A') \xrightarrow{R_\circ(f)} \mathbf{A}(A', A')$ epi $\Rightarrow \exists\ r \in \mathbf{A}(A, A') \,.\, r \circ f = 1_{A'}$.
Conv., $f^{\mathsf{op}} r^{\mathsf{op}} = 1_{A'} \Rightarrow \forall\ A''\,,\ \exists A''(f^{\mathsf{op}}) \circ \exists A''(r^{\mathsf{op}}) = \exists A''(1_{A'})$
$\Rightarrow R_\circ(f)_{A''} \circ R_\circ(r)_{A''} = 1_{\mathbf{A}(A', A'')} \Rightarrow R_\circ(f)_{A''}$ surj., epi; so $R_\circ(f)$ epi.

$\forall\ A''\,,\ \mathbf{A}(A, A'') \xrightarrow{R_\circ(f)} \mathbf{A}(A', A'')$ mono $\Leftrightarrow h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$ $\square$.

Theorem: In $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$,

U is ...	iff ε_A ...
full	has retract
faithful	is epi
full, faithful	is iso

Proof. Natural transformation $\alpha\colon \mathbf{A}(A, \_) \to \mathbf{A}(FUA, \_)$ with component $\alpha_{A'}\colon \mathbf{A}(A, A') \xrightarrow{U_{A,A'}} \mathbf{X}(UA, UA') \xrightarrow{\varphi^{-1}_{UA,A'}} \mathbf{A}(FUA, A')$.
Under Yoneda Lemma, $\alpha \mapsto \alpha_A(1_A) = \varepsilon_A$. Then by Lemma 2:
ε_A split mono $\Leftrightarrow$ α epi $\Leftrightarrow$ $\forall A'\,,\ \alpha_{A'}$ surj. $\Leftrightarrow$ $\forall A'\,,\ U_{A,A'}$ surj;
ε_A epi $\Leftrightarrow$ α mono $\Leftrightarrow$ $\forall A'\,,\ \alpha_{A'}$ mono $\Leftrightarrow$ $\forall A'\,,\ U_{A,A'}$ inj;. $\square$

20. Category equivalence

Equivalence: Full, faithful, essentially surjective functor $F\colon \mathbf{X} \to \mathbf{A}$.

Recall **essentially surjective:** $\forall\ A \in \mathbf{A}_0\,,\ \exists\ X \in \mathbf{X}\,.\ \varepsilon_A\colon FX \cong A\,.$

Preorder: Set $(Q, \leq)$ with reflexive transitive relation $\leq$ on set Q,
or a small category with $\forall\ x, y \in Q\,,\ |Q(x, y)| \leq 1$.

Define α on Q by $x\ \alpha\ y \ \Leftrightarrow\ x \leq y$ and $y \leq x$, an equivalence relation.
Set P of equivalence class representatives: $\forall\ q \in Q\,,\ \exists\ p \in P\,.\ p \cong q\,.$
Inclusion functor $F\colon (P, \leq) \hookrightarrow (Q, \leq)$ is an equivalence.
"Election" functor $U\colon (Q, \leq) \to (P, \leq)$ chooses representatives.
Then $\forall\ q \in Q\,,\ \varepsilon_q\colon FUq \cong q$, isomorphic counit of an adjunction.
Note $(P, \leq)$ is a poset — antireflexive!

Adjoint equivalence: $\begin{tikzcd} \mathbf{A} \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$, $\mathbf{A} \xrightarrow{U} \mathbf{X}$ with unit, counit iso.

Equivalence: $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$ with $1_{\mathbf{X}} \to UF, \quad FU \to 1_{\mathbf{A}}$ iso.

Theorem: Functor $F\colon \mathbf{X} \to \mathbf{A}$. TFAE: (a) F is an equivalence;
(b) F is part of an adjoint equivalence of categories;
(c) F is part of an equivalence of categories.

(a)$\Rightarrow$(b): $\forall\ A \in \mathbf{A}_0\,,\ \exists\ UA \in \mathbf{X}\,.\ \varepsilon_A\colon FUA \cong A\,.$ Full, faithful $F \Rightarrow$
$\forall\ f \in \mathbf{A}(FX, A)\,,\ \exists_!\ \varphi_{X,A}f \in \mathbf{X}(X, UA)\,.\ F\varphi_{X,A}f = \varepsilon_A^{-1} \circ f,\ \ldots$
[Complete the adjunction, dual to the construction for linear algebra.]

(c)$\Rightarrow$(a): Need F full and faithful.

F faithful: $\begin{tikzcd} X_1 \arrow[r, "\eta_{X_1}"] \arrow[d, "g"'] & UFX_1 \arrow[d, "UFg"] \\ X_2 \arrow[r, "\eta_{X_2}"'] & UFX_2 \end{tikzcd}$ and U faithful: $\begin{tikzcd} FUA_1 \arrow[r, "\varepsilon_{A_1}"] \arrow[d, "k"'] & A_1 \arrow[d, "FUk"] \\ FUA_2 \arrow[r, "\varepsilon_{A_2}"] & A_2 \end{tikzcd}$

F full: For $h \in \mathbf{A}(FX_1, FX_2)$, want $h = Ff$ for $f \in \mathbf{X}(X_1, X_2)$.

Have $\begin{tikzcd} X_1 \arrow[r, "\eta_{X_1}"] \arrow[d, "f"'] & UFX_1 \arrow[d, "Uh"] \\ X_2 \arrow[r, "\eta_{X_2}"'] & UFX_2 \end{tikzcd}$ for $f = \eta_{X_2}^{-1} \circ Uh \circ \eta_{X_1}$ and $\begin{tikzcd} X_1 \arrow[r, "\eta_{X_1}"] \arrow[d, "f"'] & UFX_1 \arrow[d, "UFf"] \\ X_2 \arrow[r, "\eta_{X_2}"'] & UFX_2 \end{tikzcd}$

so $Uh = UFf$. Then U faithful gives $h = Ff$. $\square$

Corollary: Essentially surjective $K\colon \mathbf{A} \hookrightarrow \mathbf{B}$ gives a reflection.

21. Typical equivalences

- **Skeleton** S of C: unique representative for each isomorphism class.
Like poset $(P, \leq)$ induced in preorder $(Q, \leq)$,
essentially surjective $K \colon S \hookrightarrow C$ has reflection $L \colon C \to S$.

Ex: $\big\{\{i \in \mathbb{N} \mid i < n\} \,\big|\, n \in \mathbb{N}\big\}$ as object set of skeleton of **FinSet**.

- **Morita equivalence:** Ring R, ring R_n^n of $n \times n$-matrices over R.

$$\begin{tikzcd} \mathbf{Mod}_R \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_{R_n^n} \arrow[l, bend right, "F"'] \end{tikzcd} \quad \text{with } U \colon M \to \overbrace{M \oplus \cdots \oplus M}^{n},$$

$$F \colon [R_n^n \to \mathrm{End}(N)] \mapsto [R \to R_n^n \to \mathrm{End}(N)].$$

Concrete category:
Category of sets with structure (algebraic, topological,...)
and structure-preserving functions (homomorphisms, continuous,...).

- **Duality:** Equivalence $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X}^{\mathsf{op}} \arrow[l, bend right, "F"'] \end{tikzcd}$ of concrete categories.

Dualizing object: Set T with structure $T \in \mathbf{A}$ or $T \in \mathbf{X}$,
where: $\forall\ A \in \mathbf{A}_0\,,\ \mathbf{A}(A, T) \leq \mathbf{Set}(A, T) = T^A \in \mathbf{X}$
and: $\forall\ X \in \mathbf{X}_0\,,\ \mathbf{X}(X, T) \leq \mathbf{Set}(X, T) = T^X \in \mathbf{A}$.

Then $U = \mathbf{A}(\_\_, T)$ and $F = \mathbf{X}(\_\_, T)$.

Example: Category $\mathcal{L}_{\mathsf{fin}}$ of fin.-dim. vector spaces over a field K.
Then $\mathbf{A} = \mathbf{X} = \mathcal{L}_{\mathsf{fin}}$, $T = K$ and $\varepsilon_V^{-1} \colon V \to V^{**}; v \mapsto [f \mapsto f(v)]$.

Example: Fourier transforms, **Pontryagin duality**.

Then $\mathbf{A} = \mathbf{Ab}$, $\mathbf{X} = \mathbf{CAb}$ (compact abelian groups),
and $T = (\mathbb{R}/\mathbb{Z}, +, 0)$ "1-dimensional torus" or $(S^1, \cdot, 1)$ "circle group".
$\widehat{A} := UA = \mathbf{Ab}(A, T)$, the group of **characters** $\chi \colon A \to T$.

$$\varepsilon_A^{-1} \colon A \to FUA; a \mapsto [\chi \mapsto \chi(a)].$$

Example: Category $\mathbf{A}$ of finite Boolean algebras, $\mathbf{X} = \mathbf{FinSet}$,
dualizing object $T = \mathbf{2} := \{0, 1\}$, so power set FX (char. fns.).

$$\eta_X \colon X \to UFX; x \mapsto [\chi \mapsto \chi(x)].$$

Note: Can extend from a category $\mathbf{A}_{\mathsf{f.g.}}$ of finitely generated algebras to a category $\mathbf{A}$ of all algebras: treat as colimits of f.g. algebras, which will dualize to limits of $\mathbf{X}_{\mathsf{f.g.}}$-objects.

