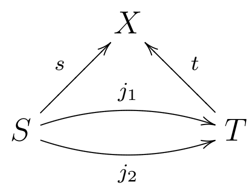
40. Subobjects and subobject classifiers

For object X of category $\mathbf{C}$, define $\mathbf{Presub}(X)$ as the full subcategory of $(\mathbf{C} \downarrow X)$ whose defining morphisms $s\colon S \to X$ are monomorphisms.

Lemma: $\mathbf{Presub}(X)$ is a preorder:

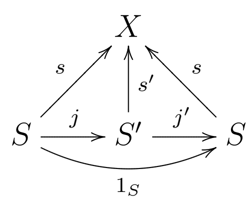


$\Rightarrow j_1 = j_2.$

Skeleton poset $\mathbf{Sub_C}(X)$ or just $\mathbf{Sub}(X)$ consists of [2nd-order concept]

subobjects of X: Equivalence classes of monomorphisms $s\colon S \to X$.

Note



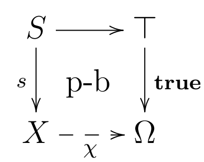
$\Rightarrow j' \circ j = 1_S,$ similarly $j \circ j' = 1_{S'}$.

Well-powered category $\mathbf{C}$: Each object has a *set* of subobjects.

Now assume $\mathbf{C}$ has finite limits,
so terminal $\top$ giving **elements** $\top \to X$ of objects X.

Subobject classifier $\top \xrightarrow{\mathbf{true}} \Omega$ in $\mathbf{C}$ [makes subobjects first-order!]:

$\forall\ X \in \mathbf{C}_0\,,\ \forall\ (S \overset{s}{\to} X) \in \mathbf{Sub_C}(X)\,, \exists_!\ \chi\,.$

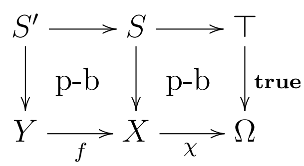


Example: $\top \xrightarrow{\mathbf{true}} \{\mathbf{false}, \mathbf{true}\}$ in **Set** with $S = \chi^{-1}\{\mathbf{true}\} \subseteq X$.

Also works in category $\mathbf{Set}^G$ of G-sets, for any group G,
with trivial action of G on Ω.

Proposition: If $\mathbf{C}$ is well-powered and locally small,
$\mathbf{Sub_C} \cong \mathbf{C}(\_\_, \Omega) = \exists\Omega \in \widehat{\mathbf{C}}_0.$

Proof.



□

Remark: S' is the inverse image of S under $f\colon Y \to X$.