34. Adjunctions yield monads

Adjunction $(F\colon \mathbf{X} \to \mathbf{A}, U\colon \mathbf{A} \to \mathbf{X}, \eta\colon 1_{\mathbf{X}} \to UF, \varepsilon\colon FU \to 1_{\mathbf{A}})$.

Triangular identities $1_F = \varepsilon F \bullet F\eta$ and $1_U = U\varepsilon \bullet \eta U$.

Trace in X: UF-coalgs. $\eta_X\colon X \to UFX$, $\mu := U\varepsilon F\colon UFUF \to UF$.

Proposition: Unital law
$$\begin{array}{ccc} UFUF & \xleftarrow{UF\eta} & UF \\ {\scriptstyle \eta UF}\uparrow & \searrow^{U\varepsilon F} & \Vert \\ UF & = = = & UF \end{array}$$

Proof. Triangular
$$\begin{array}{ccc} FUF & \xleftarrow{F\eta} & F \\ & {}_{\varepsilon F}\searrow & \Vert \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} UFU & & \\ {\scriptstyle \eta U}\uparrow & \searrow^{U\varepsilon} & \\ U & = = = & U \end{array} \qquad \square$$

Proposition: $\mu\colon UFUF \to UF$ associative.

Proof. Need
$$\begin{array}{ccc} UFUFUF & \xrightarrow{UFU\varepsilon F} & UFUF \\ {\scriptstyle U\varepsilon FUF}\downarrow & & \downarrow{\scriptstyle U\varepsilon F} \\ UFUF & \xrightarrow[U\varepsilon F]{} & UF \end{array} \quad \text{or} \quad \begin{array}{ccc} FUFU & \xrightarrow{FU\varepsilon} & FU \\ {\scriptstyle \varepsilon FU}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ FU & \xrightarrow[\varepsilon]{} & 1_{\mathbf{A}} \end{array} .$$

Naturality:
$$\begin{array}{c} FUA \\ {\scriptstyle \varepsilon_A}\downarrow \\ A \\ \ldots \text{in } \mathbf{A} \end{array} \quad \Bigg| \quad \begin{array}{ccc} FUFUA & \xrightarrow{\varepsilon_{FUA}} & FUA \\ {\scriptstyle FU\varepsilon_A}\downarrow & & \downarrow{\scriptstyle \varepsilon_A} \\ FUA & \xrightarrow[\varepsilon_A]{} & A \\ & \ldots \text{in } \mathbf{A} & \end{array} \qquad \square$$

Theorem: Adjunction $(F, U, \eta, \varepsilon)$ gives monad $(UF, U\varepsilon F, \eta)$ on $\mathbf{X}$.

Example: Free monoid adjunction, for set or alphabet X.

Then UFX or X^* is the set of words or lists $\langle x_1 \ldots x_r \rangle$ in the alphabet.

Coalgebra $\eta_X\colon X \to UFX$; letter $x \mapsto$ one-letter "word" or list $\langle x \rangle$.

Multiplication $\mu_X = U\varepsilon_{FX}\colon UFUFX \to UFX$;

list of words or lists $\mapsto$ concatenation of list:

$$\langle\langle x_{11} \ldots x_{1r_1} \rangle \ldots \langle x_{s1} \ldots x_{sr_s} \rangle\rangle \mapsto \langle x_{11} \ldots x_{1r_1} \ldots x_{s1} \ldots x_{sr_s} \rangle$$

— removes inner brackets.