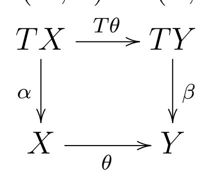
33. Endofunctors, (co)algebras, monads

Endofunctor category $\mathbf{X}^{\mathbf{X}} = [\mathbf{X}, \mathbf{X}]$ of category $\mathbf{X}$.

Algebra (X, α) for an endofunctor $T : \mathbf{X} \to \mathbf{X}$ is given
by a **structure map** $\alpha : TX \to X$ in $\mathbf{X}(TX, X)$ for some $X \in \mathbf{X}_0$.

Algebra (homo)morphism $\theta : (X, \alpha) \to (Y, \beta)$
is given by commuting diagram

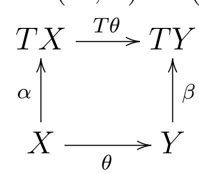


in $\mathbf{X}$.

Example: Finite subset endofunctor $\mathcal{P}_{\mathsf{fin}} : \mathbf{Set} \to \mathbf{Set}$,
bounded semilattice (commutative, idempotent monoid) $(X, \cdot, 1)$,
structure map $\alpha : \mathcal{P}_{\mathsf{fin}} X \to X; \{x_1, \dots, x_r\} \mapsto x_1 \cdot \ldots \cdot x_r \cdot 1$.

Coalgebra (X, α) for an endofunctor $T : \mathbf{X} \to \mathbf{X}$ is given
by a **structure map** $\alpha : X \to TX$ in $\mathbf{X}(X, TX)$ for some $X \in \mathbf{X}_0$.

Coalgebra (homo)morphism $\theta : (X, \alpha) \to (Y, \beta)$
is given by commuting diagram

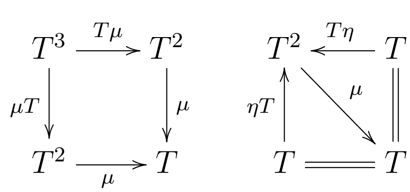


in $\mathbf{X}$.

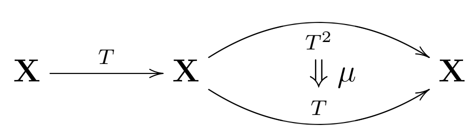
Example: Coalgebra with structure map $\alpha : X \to \mathcal{P}_{\mathsf{fin}} X$
represents a non-deterministic dynamical system.

Endofunctors form a monoidal category $\big(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}}\big)$.

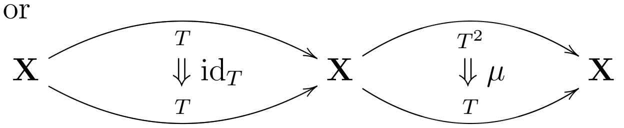
Monad on $\mathbf{X}$ is a monoid $(T, \mu : T^2 \to T, \eta : 1_{\mathbf{X}} \to T)$ in $\big(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}}\big)$:



Here, μT is:



or



$= \mu \circ \mathrm{id}_T$, **whiskering**.

Will get various kinds of algebras from monads.