39. Dagger categories

Dagger category C has a contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$, with: $X^\dagger = X$ for $X \in \mathbf{C}_0$, **adjoint** $f^\dagger\colon Y \to X$ of $f \in \mathbf{C}(X,Y)$, $f^{\dagger\dagger} = f$.

Example: One-object linear categories $\mathbb{C}, \mathbb{H}$ with $x^\dagger = \overline{x}$.

Morphism f in dagger category $\mathbf{C}$ is:

- **Hermitian** or **self-adjoint** if $f^\dagger = f$;
- **unitary** if invertible and $f^\dagger = f^{-1}$.

Dagger monoidal category: $\dagger$ is a strict monoidal functor.

Dagger compact closed category: $\forall\, X \in \mathbf{C}_0\,,\ \mathrm{coev}_X = (\mathrm{ev}_X)^\dagger$.

Lemma: $\forall\, f \in \mathbf{C}_1\,,\ f^{\dagger *} = f^{*\dagger}$. (Necessary, not sufficient, for DCCC.)

Biproduct dagger compact closed category: $\forall\, X \in \mathbf{C}_0\,,\ \pi_X^\dagger = \iota_X$.

Example: Category **FDHilb** of finite-dimensional Hilbert spaces, with $\forall\, x \in X\,,\ \forall\, y \in Y\,,\ \langle f(x) \mid y\rangle = \langle x \mid f^\dagger(y)\rangle$ for $f\colon X \to Y$.

Example: Rel with $R^* = R^\dagger$ as the converse relation.

Information theory: A **bit** in a BDCCC is $\mathbf{2} := \mathbf{1} \oplus \mathbf{1}$.

Examples: $\{0,1\}$ in **Rel**, or **qubit** $\mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2$ in **FDHilb**.

Extract information from $[\mathbf{1},\mathbf{1}]$,
e.g., **false** $= \emptyset$ and **true** $= 1_{\mathbf{1}}$ in **Rel**, or scalar $1 \mapsto c$ in **FDHilb**.

Trace of $f \in [X,X] = X^* \otimes X$ is

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{\tau} X^* \otimes X \xrightarrow{1_{X^*}\otimes f} X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1}.$$

Example in **FDHilb**:

$$\textstyle\sum_j e_j \otimes \widehat{e}_j \mapsto \sum_j \widehat{e}_j \otimes e_j \mapsto \sum_j \widehat{e}_j \otimes f(e_j) \mapsto \sum_k \sum_j \widehat{e}_j \otimes f_{jk} e_k \mapsto \sum_j f_{jj}$$

Positive endomorphism $f\colon X \to X$ if $\exists\, g\colon X \to Y\,.\ f = g^\dagger \circ g$.

Examples: In **Rel**, $x\ R\ y \Rightarrow y\ R\ x$ and $x\ R\ x$.
In **FDHilb**, $\forall\, x \in X\,,\ \langle f(x) \mid x\rangle \geq 0$.

Complete positivity of $f\colon [X,X] \to [Y,Y]$ or $f\colon X^* \otimes X \to Y^* \otimes Y$:

$\forall\, Z \in \mathbf{C}_0\,,\ \forall$ positive $g\colon \mathbf{1} \to Z^* \otimes X^* \otimes X \otimes Z\,,$

$\mathbf{1} \xrightarrow{g} Z^* \otimes X^* \otimes X \otimes Z \xrightarrow{1_{Z^*}\otimes f \otimes 1_Z} Z^* \otimes Y^* \otimes Y \otimes Z$ is positive.