35. Eilenberg-Moore algebras

Does a monad (T, μ, η) on $\mathbf{X}$ give adjunction $\mathbf{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{X}$ ($\perp$) ?

- Monoid $(M, m\colon M^2 \to M, e : \top \to M)$ is a monoid in $(\mathbf{Set}, \times, \top)$.

M**-sets** X have **action** $a\colon M \times X \to X$ with

associativity: $\begin{array}{ccc} M^2 \times X & \xrightarrow{1_M \times a} & M \times X \\ {\scriptstyle m \times 1_X}\downarrow & & \downarrow{\scriptstyle a} \\ M \times X & \xrightarrow[a]{} & X \end{array}$, unitality: $\begin{array}{ccc} X & \xrightarrow{e \times 1_X} & M \times X \\ & \searrow\!\!= & \downarrow{\scriptstyle a} \\ & & X \end{array}$

morphisms: $\begin{array}{ccc} M \times X_1 & \xrightarrow{1_M \times f} & M \times X_2 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$, category $\mathbf{Set}^M$ of M-sets.

Free $FX = (M^2 \times X \xrightarrow{m \times 1_X} M \times X)$, adjoint $U(M \times X \xrightarrow{a} X) = X$.

Unit $\eta_X\colon X \to M \times X; x \mapsto (e, x)$, counit $\varepsilon_a = a$.

Gives a model endofunctor $T\colon X \mapsto M \times X$ on $\mathbf{Set}$.

- Monad $(T, \mu\colon T^2 \to T, \eta\colon 1_{\mathbf{X}} \to T)$ is a monoid in $(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}})$.

Eilenberg-Moore algebra $a\colon TX \to X$ in $\mathbf{X}(TX, X)$ for $X \in \mathbf{X}_0$:

with associativity: $\begin{array}{ccc} T^2X & \xrightarrow{Ta} & TX \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle a} \\ TX & \xrightarrow[a]{} & X \end{array}$ and unitality: $\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & \searrow\!\!= & \downarrow{\scriptstyle a} \\ & & X \end{array}$,

morphisms: $\begin{array}{ccc} TX_1 & \xrightarrow{Tf} & TX_2 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$, cat. $\mathbf{X}^T$ of Eilenberg-Moore algebras.

Forgetful $U^T\colon \mathbf{X}^T \to \mathbf{X}; (TX \xrightarrow{a} X) \mapsto X$.

Free $F^T\colon \mathbf{X} \to \mathbf{X}^T; X \mapsto (T^2X \xrightarrow{\mu_X} TX)$. Note $\boxed{U^T F^T X = TX}$

Unit $\eta^T_X\colon X \xrightarrow{\eta_X} TX$, counit $\varepsilon^T_a\colon (T^2X \xrightarrow{Ta} TX) \xrightarrow{a} (TX \xrightarrow{a} X)$

Eilenberg-Moore adjunction $\left(F^T, U^T, \eta^T, \varepsilon^T\right)$, yields monad (T, μ, η).