22. Preservation, reflection and creation

Diagram $D\colon J \to \mathbf{A}$, functor $G\colon \mathbf{A} \to \mathbf{B}$.

$$\begin{tikzcd} J \arrow[d, "D"'] & \\ \mathbf{A} \arrow[r, "G"'] & \mathbf{B} \end{tikzcd}$$

G **preserves** J-limits if it "pushes limits forward":
Diagram $D\colon J \to \mathbf{A}$ has a limit $\big[\varprojlim D \xrightarrow{\pi_j} D_j\big]$
implies $GD\colon J \to \mathbf{B}$ has a limit $\big[G(\varprojlim D) \xrightarrow{G\pi_j} GD_j\big]$.

G **reflects** J-limits if it "pulls limits back":
Diagram $GD\colon J \to \mathbf{B}$ has a limit of the **image form** $\big[GL \xrightarrow{G\pi_j} GD_j\big]$
implies $D\colon J \to \mathbf{A}$ already had a limit $\big[\varprojlim D = L \xrightarrow{\pi_j} D_j\big]$.

G **creates** J-limits if it both preserves and reflects,
and if $\varprojlim GD$ exists, then it exists in the image form.

Corresponding definitions for colimits.

Example: $U\colon \mathbf{Grp} \to \mathbf{Set}$ preserves, reflects limits, directed colimits.
[Consider "pointwise" structure on the underlying sets.]
Doesn't preserve or reflect general colimits.

Example: $U\colon \mathbf{Top} \to \mathbf{Set}$ preserves, but doesn't reflect, limits:

$$\begin{tikzcd} E \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\text{p-b}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd} \qquad \text{in } \mathbf{Top}$$

means $E = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$ has the subspace topology.

Example:
Full and faithful $K\colon \mathbf{Ab} \hookrightarrow \mathbf{Grp}$ preserves limits, but not colimits.

Theorem: Full and faithful $G\colon \mathbf{A} \to \mathbf{B}$ reflects limits and colimits.

Example: In $\mathbf{Ab}$, coproduct $C_2 + C_3$ or $C_2 \oplus C_3$ is C_6.
In $\mathbf{Grp}$, coproduct $C_2 + C_3$ or $C_2 * C_3$ is the **modular group** $\mathsf{PSL}_2(\mathbb{Z})$.
Doesn't violate $K\colon \mathbf{Ab} \hookrightarrow \mathbf{Grp}$ reflecting colimits: $\mathsf{PSL}_2(\mathbb{Z}) \neq KC_6$.

23. Preservation and adjunction

Diagram $D\colon J \to \mathbf{A}$, adjoint functors F, U:

$$\begin{tikzcd} J \arrow[d, "D"'] & \\ \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Suppose limit $\left[\varprojlim D \xrightarrow{\pi_j} D_j\right]$ exists:

$$\begin{tikzcd} A \arrow[rr, "\kappa_j"] \arrow[dr, "r=\varprojlim\kappa"'] & & D_j \\ & \varprojlim D \arrow[ur, "\pi_j"'] & \end{tikzcd}$$

Thus $\mathbf{A}^J(\Delta A, D) \cong \mathbf{A}(A, \varprojlim D)$.
$\qquad \kappa_j = \pi_j \circ r \mapsto \qquad r$

Theorem: Right adjoints preserve limits.

Proof. UD has limit $U \varprojlim D$:
$\mathbf{X}^J(\Delta X, UD) \cong \mathbf{A}^J\big(\Delta(FX), D\big) \cong \mathbf{A}(FX, \varprojlim D) \cong \mathbf{X}(X, U\varprojlim D)$. $\square$

Corollary: Left adjoints preserve colimits.

Example: In $\mathcal{L}(FX, V) \cong \mathbf{Set}(X, UV)$,
have $U(V_1 \oplus V_2) = V_1 \times V_2$ and $F(X_1 + X_2) = FX_1 \oplus FX_2$.

Example: Multiplicity of the **Euler φ-function** or **totient function**

$$\varphi(n) = |\{r \mid 1 \le r \le n \ \text{ and } \ \gcd(r,n) = 1\}| = \big|(\mathbb{Z}/_n, \times, 1)^*\big|.$$

Recall group of Units functor $U\colon \mathbf{Mon} \to \mathbf{Grp}$; $(M, \cdot, 1) \mapsto (M^*, \cdot, ^{-1}, 1)$
is right adjoint to Forgetful $F\colon \mathbf{Grp} \to \mathbf{Mon}$; $(G, \cdot, ^{-1}, 1) \mapsto (G, \cdot, 1)$.

For coprime m, n, have $(\mathbb{Z}/_{mn}, \times, 1) \cong (\mathbb{Z}/_m, \times, 1) \times (\mathbb{Z}/_n, \times, 1)$.
Then $\varphi(mn) = \big|(\mathbb{Z}/_{mn}, \times, 1)^*\big| = \Big|\big[(\mathbb{Z}/_m, \times, 1) \times (\mathbb{Z}/_n, \times, 1)\big]^*\Big|$
$= \big|(\mathbb{Z}/_m, \times, 1)^* \times (\mathbb{Z}/_n, \times, 1)^*\big| = \big|(\mathbb{Z}/_m, \times, 1)^*\big| \times \big|(\mathbb{Z}/_n, \times, 1)^*\big| = \varphi(m)\varphi(n)$.

Example: Equivalence $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$

implies $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$ and $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\top"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$,

so F and U preserve limits and colimits.

24. Heyting algebras and topologies

Preorder $(P, \leq)$ with all finite products, sufficiently including:

The empty product (terminal object) $\top$ with $\forall\, x \in P\,,\; x \leq \top$;

The (comm., assoc.) **meet** or g.l.b with $\forall\, x, y \in P\,,\; x \leftarrow x \cdot y \rightarrow y$.

For each fixed a in P, functor $S(a)\colon (P, \leq) \rightarrow (P, \leq); x \mapsto (x \cdot a)$.

Suppose each $S(a)$ has a right adjoint $R(a)\colon z \mapsto (a \multimap z)$:

$$\forall\, x, y, z \in P\,,\; \boxed{x \cdot y \leq z \;\Leftrightarrow\; x \leq y \multimap z} \qquad (*)$$

Example: Propositions, "and" is product; "deduce q from p" is $p \rightarrow q$. Then $p \multimap q$ would be proposition "p implies q".

Bounded lattice: poset, finite products, coproducts, $0 = \perp$, $1 = \top$.

Complete lattice: poset with all products and coproducts.

Heyting algebra is a bounded lattice with the adjunctions $(*)$.

Prop: Heyting algebras are distributive: $S(a)$ preserves coproducts.

Prop: Complete Heyting algebras are completely distributive. By $(*)$, have $y \multimap z = \sum\{x \mid x \cdot y \leq z\}$.

Example: Boolean algebra with implication $p \multimap q = p \rightarrow q = (\neg p) \vee q$

Negation (pseudocomplement) $\neg x := x \multimap 0$ in any Heyting algebra.

Example: $\{0 \leq \frac{1}{2} \leq 1\}$, where $\frac{1}{2} \multimap 0 = \max\{x \mid x \cdot \frac{1}{2} \leq 0\} = 0$.
Then $\neg\neg\frac{1}{2} = \neg 0 = 1 \neq \frac{1}{2}$; "Law of the excluded middle" does not hold.

Regular elements $x = \neg\neg x$ in Heyting algebra form Boolean algebra.

Topology: In any topological space $(X, \mathcal{O})$, the subset $\mathcal{O}$ of 2^X comprising the open sets forms a complete Heyting algebra. Unions in 2^X, but infinite intersections differ, take interior. Here $P \multimap Q = [(X \smallsetminus P) \cup Q]^\circ$

- **Indiscrete topology** $\mathcal{O} = \{\emptyset, X\}$
- **Discrete topology** $\mathcal{O} = 2^X$
- **Alexandrov topology** of poset $(P, \leq)$ is the set of all downsets.
- **Cofinite topology** of set X has $\mathcal{O} = \{\emptyset\} \cup \{S \subseteq X \mid X \smallsetminus S \text{ finite}\}$
- For monoid M and an M-set X, take $\mathcal{O}$ as the set of M-subsets. If M is a group, get a Boolean algebra.

25. Currying

Heyting algebra: $\forall\, x, y, z \in P_0\,,\ \ P(x \cdot y, z) \cong P(x, y \multimap z)$
In particular, $\forall\, y, z \in P_0\,,\ \ P(y, z) \cong P(1, y \multimap z)$.

Currying: $\forall\, X, Y, Z \in \mathbf{Set}_0\,,\ \mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}\big(X, \mathbf{Set}(Y, Z)\big)$
In particular, $\forall\, Y, Z \in \mathbf{Set}_0\,,\ \mathbf{Set}(Y, Z) \cong \mathbf{Set}\big(\top, \mathbf{Set}(Y, Z)\big)$.

