36. The Kleisli category of a monad

Alternative adjunction $\mathbf{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{X}$ ($\perp$) from monad (T,μ,η) on $\mathbf{X}$.

Kleisli category $\mathbf{X}_T$ of monad (T,μ,η) on $\mathbf{X}$:
$(\mathbf{X}_T)_0 = \mathbf{X}_0$, $\mathbf{X}_T(X,Y) = \mathbf{X}(X,TY)$, identities $1_X = \eta_X \colon X \to TX$,
composition $(Y \xrightarrow{g} TZ) \circ (X \xrightarrow{f} TY) = (X \xrightarrow{f} TY \xrightarrow{Tg} T^2 Z \xrightarrow{\mu_Z} TZ)$.

Adjunction $\mathbf{X}_T(F_T X, Y) = \mathbf{X}_T(X,Y) = \mathbf{X}(X,TY) = \mathbf{X}(X, U_T Y)$
with left adjoint $F_T(X \xrightarrow{f} Y) = (X \xrightarrow{f} Y \xrightarrow{\eta_Y} TY)$,
right adjoint $U_T(Y \xrightarrow{g} TZ) = (TY \xrightarrow{Tg} T^2 Z \xrightarrow{\mu_Z} TZ)$, unit $X \xrightarrow{\eta_X} TX$,
counit $\varepsilon_Y = TY \xrightarrow{1_{TY}} TY$, and monad $(U_T F_T, U_T \varepsilon F_T, \eta) = (T,\mu,\eta)$.

Power set monad $(\mathcal{P},\mu,\eta)$ with $\eta_X \colon x \mapsto \{x\}$ and set family union
$\mu_X \colon \{\{x,\dots\},\{y,\dots\},\dots\} \mapsto \{x,\dots,y,\dots,\dots\}$ — like with lists.

Category Rel of relations $X \xrightarrow{R} Y = \{(x,y) \mid x\ R\ y\}$ on sets:
Have $\mathbf{Rel}_0 = \mathbf{Set}_0$,
with 1_X as the identity function or equality relation on a set X.
Relation product $(X \xrightarrow{R} Y \xrightarrow{S} Z) := \{(x,z) \mid \exists\ t \in Y\ .\ x\ R\ t\ S\ z\}$.

Theorem: Category **Rel** is the Kleisli category for $(\mathcal{P},\mu,\eta)$.

Proof. $X \xrightarrow{R} Y$ gives $\mathbf{Set}_{\mathcal{P}}$-morphism $X \xrightarrow{R} \mathcal{P}Y; x \mapsto \{y \mid x\ R\ y\}$.
Kleisli identity is $\eta_X \colon X \to \mathcal{P}X; x \mapsto \{x\} = \{x' \mid x = x'\}$,
and Kleisli composition gives the relation product:

$$X \xrightarrow[R]{} \mathcal{P}Y \xrightarrow[\mathcal{P}S]{} \mathcal{P}^2 Z \xrightarrow[\mu_Z]{} \mathcal{P}Z$$

$$x \longmapsto \{t \mid x\ R\ t\} \longmapsto \{\{z \mid t\ S\ z\} \mid x\ R\ t\} \longmapsto \{z \mid \exists\ t\ .\ x\ R\ t\ S\ z\}$$

$$E \longmapsto \{\{z \mid e\ S\ z\} \mid e \in E\}$$

□