42. Elementary and Grothendieck topoi

Elementary topos: Category $\mathbf{C}$ with finite limits and a power object.

Properties: Finite colimits, subobject classifier, Cartesian closed.

Examples: Presheaf categories $\widehat{D} = \mathbf{Set}^{D^{\mathsf{op}}}$ for small D.

Grothendieck topos: $\mathcal{E}$ with reflective full $K\colon \mathcal{E} \hookrightarrow \widehat{D}$, for some D,
where the left adjoint $L\colon \widehat{D} \to \mathcal{E}$ preserves finite limits.

Example: Sheaves — the presheaves $F \in \mathcal{E}_0 \subseteq \widehat{D}$,
where $D = (\mathcal{O}, \subseteq)$ for a topological space $(X, \mathcal{O})$, satisfying:

For each open cover $U = \bigcup_{i \in I} U_i$ of each $U \in \mathcal{O}$, require equalizer

$$\begin{array}{ccccc}
 & & FU_i & \xrightarrow{F(U_i \cap U_j \subseteq U_i)} & F(U_i \cap U_j) \\
 & & \uparrow{\scriptstyle \pi_i} & & \uparrow{\scriptstyle \pi_{i,j}} \\
FU & \overset{e}{\dashrightarrow} & \prod_{k \in I} FU_k & \underset{q}{\overset{p}{\dashrightarrow\!\!\!\dashrightarrow}} & \prod_{k,l \in I} F(U_k \cap U_l)\,. \\
 & & \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle \pi_{i,j}} \\
 & & FU_j & \xrightarrow[F(U_i \cap U_j \subseteq U_j)]{} & F(U_i \cap U_j)
\end{array}$$

Typically, $F(V \subseteq U)\colon FU \to FV; f \mapsto f|_V$ (restriction of functions).

Equalizer condition means match of FU_i and FU_j on $F(U_i \cap U_j)$.

Elementary definition: A **topos** is a category $\mathbf{C}$ with the following.

(a) A terminal object $\top$.

(b) Pullback of each $X \to B \leftarrow Y$.

(c) Monic $\top \xrightarrow{\mathbf{true}} \Omega$, and $\forall$ monic $S \xrightarrow{s} X$, $\exists_!\ \chi$.

$$\begin{array}{ccc}
S & \longrightarrow & \top \\
{\scriptstyle s}\downarrow & \text{p-b} & \downarrow{\scriptstyle \mathbf{true}} \\
X & \underset{\chi}{\dashrightarrow} & \Omega
\end{array}$$

(d) $\forall\ Y$, $\exists\ (\ni_Y \colon \mathcal{P}Y \times Y \to \Omega)$. $\forall\ (\rho\colon X \times Y \to \Omega)$,

$\exists$ unique $\begin{array}{c} X \\ {\scriptstyle r}\downarrow \\ \mathcal{P}Y \end{array}$ such that $\begin{array}{ccc} X \times Y & \xrightarrow{\rho} & \Omega \\ {\scriptstyle r \times 1_Y}\downarrow & & \| \\ \mathcal{P}Y \times Y & \xrightarrow[\ni_Y]{} & \Omega \end{array}$

Lawvere: First-order theory of topoi as a foundation for mathematics.