Tensor product: $\forall\, X, Y, Z \in \mathcal{L}_0\,,\ \mathcal{L}(X \otimes Y, Z) \cong \mathcal{L}\big(X, \mathcal{L}(Y, Z)\big)$
In particular, $\forall\, Y, Z \in \mathcal{L}_0\,,\ \mathcal{L}(Y, Z) \cong \mathcal{L}\big(K, \mathcal{L}(Y, Z)\big)$,
"linear spaces" as modules over commutative ring K., e.g., $\mathbb{Z}$ for $\mathbf{Ab}$.

Note $\mathcal{L}\big(X, \mathcal{L}(Y, Z)\big) \subseteq \mathbf{Set}\big(X, \mathbf{Set}(Y, Z)\big) \cong \mathbf{Set}(X \times Y, Z)$, so
$\mathcal{L}\big(X, \mathcal{L}(Y, Z)\big)$ tracks the **bilinear** maps $X \times Y \to Z$.

In all three cases, $\cong$ is a natural isomorphism of sets, so on the left hand side of the lower $\cong$ is a hom-set of the locally small category.

Strict symmetric monoidal category $(\mathbf{C}, \otimes, I)$: $X \otimes Y = Y \otimes X$,
$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$, and $I \otimes X = X = X \otimes I$.

E.g: Heyting algebra $(P, \cdot, 1)$, Cartesian $(\mathbf{Set}, \times, \top)$, linear $(\mathcal{L}, \otimes, K)$.

Closed monoidal category: Adjunction $\mathbf{C}(X \otimes Y, Z) \cong \mathbf{C}(X, [Y, Z])$
with **internal hom**-object $[Y, Z]$, set isom. $\mathbf{C}(Y, Z) \cong \mathbf{C}(I, [Y, Z])$.

Bifunctors: monoidal product $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$
and internal hom $[\_\_\,, \_\_] \colon \mathbf{C}^{\mathsf{op}} \times \mathbf{C} \to \mathbf{C}$.

Heyting algebras: monoid product $\to$ adjunction $\to$ internal hom.
Linear spaces: internal hom $\to$ adjunction $\to$ monoid product.

Note $\_\_ \otimes Y$ a left adjoint $\Rightarrow$ preserves coproducts $\Rightarrow$ distributivity:

$$(X + X') \otimes Y = (X \otimes Y) + (X' \otimes Y) \text{ or } \Big(\sum X_i\Big) \otimes Y = \sum (X_i \otimes Y)$$

Also $[Y, \_\_]$ a right adjoint $\Rightarrow$ preserves products $\Rightarrow$ "exponentiation":

$$[Y, Z_1 \times Z_2] = [Y, Z_1] \times [Y, Z_2] \text{ or } [Y, \textstyle\prod Z_i] = \prod [Y, Z_i]$$

Compare $\mathbf{C}(Y, \prod Z_i) \cong \prod \mathbf{C}(Y, Z_i)$

Arithmetic: $(l + l') \cdot m = l \cdot m + l' \cdot m$ and $(n_1 \cdot n_2)^m = n_1^m \cdot n_2^m$
in the skeleton $(\mathbb{N}, \cdot, 1)$ of $(\mathbf{FinSet}, \times, \top)$.

26. Enriched categories

Bicomplete category: All limits and colimits.

Base category: bicomplete symmetric monoidal category $(\mathbf{B}, \otimes, I)$, e.g., $(\mathbf{Set}, \times, \top)$, $(\mathcal{L}, \otimes, K)$, poset $\Big(([0,\infty], \geq\ ), +, 0\Big)$ with $x+\infty = \infty$.

B-enriched category: quiver C with $\forall\ x, y \in C_0\,,\ C(x,y) \in \mathbf{B}_0$ and:

- **composition:** $\forall\ x, y, z \in C_0\,, \circ \in \mathbf{B}\big(C(x,y) \otimes C(y,z), C(x,z)\big)$
- **identities:** $\forall\ x \in C_0\,, j_x \in \mathbf{B}(I, C(x,x))$ with commuting:

$$\begin{tikzcd}
C(w,x) \otimes C(x,y) \otimes C(y,z) \arrow[r, "\circ\otimes 1"] \arrow[d, "1\otimes\circ"'] & C(w,y) \otimes C(y,z) \arrow[d, "\circ"] \\
C(w,x) \otimes C(x,z) \arrow[r, "\circ"'] & C(w,z)
\end{tikzcd}$$

and

$$\begin{tikzcd}
C(x,y) \arrow[r, "j_x\otimes 1"] \arrow[d, "1\otimes j_y"'] \arrow[dr, equal] & C(x,x) \otimes C(x,y) \arrow[d, "\circ"] \\
C(x,y) \otimes C(y,y) \arrow[r, "\circ"'] & C(x,y)
\end{tikzcd}$$

[recall $B = I \otimes B$, etc.]

... for $w, x, y, z \in C_0$.

Locally small category is enriched over $(\mathbf{Set}, \times, \top)$.

Pre-additive category is enriched over $(\mathbf{Ab}, \otimes, \mathbb{Z})$.

Linear category $\mathcal{L}$ is enriched over $(\mathcal{L}, \otimes, K)$.

Closed monoidal category is enriched over itself.

Preorder is enriched over the Boolean algebra $\mathbf{2} = \big(\{\bot < \top\}, \wedge, \top\big)$.

Directed metric spaces are enriched over $([0,\infty], +, 0)$.
Thus $d(x,y) \in [0,\infty]$ for $x, y \in C_0$,
and composition means $d(x.y) + d(y,z) \geq d(x,z)$.

If the symmetric monoidal $(\mathbf{B}, \otimes, I)$ is closed, can "impoverish" the enriched category C to $C_\circ$ with $C_\circ(x,y) = \mathbf{B}(I, C(x,y))$ for $x, y \in C_0$.

27. Copowers and free enriched categories

For a set S and an object b of a cocomplete category B, the colimit of the constant diagram $S \to \{b\}$ is the **copower** or **multiple**

$$S \cdot b = \sum_{s \in S} b, \text{ with insertions } \iota_s \colon b \to S \cdot b \text{ for } s \in S.$$

Example: For $X \in \mathbf{Set}$, have $\iota_s \colon X \to S \times X = S \cdot X; x \mapsto (s, x)$.

Example: For $V \in \mathcal{L}$, have $S \cdot V = \overbrace{V \oplus \ldots \oplus V}^{|S| \text{ copies}}$ for S finite.

For arbitrary S, have power $V^S = \mathbf{Set}(S, UV) \cong \mathcal{L}(FS, V) \in \mathcal{L}_0$,
and copower $S \cdot V = \big\{ f \colon S \to V \;\big|\; \infty > |\{s \in S \mid f(s) \neq 0\}| \big\}$,
a subobject of V^S, proper if S is infinite.

Category C, bicomplete closed symmetric monoidal base category $(\mathbf{B}, \otimes, I)$.

Free B-enriched category $\mathbf{B}C$ on C: left adjoint to impoverishment.

Object class $\mathbf{B}C_0 = C_0$.

For $x, y \in \mathbf{B}C_0 := C_0$, define $\mathbf{B}C(x,y) := C(x,y) \cdot I = \sum\limits_{f \in C(x,y)} I$.

For $x \in C_0$, define $j_x = \iota_{1_x} \colon I \to C(x,x) \cdot I$.

For $x, y, z \in C_0$, distributivity and unitality give $\mathbf{B}C(x,y) \otimes \mathbf{B}C(y,z) = \sum_{f \in C(x,y)} I \otimes \sum_{g \in C(y,z)} I = \sum_{f \in C(x,y)} \sum_{g \in C(y,z)} I \otimes I = \sum_{(f,g) \in C(x,y) \times C(y,z)} I$.

Then have composition

$$\begin{tikzcd} \sum_{C(x,y) \times C(y,z)} I \arrow[r, dashed, "\circ"] & \sum_{C(x,z)} I \\ I \arrow[u, "\iota_{(f,g)}"] \arrow[ur, "\iota_{g \circ f}"'] & \end{tikzcd}$$

Example: For a category C, and Boolean algebra $\mathbf{2}$,
the free $\mathbf{2}$-enriched $\mathbf{2}C$ is the preorder
obtained by "forgetting arrow labels" of C.

Group rings: For linear $(\mathcal{L}, \otimes, K)$,
and a one-object group $G = G_1$ on $G_0 = \{*\}$,
the *group ring* over K is the one-object free $\mathcal{L}$-category $\mathcal{L}G$,
with morphism set $G \cdot K$.

Standard Hopf algebra notation: $\eta_G = j_* = \iota_{1_*} \colon K \to G \cdot K$.

28. Pointed sets, kernels, and cokernels

Pointed set X_e has chosen element e, so $e\colon \top \to X$ with image $\{e\}$.

Category of pointed sets is the slice category $(\top \downarrow \mathbf{Set})$.
Internal hom $[X_e, Y_d] = [X, Y]_{\{d\}}$ with constant $d\colon X \to Y; x \mapsto d$.

Currying: $(\top \downarrow \mathbf{Set})(X_e \wedge Y_d, Z_c) \cong (\top \downarrow \mathbf{Set})\big(X_e, [Y_d, Z_c]\big)$ with the **smash product** $X_e \wedge Y_d = \Big(\big((X \smallsetminus \{e\}) \times (Y \smallsetminus \{d\})\big) \cup \{(e,d)\}\Big)_{\{(e,d)\}}$.

