38. Monoidal functors

Monoidal categories $(\mathbf{C}, \otimes, \mathbf{1})$, $(\mathbf{C}', \otimes, \mathbf{1}')$.

Monoidal functor $F\colon (\mathbf{C}, \otimes, \mathbf{1}) \to (\mathbf{C}', \otimes, \mathbf{1}')$
with natural transformations $\mu_{X,Y}\colon F(X) \otimes F(Y) \to F(X \otimes Y)$
and $\mathbf{C}'$-morphism $\epsilon\colon \mathbf{1}' \to F(\mathbf{1})$ such that:

$$
\begin{array}{ccc}
F(X) \otimes \big(F(Y) \otimes F(Z)\big) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & \big(F(X) \otimes F(Y)\big) \otimes F(Z) \\
\Big\downarrow {\scriptstyle 1_{F(X)} \otimes \mu_{Y,Z}} & & \Big\downarrow {\scriptstyle \mu_{X,Y} \otimes 1_{F(Z)}} \\
F(X) \otimes F(Y \otimes Z) & & F(X \otimes Y) \otimes F(Z) \\
\Big\downarrow {\scriptstyle \mu_{X,Y \otimes Z}} & & \Big\downarrow {\scriptstyle \mu_{X \otimes Y, Z}} \\
F\big(X \otimes (Y \otimes Z)\big) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F\big((X \otimes Y) \otimes Z\big),
\end{array}
$$

— **associativity**, and **unitality**:

$$
\begin{array}{ccc}
F(X) \otimes \mathbf{1}' & \xrightarrow{\rho'_{F(X)}} & F(X) \\
\Big\downarrow {\scriptstyle 1_{F(X)} \otimes \epsilon} & & \Big\uparrow {\scriptstyle F\rho_X} \\
F(X) \otimes F(\mathbf{1}) & \xrightarrow[\mu_{X,\mathbf{1}}]{} & F(X \otimes \mathbf{1})
\end{array}
\quad , \quad
\begin{array}{ccc}
F(X) & \xleftarrow{\lambda'_{F(X)}} & \mathbf{1}' \otimes F(X) \\
\Big\uparrow {\scriptstyle F\lambda_X} & & \Big\downarrow {\scriptstyle \epsilon \otimes 1_{F(X)}} \\
F(\mathbf{1} \otimes X) & \xleftarrow[\mu_{\mathbf{1},X}]{} & F(\mathbf{1}) \otimes F(X).
\end{array}
$$

Example: Underlying set functor $U\colon (\mathcal{L}, \otimes, K) \to (\mathbf{Set}, \times, \{1\})$.

Here $\epsilon\colon \{1\} \to K; 1 \mapsto 1$, and $\mu_{X,Y}\colon UX \times UY \to U(X \otimes Y)$ is the usual quotient by relations $(k_1x_1 + k_2x_2, y) \overset{!}{=} k_1(x_1, y) + k_2(x_2, y)$, etc.

$\left\{\begin{matrix}\textbf{Strong} \\ \textbf{Strict}\end{matrix}\right\}$ **monoidal functor:** ϵ and the $\mu_{X,Y}$ are $\left\{\begin{matrix}\text{isomorphisms.} \\ \text{identities.}\end{matrix}\right\}$

Example: Free vector space functor $F\colon (\mathbf{Set}, \times, \{1\}) \to (\mathcal{L}, \otimes, K)$.

Here $F(X \times Y) = F(X) \otimes F(Y)$ and $F\{1\} = K$, so strong, strict.

Braided monoidal functor:

$$
\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\sigma'_{F(X),F(Y)}} & F(Y) \otimes F(X) \\
\Big\downarrow {\scriptstyle \mu_{X,Y}} & & \Big\downarrow {\scriptstyle \mu_{Y,X}} \\
F(X \otimes Y) & \xrightarrow[F\sigma_{X,Y}]{} & F(Y \otimes X),
\end{array}
$$

Symmetric monoidal functor if $(\mathbf{C}, \otimes, \mathbf{1})$, $(\mathbf{C}', \otimes, \mathbf{1}')$ symmetric.

Example: ${}^*\colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ in a compact closed category $(\mathbf{C}, \otimes, \mathbf{1})$.