

33. ENDOFUNCTORS, (CO)ALGEBRAS, MONADS

Endofunctor category $\mathbf{X}^{\mathbf{X}} = [\mathbf{X}, \mathbf{X}]$ of category \mathbf{X} .

Algebra (X, α) for an endofunctor $T: \mathbf{X} \rightarrow \mathbf{X}$ is given by a **structure map** $\alpha: TX \rightarrow X$ in $\mathbf{X}(TX, X)$ for some $X \in \mathbf{X}_0$.

Algebra (homo)morphism $\theta: (X, \alpha) \rightarrow (Y, \beta)$

is given by commuting diagram $TX \xrightarrow{T\theta} TY$ in \mathbf{X} .

$$\begin{array}{ccc} TX & \xrightarrow{T\theta} & TY \\ \alpha \downarrow & & \downarrow \beta \\ X & \xrightarrow{\theta} & Y \end{array}$$

Example: Finite subset endofunctor $\mathcal{P}_{\text{fin}}: \mathbf{Set} \rightarrow \mathbf{Set}$, bounded semilattice (commutative, idempotent monoid) $(X, \cdot, 1)$, structure map $\alpha: \mathcal{P}_{\text{fin}}X \rightarrow X; \{x_1, \dots, x_r\} \mapsto x_1 \cdot \dots \cdot x_r \cdot 1$.

Coalgebra (X, α) for an endofunctor $T: \mathbf{X} \rightarrow \mathbf{X}$ is given by a **structure map** $\alpha: X \rightarrow TX$ in $\mathbf{X}(X, TX)$ for some $X \in \mathbf{X}_0$.

Coalgebra (homo)morphism $\theta: (X, \alpha) \rightarrow (Y, \beta)$

is given by commuting diagram $TX \xrightarrow{T\theta} TY$ in \mathbf{X} .

$$\begin{array}{ccc} TX & \xrightarrow{T\theta} & TY \\ \alpha \uparrow & & \uparrow \beta \\ X & \xrightarrow{\theta} & Y \end{array}$$

Example: Coalgebra with structure map $\alpha: X \rightarrow \mathcal{P}_{\text{fin}}X$ represents a non-deterministic dynamical system.

Endofunctors form a monoidal category $(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}})$.

Monad on \mathbf{X} is a monoid $(T, \mu: T^2 \rightarrow T, \eta: 1_{\mathbf{X}} \rightarrow T)$ in $(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}})$:

$$\begin{array}{ccc} \begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \mu T \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} & \begin{array}{ccc} T^2 & \xleftarrow{T\eta} & T \\ \eta T \uparrow & \searrow \mu & \parallel \\ T & \xlongequal{\quad} & T \end{array} & \text{Here, } \mu T \text{ is:} \\ & & \mathbf{X} \xrightarrow{T} \mathbf{X} \begin{array}{c} \curvearrowright \begin{array}{c} T^2 \\ \downarrow \mu \\ T \end{array} \curvearrowleft \\ \mathbf{X} \end{array} \end{array}$$

or

$$\mathbf{X} \begin{array}{c} \curvearrowright \begin{array}{c} T \\ \downarrow \text{id}_T \\ T \end{array} \curvearrowleft \\ \mathbf{X} \end{array} \begin{array}{c} \curvearrowright \begin{array}{c} T^2 \\ \downarrow \mu \\ T \end{array} \curvearrowleft \\ \mathbf{X} \end{array} = \mu \circ \text{id}_T, \text{ whiskering.}$$

Will get various kinds of algebras from monads.