Suppose category C has a zero object 0, e.g., $(\top \to \top)$ in $(\top \downarrow \mathbf{Set})$.

Zero morphism in $C(x,y)$ is the composite $(x \xrightarrow{0} y) = (x \to 0 \to y)$.

Kernel: $\operatorname{Ker} f \xrightarrow{\ker f} x$ is the equalizer of $\begin{tikzcd} x \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & y \end{tikzcd}$.

Cokernel: $y \xrightarrow{\operatorname{coker} f} \operatorname{Coker} f$ is the coequalizer of $\begin{tikzcd} x \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & y \end{tikzcd}$.

Lemma: $\operatorname{Ker} f \xrightarrow{\ker f} x$ is mono; and dually $y \xrightarrow{\operatorname{coker} f} \operatorname{Coker} f$ is epi.

Proof. $\forall\ z \xrightarrow{r, r'} \operatorname{Ker} f$, $(\ker f) \circ r = (\ker f) \circ r' =: \kappa_x$

$$\Rightarrow \quad \begin{tikzcd} \operatorname{Ker} f \arrow[r, "\ker f"] & x \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & y \\ z \arrow[u, dashed, "{r,r'}"] \arrow[ur, "\kappa_x"'] & & \end{tikzcd} \qquad \Rightarrow\ r = r'. \quad \square$$

Example: For $f \in (\top \downarrow \mathbf{Set})(X_d, Y_e)$, have $\operatorname{Ker} f = \big(f^{-1}\{e\}\big)_d \hookrightarrow X_d$.

Object c of C with zero, (co)kernels, preorders $\big(\partial_1^{-1}\{c\}, \big|\,\big)$ and $\big(\partial_0^{-1}\{c\}, \big|^{\mathsf{op}}\big)$.

$$\begin{tikzcd} \operatorname{Ker} g \arrow[r, "\ker g"] & c \\ & d \arrow[u, "f"'] \arrow[ul] \end{tikzcd} \qquad \boxed{\text{adjunction}} \qquad \begin{tikzcd} c \arrow[r, "\operatorname{coker} f"] \arrow[d, "g"'] & \operatorname{Coker} f \arrow[dl] \\ b & \end{tikzcd}$$

$(\ker g) \,\big|\, f \Leftrightarrow \quad \Leftrightarrow g \circ f = 0 \Leftrightarrow \quad \Leftrightarrow (\operatorname{coker} f) \,\big|\, g$

So: $\boxed{\ker g = \ker \operatorname{coker} \ker g}$ and $\boxed{\operatorname{coker} f = \operatorname{coker} \ker \operatorname{coker} f}$

29. Factorization of morphisms, abelian categories

First Isom. Thm. for sets:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "e"'] & Y & \text{so } f = m \circ e\,, \\ & f(X) \arrow[u, "m"'] & m \text{ mono}, e \text{ epi} \end{tikzcd}$$

Abelian category [Freyd]: (A0) $\mathbf{A}$ has a zero object;
(A1) For $A, B \in \mathbf{A}_0$, product $A \times B$ and coproduct $A + B$ exist;
(A2) For $A, B \in \mathbf{A}_0$ and $f \in \mathbf{A}(A, B)$,

have kernel $\boxed{\operatorname{Ker} f \xrightarrow{\ker f} A}$ and cokernel $\boxed{B \xrightarrow{\operatorname{coker} f} \operatorname{Coker} f}$

(A3) Every monomorphism is a kernel; every epimorphism is a cokernel.

Image: $[\operatorname{Im} f \xrightarrow{\operatorname{im} f} B] := [\operatorname{Ker}(\operatorname{coker} f) \xrightarrow{\ker \operatorname{coker} f} B]\,,$
a monomorphism, smallest subobject of B that divides f.

Coimage: $[A \xrightarrow{\operatorname{coim} f} \operatorname{Coim} f] := [A \xrightarrow{\operatorname{coker} \ker f} \operatorname{Coker}(\ker f)]\,,$
an epimorphism, smallest quotient of A that divides f.

Factorization $[A \xrightarrow{f} B] = [A \xrightarrow{q} \operatorname{Im} f \xrightarrow{\operatorname{im} f} B]$ with q an epimorphism.
Indeed, $\operatorname{coker} q \neq 0$ would mean f divided by a smaller subobject of B.

Theorem: $f \colon A \to B$ mono and epi $\Rightarrow$ f is an isomorphism.

Proof. Have $B \xrightarrow{\operatorname{coker} f} 0$ since f epi, and $1_B = \ker \operatorname{coker} f$.
Since f is mono, $A \xrightarrow{f} B$ is also a kernel of $\operatorname{coker} f$ [and so $A \cong B$].
Thus f has a section: $f \circ s = 1_B$. Dually, it has a retraction: $r \circ f = 1_A$.
Since $r = r \circ 1_B = r \circ f \circ s = 1_A \circ s = s$, have f invertible. $\square$

Corollary: $\operatorname{Im} f \cong \operatorname{Coim} f$, and $\boxed{f = \operatorname{im} f \circ \operatorname{coim} f}$

See `https://math.stackexchange.com/questions/3268091/coimage-and-image-in-abelian-categories`

30. Enriching abelian categories

Abelian category $\mathbf{A}$ (Freyd's definition).

Matrices:
$$\begin{tikzcd} X & X \times Y \arrow[l] \arrow[r] & Y \\ & A \arrow[ul, bend left, "f"] \arrow[u, "{[f\ \ g]}"'] \arrow[ur, bend right, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[dr, bend right, "f"'] & X + Y \arrow[d, "{\begin{bmatrix} f \\ g \end{bmatrix}}"] & Y \arrow[l] \arrow[dl, bend left, "g"] \\ & A & \end{tikzcd}$$

Exact: $0 \to X \xrightarrow{\iota_X} X + Y \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} Y \to 0,\ \ 0 \to X \xrightarrow{[1\ \ 0]} X \times Y \xrightarrow{\pi_Y} Y \to 0$

Theorem: $0 \to X + Y \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} X \times Y \to 0$ exact,
so $X + Y \cong X \times Y =: X \oplus Y$, **biproduct**.

Diagonal: $\Delta\colon X \xrightarrow{[1\ \ 1]} X \oplus X$. **Summation:** $\Sigma\colon X \oplus X \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} X$.

For $f, g \in \mathbf{A}(A, B)$, define
$$\begin{tikzcd} A \arrow[r, "f +_L g"] \arrow[d, "\Delta"'] & B \\ A \oplus A \arrow[ur, "{\begin{bmatrix} f \\ g \end{bmatrix}}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "f +_R g"] \arrow[dr, "{[f\ \ g]}"'] & B \\ & B \oplus B \arrow[u, "\Sigma"'] \end{tikzcd}.$$

Proposition: $0 +_L f = f = f +_L 0, \quad 0 +_R f = f = f +_R 0.$

Proposition: $(f +_L g) +_R (h +_L k) = (f +_R h) +_L (g +_R k)$.

Proof. Both sides are $A \xrightarrow{\Delta} A \oplus A \xrightarrow{\begin{bmatrix} f & h \\ g & k \end{bmatrix}} B \oplus B \xrightarrow{\Sigma} B$. $\square$

Theorem: $+_L = +_R$, commutative and associative.

Proof. Setting $g = h = 0$, have $f +_R h = f +_L h =: f + h$.
Setting $h = 0$, have $(f + g) + k = f + (g + k)$.
Setting $f = k = 0$, have $g + h = h + g$. $\square$

Theorem: $\mathbf{A}(A, B)$ is an abelian group.

Proof. For $f\colon A \to B$, have $A \oplus A \xrightarrow{\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}} B \oplus B$ monic and epic, so an isomorphism with inverse $B \oplus B \xrightarrow{\begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix}} A \oplus A$, then $f + g = 0$. $\square$

31. Categories and 2-categories

Category **Cat** of (small) categories;
with $\mathbf{Cat}(D, C) = C^D =: [D, C]$ (functor category).

Cartesian closed monoidal $(\mathbf{Cat}, \times, \mathbf{1})$, **terminal category** $\mathbf{1} = \bullet\circlearrowleft$ as the monoidal unit, and Currying $\mathbf{Cat}(A \times B, C) \cong \mathbf{Cat}(A, [B, C])$.

(Strict) **2-category:** (1-)category $\mathbf{C}$ enriched over **Cat**.

- **0-cells:** objects $A, B, \ldots$ of $\mathbf{C}$.
- **1-cells:** morphisms of $\mathbf{C}$, i.e., objects in the categories $\mathbf{C}(A, B)$ or elements of the sets $\mathbf{Cat}\big(\mathbf{1}, \mathbf{C}(A, B)\big)$.
- **2-cells:** morphisms in the categories $\mathbf{C}(A, B)$.

Example: 2-category **Cat**. Categories as 0-cells. Functors as 1-cells.

