37. Compact closed categories

Compact closed category: Symmetric, monoidal $(\mathbf{C}, \otimes, \mathbf{1})$ with:

- Contravariant **duality functor** $^*\colon \mathbf{C} \to \mathbf{C}$;
- **Evaluation** natural transformation $\mathrm{ev}_X\colon X^* \otimes X \to \mathbf{1}$; and
- **Coevaluation** natural transformation $\mathrm{coev}_X\colon \mathbf{1} \to X \otimes X^*$,

yanking conditions

$$\Big(X \xrightarrow{\mathrm{coev}_X \otimes 1_X} X \otimes X^* \otimes X \xrightarrow{1_X \otimes \mathrm{ev}_X} X\Big) = 1_X$$

and

$$\Big(X^* \xrightarrow{1_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes 1_{X^*}} X^*\Big) = 1_{X^*}$$

Lemma: Internal hom $[X, Y] = X^* \otimes Y$.

Example: $(\mathcal{L}_{\mathsf{fin}}, \otimes, K)$ with duality $X^* = \mathcal{L}(X, K)$.
If X has basis $\{e_1, \ldots, e_n\}$,
and X^* has dual basis $\{\widehat{e}_1, \ldots, \widehat{e}_n\}$ with $\widehat{e}_i(e_j) = \delta_{ij}$,
then $\mathrm{coev}\colon K \to X \otimes X^*; 1 \mapsto \sum_{j=1}^n e_j \otimes \widehat{e}_j$.

Yanking: $e_i \mapsto \sum_{j=1}^n e_j \otimes \widehat{e}_j \otimes e_i \mapsto \sum_{j=1}^n e_j \otimes \widehat{e}_j(e_i) = \sum_{j=1}^n e_j \delta_{ji} = e_i$

and $\widehat{e}_i \mapsto \widehat{e}_i \otimes \sum_{j=1}^n e_j \otimes \widehat{e}_j \mapsto \sum_{j=1}^n \widehat{e}_i(e_j) \otimes \widehat{e}_j = \sum_{j=1}^n \delta_{ij} \widehat{e}_j = \widehat{e}_i$.

Lemma: For Y with basis $\{d_1, \ldots, d_m\}$,
morphism $e_i \mapsto d_j$ corresponds to tensor $\widehat{e}_i \otimes d_j$.

Example: Relation category **Rel**, biproduct is the disjoint union with

$$X \ni x \underset{\pi_X}{\overset{\iota_X}{\rightleftarrows}} x \in X + Y \ni y \underset{\pi_Y}{\overset{\iota_Y}{\leftrightarrows}} y \in Y$$

Monoidal category $\big(\mathbf{Rel}, \times, \{0\}\big)$, compact closed with $X^* = X$.

Evaluation $\{((x, x), 0) \mid x \in X\}$, coevaluation $\{(0, (x, x)) \mid x \in X\}$.

First yanking condition:
relation product of $\{(x, (x', x', x)) \mid x, x' \in X\}$
with $\{((x', x, x), x') \mid x, x' \in X\}$ is $\{(x, x) \mid x \in X\}$.

Second is similar.