Natural transformations $\tau \colon F \to G$ as 2-cells
$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & B \end{tikzcd}$$

Horizontal 2-cell composition: $\mathbf{Cat}(A, B) \times \mathbf{Cat}(B, C) \xrightarrow{\circ} \mathbf{Cat}(A, C)$;

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & B \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\tau'"] & C \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[r, bend left, "F'\circ F"] \arrow[r, bend right, "G'\circ G"'] \arrow[r, phantom, "\Downarrow\tau'\circ\tau"] & C \end{tikzcd}$$

with
$$\begin{tikzcd} F' \circ Fa \arrow[r, "\tau'_{Fa}"] \arrow[d, "F'\tau_a"'] \arrow[dr, "(\tau'\circ\tau)_a"] & G' \circ Fa \arrow[d, "G'\tau_a"] \\ F' \circ Ga \arrow[r, "\tau'_{Ga}"'] & G' \circ Ga \end{tikzcd}$$

Identity: $\mathbf{Cat}(\mathbf{1}, \mathbf{Cat}(C, C)) \ni j_C \colon \bullet\circlearrowleft \;\mapsto$
$$\begin{tikzcd} C \arrow[r, bend left, "1_C"] \arrow[r, bend right, "1_C"'] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & C \end{tikzcd}$$

Vertical 2-cell composition on $\mathbf{Cat}(A, B)$:
$$\begin{tikzcd} Fa \arrow[rr, "(\tau\bullet\sigma)_a"] \arrow[dr, "\sigma_a"'] & & Ha \\ & Ga \arrow[ur, "\tau_a"'] & \end{tikzcd}$$

Entropic or interchange law: $(\tau' \bullet \sigma') \circ (\tau \bullet \sigma) = (\tau' \circ \tau) \bullet (\sigma' \circ \sigma)$,
as bifunctorial horizontal composition respects vertical composition.

$(n+1)$**-category:** an n-category enriched over **Cat**.

32. The braid category

Monoid $(M, \nabla\colon M \times M \to M, \eta\colon \top \to M)$ in $\mathbf{C}$ with products.

Monoidal category $(\mathbf{C}, \otimes, \mathbf{I})$ is a monoid in $(\mathbf{Cat}, \times, \mathbf{1})$.

Sequence $(G_n \mid n \in \mathbb{N})$ **of groups**
with trivial G_0, and homomorphisms $\rho_{m,n}\colon G_m \times G_n \to G_{m+n}$

satisfying
$$\begin{tikzcd} G_l \times G_m \times G_n \arrow[r, "\rho_{l,m}\times 1"] \arrow[d, "1\times\rho_{m,n}"'] & G_{l+m} \times G_n \arrow[d, "\rho_{l+m,n}"] \\ G_l \times G_{m+n} \arrow[r, "\rho_{l,m+n}"'] & G_{l+m+n} \end{tikzcd}$$
. Category $\mathbf{G}$ with $\mathbf{G}_0 = \mathbb{N}$

and $\mathbf{G}(m,n) = G_m$ for $m = n$ and $\emptyset$ for $m \neq n$.

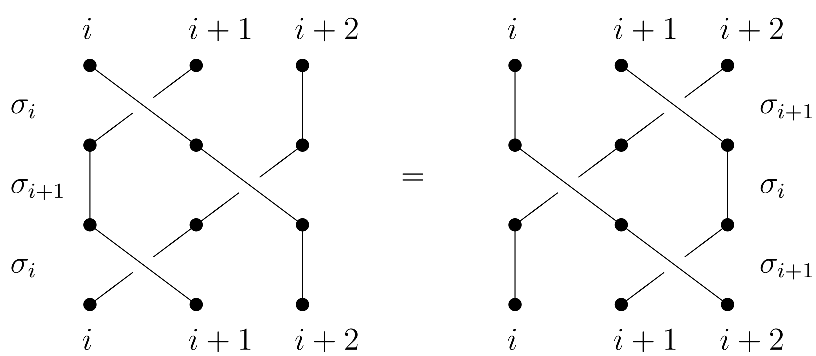
Monoidal category $(\mathbf{G}, +, 0)$ is $(\mathbb{N}, +, 0)$ at the object level,
with $+ = \bigcup \big(\rho_{m,n}\colon G_m \times G_n \to G_{m+n}\big)$ on morphisms.

Symmetric groups $S_n =$
$\langle \tau_1, \ldots, \tau_{n-1} \mid \tau_i^2 = 1,\ \tau_i\tau_j = \tau_j\tau_i \text{ for } |i-j| > 1,\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}\rangle$.

General linear groups $\mathsf{GL}(K)$ with $\mathsf{GL}_n(K)$.

Braid groups
$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle$
give the **braid category B**.

Braid relation:



First string on top layer, second in middle, third on bottom layer.

- "Third Reidemeister move" in knot theory terms.
- "Yang-Baxter equation" in physics.

33. Endofunctors, (co)algebras, monads

Endofunctor category $\mathbf{X}^{\mathbf{X}} = [\mathbf{X}, \mathbf{X}]$ of category $\mathbf{X}$.

Algebra (X, α) for an endofunctor $T \colon \mathbf{X} \to \mathbf{X}$ is given
by a **structure map** $\alpha \colon TX \to X$ in $\mathbf{X}(TX, X)$ for some $X \in \mathbf{X}_0$.

Algebra (homo)morphism $\theta \colon (X, \alpha) \to (Y, \beta)$
is given by commuting diagram

$$\begin{tikzcd} TX \arrow[r, "T\theta"] \arrow[d, "\alpha"'] & TY \arrow[d, "\beta"] \\ X \arrow[r, "\theta"'] & Y \end{tikzcd}$$

in $\mathbf{X}$.

Example: Finite subset endofunctor $\mathcal{P}_{\mathsf{fin}} \colon \mathbf{Set} \to \mathbf{Set}$,
bounded semilattice (commutative, idempotent monoid) $(X, \cdot, 1)$,
structure map $\alpha \colon \mathcal{P}_{\mathsf{fin}} X \to X; \{x_1, \dots, x_r\} \mapsto x_1 \cdot \ldots \cdot x_r \cdot 1$.

Coalgebra (X, α) for an endofunctor $T \colon \mathbf{X} \to \mathbf{X}$ is given
by a **structure map** $\alpha \colon X \to TX$ in $\mathbf{X}(X, TX)$ for some $X \in \mathbf{X}_0$.

Coalgebra (homo)morphism $\theta \colon (X, \alpha) \to (Y, \beta)$
is given by commuting diagram

$$\begin{tikzcd} TX \arrow[r, "T\theta"] & TY \\ X \arrow[r, "\theta"'] \arrow[u, "\alpha"] & Y \arrow[u, "\beta"'] \end{tikzcd}$$

in $\mathbf{X}$.

Example: Coalgebra with structure map $\alpha \colon X \to \mathcal{P}_{\mathsf{fin}} X$
represents a non-deterministic dynamical system.

Endofunctors form a monoidal category $\big(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}}\big)$.

Monad on $\mathbf{X}$ is a monoid $(T, \mu \colon T^2 \to T, \eta \colon 1_{\mathbf{X}} \to T)$ in $\big(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}}\big)$:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[dr, "\mu"] & T \arrow[l, "T\eta"'] \arrow[d, equal] \\ T \arrow[u, "\eta T"] \arrow[r, equal] & T \end{tikzcd}$$

Here, μT is:

$$\begin{tikzcd} \mathbf{X} \arrow[r, "T"] & \mathbf{X} \arrow[r, bend left, "T^2"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow \mu"] & \mathbf{X} \end{tikzcd}$$

or

$$\begin{tikzcd} \mathbf{X} \arrow[r, bend left, "T"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_T"] & \mathbf{X} \arrow[r, bend left, "T^2"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow \mu"] & \mathbf{X} \end{tikzcd} = \mu \circ \mathrm{id}_T,$$ **whiskering**.

Will get various kinds of algebras from monads.

34. Adjunctions yield monads

Adjunction $(F\colon \mathbf{X} \to \mathbf{A}, U\colon \mathbf{A} \to \mathbf{X}, \eta\colon 1_{\mathbf{X}} \to UF, \varepsilon\colon FU \to 1_{\mathbf{A}})$.

Triangular identities $1_F = \varepsilon F \bullet F\eta$ and $1_U = U\varepsilon \bullet \eta U$.

Trace in X: UF-coalgs. $\eta_X\colon X \to UFX$, $\mu := U\varepsilon F\colon UFUF \to UF$.

Proposition: Unital law

$$\begin{tikzcd} UFUF \arrow[dr, "U\varepsilon F"] & UF \arrow[l, "UF\eta"'] \arrow[d, equal] \\ UF \arrow[u, "\eta UF"] \arrow[r, equal] & UF \end{tikzcd}$$

Proof. Triangular

$$\begin{tikzcd} FUF \arrow[dr, "\varepsilon F"'] & F \arrow[l, "F\eta"'] \arrow[d, equal] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} UFU \arrow[dr, "U\varepsilon"] & \\ U \arrow[u, "\eta U"] \arrow[r, equal] & U \end{tikzcd}$$

□

Proposition: $\mu\colon UFUF \to UF$ associative.

Proof. Need

$$\begin{tikzcd} UFUFUF \arrow[r, "UFU\varepsilon F"] \arrow[d, "U\varepsilon FUF"'] & UFUF \arrow[d, "U\varepsilon F"] \\ UFUF \arrow[r, "U\varepsilon F"'] & UF \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} FUFU \arrow[r, "FU\varepsilon"] \arrow[d, "\varepsilon FU"'] & FU \arrow[d, "\varepsilon"] \\ FU \arrow[r, "\varepsilon"'] & 1_{\mathbf{A}} \end{tikzcd}.$$

Naturality:

$$\begin{tikzcd} FUA \arrow[d, "\varepsilon_A"'] \\ A \end{tikzcd} \qquad \begin{tikzcd} FUFUA \arrow[r, "\varepsilon_{FUA}"] \arrow[d, "FU\varepsilon_A"'] & FUA \arrow[d, "\varepsilon_A"] \\ FUA \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

. . . in $\mathbf{A}$ | . . . in $\mathbf{A}$ □

Theorem: Adjunction $(F, U, \eta, \varepsilon)$ gives monad $(UF, U\varepsilon F, \eta)$ on $\mathbf{X}$.

Example: Free monoid adjunction, for set or alphabet X.

Then UFX or X^* is the set of words or lists $\langle x_1 \dots x_r \rangle$ in the alphabet.

Coalgebra $\eta_X\colon X \to UFX$; letter $x \mapsto$ one-letter "word" or list $\langle x \rangle$.

Multiplication $\mu_X = U\varepsilon_{FX}\colon UFUFX \to UFX$;

list of words or lists $\mapsto$ concatenation of list:

$$\langle \langle x_{11} \dots x_{1r_1} \rangle \dots \langle x_{s1} \dots x_{sr_s} \rangle \rangle \mapsto \langle x_{11} \dots x_{1r_1} \dots x_{s1} \dots x_{sr_s} \rangle$$

— removes inner brackets.

35. Eilenberg-Moore algebras

Does a monad (T,μ,η) on $\mathbf{X}$ give adjunction
$$\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd} \text{ ?}$$

- Monoid $(M, m\colon M^2 \to M, e: \top \to M)$ is a monoid in $(\mathbf{Set}, \times, \top)$.

M**-sets** X have **action** $a\colon M \times X \to X$ with

associativity:
$$\begin{tikzcd} M^2 \times X \arrow[r, "1_M \times a"] \arrow[d, "m \times 1_X"'] & M \times X \arrow[d, "a"] \\ M \times X \arrow[r, "a"'] & X \end{tikzcd}$$
, unitality:
$$\begin{tikzcd} X \arrow[r, "e \times 1_X"] \arrow[dr, equal] & M \times X \arrow[d, "a"] \\ & X \end{tikzcd}$$

morphisms:
$$\begin{tikzcd} M \times X_1 \arrow[r, "1_M \times f"] \arrow[d, "a_1"'] & M \times X_2 \arrow[d, "a_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$
, category $\mathbf{Set}^M$ of M-sets.

Free $FX = (M^2 \times X \xrightarrow{m \times 1_X} M \times X)$, adjoint $U(M \times X \xrightarrow{a} X) = X$.

Unit $\eta_X\colon X \to M \times X; x \mapsto (e,x)$, counit $\varepsilon_a = a$.

Gives a model endofunctor $T\colon X \mapsto M \times X$ on $\mathbf{Set}$.

- Monad $(T, \mu\colon T^2 \to T, \eta\colon 1_{\mathbf{X}} \to T)$ is a monoid in $(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}})$.

Eilenberg-Moore algebra $a\colon TX \to X$ in $\mathbf{X}(TX, X)$ for $X \in \mathbf{X}_0$:

with associativity:
$$\begin{tikzcd} T^2X \arrow[r, "Ta"] \arrow[d, "\mu_X"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd}$$
and unitality:
$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd}$$
,

morphisms:
$$\begin{tikzcd} TX_1 \arrow[r, "Tf"] \arrow[d, "a_1"'] & TX_2 \arrow[d, "a_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$
, cat. $\mathbf{X}^T$ of Eilenberg-Moore algebras.

Forgetful $U^T\colon \mathbf{X}^T \to \mathbf{X}; (TX \xrightarrow{a} X) \mapsto X$.

Free $F^T\colon \mathbf{X} \to \mathbf{X}^T; X \mapsto (T^2X \xrightarrow{\mu_X} TX)$. Note $\boxed{U^T F^T X = TX}$

Unit $\eta^T_X\colon X \xrightarrow{\eta_X} TX$, counit $\varepsilon^T_a\colon (T^2X \xrightarrow{Ta} TX) \xrightarrow{a} (TX \xrightarrow{a} X)$

Eilenberg-Moore adjunction $\left(F^T, U^T, \eta^T, \varepsilon^T\right)$, yields monad (T,μ,η).

36. The Kleisli category of a monad

Alternative adjunction
$$\begin{tikzcd} \mathbf{A} \arrow[rr, bend right, "U"'] \arrow[rr, phantom, "\perp"] & & \mathbf{X} \arrow[ll, bend right, "F"'] \end{tikzcd}$$
from monad (T, μ, η) on $\mathbf{X}$.

Kleisli category $\mathbf{X}_T$ of monad (T, μ, η) on $\mathbf{X}$:
$(\mathbf{X}_T)_0 = \mathbf{X}_0$, $\mathbf{X}_T(X, Y) = \mathbf{X}(X, TY)$, identities $1_X = \eta_X \colon X \to TX$,
composition $(Y \xrightarrow{g} TZ) \circ (X \xrightarrow{f} TY) = (X \xrightarrow{f} TY \xrightarrow{Tg} T^2Z \xrightarrow{\mu_Z} TZ)$.

Adjunction $\mathbf{X}_T(F_TX, Y) = \mathbf{X}_T(X, Y) = \mathbf{X}(X, TY) = \mathbf{X}(X, U_TY)$
with left adjoint $F_T(X \xrightarrow{f} Y) = (X \xrightarrow{f} Y \xrightarrow{\eta_Y} TY)$,
right adjoint $U_T(Y \xrightarrow{g} TZ) = (TY \xrightarrow{Tg} T^2Z \xrightarrow{\mu_Z} TZ)$, unit $X \xrightarrow{\eta_X} TX$,
counit $\varepsilon_Y = TY \xrightarrow{1_{TY}} TY$, and monad $(U_TF_T, U_T\varepsilon F_T, \eta) = (T, \mu, \eta)$.

Power set monad $(\mathcal{P}, \mu, \eta)$ with $\eta_X \colon x \mapsto \{x\}$ and set family union
$\mu_X \colon \{\{x, \dots\}, \{y, \dots\}, \dots\} \mapsto \{x, \dots, y, \dots, \dots\}$ — like with lists.

Category Rel of relations $X \xrightarrow{R} Y = \{(x, y) \mid x\ R\ y\}$ on sets:
Have $\mathbf{Rel}_0 = \mathbf{Set}_0$,
with 1_X as the identity function or equality relation on a set X.
Relation product $(X \xrightarrow{R} Y \xrightarrow{S} Z) := \{(x, z) \mid \exists\ t \in Y\ .\ x\ R\ t\ S\ z\}$.

Theorem: Category **Rel** is the Kleisli category for $(\mathcal{P}, \mu, \eta)$.

Proof. $X \xrightarrow{R} Y$ gives $\mathbf{Set}_{\mathcal{P}}$-morphism $X \xrightarrow{R} \mathcal{P}X; x \mapsto \{y \mid x\ R\ y\}$.
Kleisli identity is $\eta_X \colon X \to \mathcal{P}X; x \mapsto \{x\} = \{x' \mid x = x'\}$,
and Kleisli composition gives the relation product:

$$X \xrightarrow[R]{} \mathcal{P}Y \xrightarrow[\mathcal{P}S]{} \mathcal{P}^2Z \xrightarrow[\mu_Z]{} \mathcal{P}Z$$

$$x \longmapsto \{t \mid x\ R\ t\} \longmapsto \{\{z \mid t\ S\ z\} \mid x\ R\ t\} \longmapsto \{z \mid \exists\ t\ .\ x\ R\ t\ S\ z\}$$

$$E \longmapsto \{\{z \mid e\ S\ z\} \mid e \in E\}$$

□

37. Compact closed categories

Compact closed category: Symmetric, monoidal $(\mathbf{C}, \otimes, \mathbf{1})$ with:

- Contravariant **duality functor** ${}^*\colon \mathbf{C} \to \mathbf{C}$;
- **Evaluation** natural transformation $\mathrm{ev}_X\colon X^* \otimes X \to \mathbf{1}$; and
- **Coevaluation** natural transformation $\mathrm{coev}_X\colon \mathbf{1} \to X \otimes X^*$,

yanking conditions

$$\Big(X \xrightarrow{\mathrm{coev}_X \otimes 1_X} X \otimes X^* \otimes X \xrightarrow{1_X \otimes \mathrm{ev}_X} X\Big) = 1_X$$

and

$$\Big(X^* \xrightarrow{1_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes 1_{X^*}} X^*\Big) = 1_{X^*}$$

Lemma: Internal hom $[X, Y] = X^* \otimes Y$.

Example: $(\mathcal{L}_{\mathsf{fin}}, \otimes, K)$ with duality $X^* = \mathcal{L}(X, K)$.
If X has basis $\{e_1, \ldots, e_n\}$,
and X^* has dual basis $\{\widehat{e}_1, \ldots, \widehat{e}_n\}$ with $\widehat{e}_i(e_j) = \delta_{ij}$,
then $\mathrm{coev}\colon K \to X \otimes X^*; 1 \mapsto \sum_{j=1}^n e_j \otimes \widehat{e}_j$.

Yanking: $e_i \mapsto \sum_{j=1}^n e_j \otimes \widehat{e}_j \otimes e_i \mapsto \sum_{j=1}^n e_j \otimes \widehat{e}_j(e_i) = \sum_{j=1}^n e_j \delta_{ji} = e_i$

and $\widehat{e}_i \mapsto \widehat{e}_i \otimes \sum_{j=1}^n e_j \otimes \widehat{e}_j \mapsto \sum_{j=1}^n \widehat{e}_i(e_j) \otimes \widehat{e}_j = \sum_{j=1}^n \delta_{ij} \widehat{e}_j = \widehat{e}_i$.

Lemma: For Y with basis $\{d_1, \ldots, d_m\}$,
morphism $e_i \mapsto d_j$ corresponds to tensor $\widehat{e}_i \otimes d_j$.

Example: Relation category **Rel**, biproduct is the disjoint union with

$$\begin{tikzcd} X \ni x \arrow[r, bend left, mapsto, "\iota_X"] & x \in X + Y \ni y \arrow[l, bend left, mapsto, "\pi_X"] \arrow[r, bend right, mapsto, "\pi_Y"'] & y \in Y \arrow[l, bend right, mapsto, "\iota_Y"'] \end{tikzcd}$$

Monoidal category $\big(\mathbf{Rel}, \times, \{0\}\big)$, compact closed with $X^* = X$.

Evaluation $\{((x, x), 0) \mid x \in X\}$, coevaluation $\{(0, (x, x)) \mid x \in X\}$.

First yanking condition:
relation product of $\{(x, (x', x', x)) \mid x, x' \in X\}$
with $\{((x', x, x), x') \mid x, x' \in X\}$ is $\{(x, x) \mid x \in X\}$.

Second is similar.

38. Monoidal functors

Monoidal categories $(\mathbf{C}, \otimes, \mathbf{1})$, $(\mathbf{C}', \otimes, \mathbf{1}')$.

Monoidal functor $F\colon (\mathbf{C}, \otimes, \mathbf{1}) \to (\mathbf{C}', \otimes, \mathbf{1})$
with natural transformations $\mu_{X,Y}\colon F(X) \otimes F(Y) \to F(X \otimes Y)$
and $\mathbf{C}'$-morphism $\epsilon\colon \mathbf{1}' \to F(\mathbf{1})$ such that:

$$\begin{tikzcd}
F(X) \otimes \big(F(Y) \otimes F(Z)\big) \arrow[r, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "1_{F(X)}\otimes\mu_{Y,Z}"'] & \big(F(X) \otimes F(Y)\big) \otimes F(Z) \arrow[d, "\mu_{X,Y}\otimes 1_{F(Z)}"] \\
F(X) \otimes F(Y \otimes Z) \arrow[d, "\mu_{X,Y\otimes Z}"'] & F(X \otimes Y) \otimes F(Z) \arrow[d, "\mu_{X\otimes Y,Z}"] \\
F\big(X \otimes (Y \otimes Z)\big) \arrow[r, "F(\alpha_{X,Y,Z})"'] & F\big((X \otimes Y) \otimes Z\big),
\end{tikzcd}$$

— **associativity**, and **unitality**:

$$\begin{tikzcd}
F(X) \otimes \mathbf{1}' \arrow[r, "\rho'_{F(X)}"] \arrow[d, "1_{F(X)}\otimes\epsilon"'] & F(X) \\
F(X) \otimes F(\mathbf{1}) \arrow[r, "\mu_{X,\mathbf{1}}"'] & F(X \otimes \mathbf{1}) \arrow[u, "F\rho_X"']
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
F(X) & \mathbf{1}' \otimes F(X) \arrow[l, "\lambda'_{F(X)}"'] \arrow[d, "\epsilon\otimes 1_{F(X)}"] \\
F(\mathbf{1} \otimes X) \arrow[u, "F\lambda_X"] & F(\mathbf{1}) \otimes F(X). \arrow[l, "\mu_{\mathbf{1},X}"]
\end{tikzcd}$$

Example: Underlying set functor $U\colon (\mathcal{L}, \otimes, K) \to (\mathbf{Set}, \times, \{1\})$.
Here $\epsilon\colon \{1\} \to K; 1 \mapsto 1$, and $\mu_{X,Y}\colon UX \times UY \to U(X \otimes Y)$ is the usual quotient by relations $(k_1x_1 + k_2x_2, y) \overset{!}{=} k_1(x_1, y) + k_2(x_2, y)$, etc.

$\left\{\begin{matrix}\textbf{Strong}\\ \textbf{Strict}\end{matrix}\right\}$ **monoidal functor:** ϵ and the $\mu_{X,Y}$ are $\left\{\begin{matrix}\text{isomorphisms.}\\ \text{identities.}\end{matrix}\right\}$

Example: Free vector space functor $F\colon (\mathbf{Set}, \times, \{1\}) \to (\mathcal{L}, \otimes, K)$.
Here $F(X \times Y) = F(X) \otimes F(Y)$ and $F\{1\} = K$, so strong, strict.

Braided monoidal functor:
$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\sigma'_{F(X),F(Y)}"] \arrow[d, "\mu_{X,Y}"'] & F(Y) \otimes F(X) \arrow[d, "\mu_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F\sigma_{X,Y}"'] & F(Y \otimes X),
\end{tikzcd}$$

Symmetric monoidal functor if $(\mathbf{C}, \otimes, \mathbf{1})$, $(\mathbf{C}', \otimes, \mathbf{1}')$ symmetric.

Example: ${}^*\colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ in a compact closed category $(\mathbf{C}, \otimes, \mathbf{1})$.

39. Dagger categories

Dagger category $\mathbf{C}$ has a contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$, with: $X^\dagger = X$ for $X \in \mathbf{C}_0$, **adjoint** $f^\dagger\colon Y \to X$ of $f \in \mathbf{C}(X,Y)$, $f^{\dagger\dagger} = f$.

Example: One-object linear categories $\mathbb{C}, \mathbb{H}$ with $x^\dagger = \overline{x}$.

Morphism f in dagger category $\mathbf{C}$ is:

- **Hermitian** or **self-adjoint** if $f^\dagger = f$;
- **unitary** if invertible and $f^\dagger = f^{-1}$.

Dagger monoidal category: $\dagger$ is a strict monoidal functor.

Dagger compact closed category: $\forall\ X \in \mathbf{C}_0\,,\ \mathrm{coev}_X = (\mathrm{ev}_X)^\dagger$.

Lemma: $\forall\ f \in \mathbf{C}_1\,,\ f^{\dagger *} = f^{*\dagger}$. (Necessary, not sufficient, for DCCC.)

Biproduct dagger compact closed category: $\forall\ X \in \mathbf{C}_0\,,\ \pi_X^\dagger = \iota_X$.

Example: Category **FDHilb** of finite-dimensional Hilbert spaces, with $\forall\ x \in X\,,\ \forall\ y \in Y\,,\ \langle f(x) \mid y \rangle = \langle x \mid f^\dagger(y) \rangle$ for $f\colon X \to Y$.

Example: **Rel** with $R^* = R^\dagger$ as the converse relation.

Information theory: A **bit** in a BDCCC is $\mathbf{2} := \mathbf{1} \oplus \mathbf{1}$.

Examples: $\{0,1\}$ in **Rel**, or **qubit** $\mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2$ in **FDHilb**.

Extract information from $[\mathbf{1}, \mathbf{1}]$,
e.g., **false** $= \emptyset$ and **true** $= 1_{\mathbf{1}}$ in **Rel**, or scalar $1 \mapsto c$ in **FDHilb**.

Trace of $f \in [X,X] = X^* \otimes X$ is

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{\tau} X^* \otimes X \xrightarrow{1_{X^*} \otimes f} X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1}.$$

Example in **FDHilb**:

$$\textstyle\sum_j e_j \otimes \widehat{e}_j \mapsto \sum_j \widehat{e}_j \otimes e_j \mapsto \sum_j \widehat{e}_j \otimes f(e_j) \mapsto \sum_k \sum_j \widehat{e}_j \otimes f_{jk} e_k \mapsto \sum_j f_{jj}$$

Positive endomorphism $f\colon X \to X$ if $\exists\ g\colon X \to Y\ .\ f = g^\dagger \circ g$.

Examples: In **Rel**, $x\ R\ y \Rightarrow y\ R\ x$ and $x\ R\ x$.
In **FDHilb**, $\forall\ x \in X\,,\ \langle f(x) \mid x \rangle \geq 0$.

Complete positivity of $f\colon [X,X] \to [Y,Y]$ or $f\colon X^* \otimes X \to Y^* \otimes Y$:

$\forall\ Z \in \mathbf{C}_0\,,\ \forall$ positive $g\colon \mathbf{1} \to Z^* \otimes X^* \otimes X \otimes Z\,,$

$\mathbf{1} \xrightarrow{g} Z^* \otimes X^* \otimes X \otimes Z \xrightarrow{1_{Z^*} \otimes f \otimes 1_Z} Z^* \otimes Y^* \otimes Y \otimes Z$ is positive.

40. Subobjects and subobject classifiers

For object X of category $\mathbf{C}$, define $\mathbf{Presub}(X)$ as the full subcategory of $(\mathbf{C} \downarrow X)$ whose defining morphisms $s \colon S \to X$ are monomorphisms.

Lemma: $\mathbf{Presub}(X)$ is a preorder:

$$\begin{tikzcd} & X & \\ S \arrow[ur, "s"] \arrow[rr, bend left, "j_1"] \arrow[rr, bend right, "j_2"'] & & T \arrow[ul, "t"'] \end{tikzcd} \qquad \Rightarrow j_1 = j_2.$$

Skeleton poset $\mathbf{Sub_C}(X)$ or just $\mathbf{Sub}(X)$ consists of [2nd-order concept]

subobjects of X: Equivalence classes of monomorphisms $s \colon S \to X$.

Note

$$\begin{tikzcd} & X & \\ S \arrow[ur, "s"] \arrow[r, "j"] \arrow[rr, bend right, "1_S"'] & S' \arrow[u, "s'"] \arrow[r, "j'"] & S \arrow[ul, "s"'] \end{tikzcd} \qquad \Rightarrow j' \circ j = 1_S, \text{ similarly } j \circ j' = 1_{S'}\,.$$

Well-powered category $\mathbf{C}$: Each object has a *set* of subobjects.

Now assume $\mathbf{C}$ has finite limits,
so terminal $\top$ giving **elements** $\top \to X$ of objects X.

Subobject classifier $\top \xrightarrow{\mathbf{true}} \Omega$ in $\mathbf{C}$ [makes subobjects first-order!]:

$\forall\ X \in \mathbf{C}_0\,,\ \forall\ (S \overset{s}{\to} X) \in \mathbf{Sub_C}(X)\,, \exists_!\ \chi\,.$

$$\begin{tikzcd} S \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\text{p-b}"] & \top \arrow[d, "\mathbf{true}"] \\ X \arrow[r, dashed, "\chi"'] & \Omega \end{tikzcd}$$

Example: $\top \xrightarrow{\mathbf{true}} \{\mathbf{false}, \mathbf{true}\}$ in $\mathbf{Set}$ with $S = \chi^{-1}\{\mathbf{true}\} \subseteq X$.

Also works in category $\mathbf{Set}^G$ of G-sets, for any group G,
with trivial action of G on Ω.

Proposition: If $\mathbf{C}$ is well-powered and locally small,
$\mathbf{Sub_C} \cong \mathbf{C}(\_\_\,, \Omega) = \exists \Omega \in \widehat{\mathbf{C}}_0$.

Proof.

$$\begin{tikzcd} S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p-b}"] & S \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p-b}"] & \top \arrow[d, "\mathbf{true}"] \\ Y \arrow[r, "f"'] & X \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

$\square$

Remark: S' is the inverse image of S under $f \colon Y \to X$.

41. Dinatural transformations and power objects

Given graph maps $F, G\colon D^{\mathsf{op}} \times D \to C$ for graph D and category C, a **dinatural transformation** $\delta\colon F \Rightarrow G$ is a "vector" $(\delta_x \mid x \in D_0)$ of **components** $\delta_x\colon F(x,x) \to G(x,x)$ in C_1 such that,

for all $f\colon x \to y$ in D_1, the hexagon of the **dinaturality diagram**

$$
\begin{tikzcd}
x \\
y \arrow[u, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[d, "f"'] \\
y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(x,x) \arrow[r, "\delta_x"] & G(x,x) \arrow[dr, "G(1_x{,}f)"] & \\
F(y,x) \arrow[ur, "F(f{,}1_x)"] \arrow[dr, "F(1_y{,}f)"'] & & & G(x,y) \\
 & F(y,y) \arrow[r, "\delta_y"'] & G(y,y) \arrow[ur, "G(f{,}1_y)"'] &
\end{tikzcd}
$$

$\boxed{\text{in } D^{\mathsf{op}}}$ $\boxed{\text{in } D}$ $\boxed{\text{in } C}$

commutes in the category C.

Example: $\delta_x^n\colon C(x,x) \to C(x,x); k \mapsto k^n$ — **Church numeral** n.

$$
\begin{tikzcd}
 & h \circ f \mapsto (h \circ f)^n \arrow[dr, mapsto] & \\
h \arrow[ur, mapsto] \arrow[dr, mapsto] & & f \circ (h \circ f)^n = (f \circ h)^n \circ f \\
 & f \circ h \mapsto (f \circ h)^n \arrow[ur, mapsto] &
\end{tikzcd}
$$

Example: For $\mathbf{C}$ with (finite limits and) a subobject classifier Ω,

$$\mathbf{Sub_C}(X \times Y) \cong \mathbf{C}(X \times Y, \Omega) \cong \mathbf{C}(X, \mathcal{P}Y)$$

defines the **power object** $\mathcal{P}Y$ of an object Y.

In $\mathbf{C}(\mathcal{P}Y \times Y, \Omega) \cong \mathbf{C}(\mathcal{P}Y, \mathcal{P}Y)$, suppose $\ni_Y \mapsto 1_{\mathcal{P}Y}$.

In **Set**, have $\ni_Y$ as characteristic function of $\{(S,y) \in \mathcal{P}Y \times Y \mid S \ni y\}$.

For $f\colon X \to Y$, define $\mathcal{P}f\colon \mathcal{P}Y \to \mathcal{P}X$ as the unique morphism making

$$
\begin{tikzcd}
 & \mathcal{P}X \times X \arrow[r, "\ni_X"] & \Omega \arrow[dr, equal] & \\
\mathcal{P}Y \times X \arrow[ur, "\mathcal{P}f \times 1_X"] \arrow[dr, "\mathcal{P}1_Y \times f"'] & & & \Omega \\
 & \mathcal{P}Y \times Y \arrow[r, "\ni_Y"'] & \Omega \arrow[ur, equal] &
\end{tikzcd}
$$

commute.

So dinatural $\boxed{\ni\colon \mathcal{P} \times 1_{\mathbf{C}} \Rightarrow \Delta\Omega}$ for $\mathcal{P}\colon \mathbf{C}^{\mathsf{op}} \to \mathbf{C}$.

42. Elementary and Grothendieck topoi

Elementary topos: Category $\mathbf{C}$ with finite limits and a power object.

Properties: Finite colimits, subobject classifier, Cartesian closed.

Examples: Presheaf categories $\widehat{D} = \mathbf{Set}^{D^{\mathsf{op}}}$ for small D.

Grothendieck topos: $\mathcal{E}$ with reflective full $K\colon \mathcal{E} \hookrightarrow \widehat{D}$, for some D,
where the left adjoint $L\colon \widehat{D} \to \mathcal{E}$ preserves finite limits.

Example: Sheaves — the presheaves $F \in \mathcal{E}_0 \subseteq \widehat{D}$,
where $D = (\mathcal{O}, \subseteq)$ for a topological space $(X, \mathcal{O})$, satisfying:

For each open cover $U = \bigcup_{i \in I} U_i$ of each $U \in \mathcal{O}$, require equalizer

$$\begin{tikzcd}
 & FU_i \arrow[r, "F(U_i \cap U_j \subseteq U_i)"] & F(U_i \cap U_j) \\
FU \arrow[r, dashed, "e"] & \prod_{k \in I} FU_k \arrow[u, "\pi_i"] \arrow[d, "\pi_j"'] \arrow[r, dashed, shift left, "p"] \arrow[r, dashed, shift right, "q"'] & \prod_{k,l \in I} F(U_k \cap U_l)\,. \arrow[u, "\pi_{i,j}"'] \arrow[d, "\pi_{i,j}"] \\
 & FU_j \arrow[r, "F(U_i \cap U_j \subseteq U_j)"'] & F(U_i \cap U_j)
\end{tikzcd}$$

Typically, $F(V \subseteq U)\colon FU \to FV; f \mapsto f|_V$ (restriction of functions).

Equalizer condition means match of FU_i and FU_j on $F(U_i \cap U_j)$.

Elementary definition: A **topos** is a category $\mathbf{C}$ with the following.

(a) A terminal object $\top$.

(b) Pullback of each $X \to B \leftarrow Y$.

(c) Monic $\top \xrightarrow{\mathbf{true}} \Omega$, and $\forall$ monic $S \xrightarrow{s} X$, $\exists_!\ \chi$.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\text{p-b}"] & \top \arrow[d, "\mathbf{true}"] \\
X \arrow[r, dashed, "\chi"'] & \Omega
\end{tikzcd}$$

(d) $\forall\ Y,\ \exists\ (\ni_Y\colon \mathcal{P}Y \times Y \to \Omega)\,.\ \forall\ (\rho\colon X \times Y \to \Omega)\,,$

$\exists$ unique
$$\begin{tikzcd}
X \arrow[d, "r"'] \\
\mathcal{P}Y
\end{tikzcd}$$
such that
$$\begin{tikzcd}
X \times Y \arrow[r, "\rho"] \arrow[d, "r \times 1_Y"'] & \Omega \arrow[d, equal] \\
\mathcal{P}Y \times Y \arrow[r, "\ni_Y"'] & \Omega
\end{tikzcd}$$

$\boxed{\text{Lawvere: First-order theory of topoi as a foundation for mathematics.}}$