32. The braid category

Monoid $(M, \nabla\colon M \times M \to M, \eta\colon \top \to M)$ in $\mathbf{C}$ with products.

Monoidal category $(\mathbf{C}, \otimes, \mathbf{I})$ is a monoid in $(\mathbf{Cat}, \times, \mathbf{1})$.

Sequence $(G_n \mid n \in \mathbb{N})$ **of groups**
with trivial G_0, and homomorphisms $\rho_{m,n}\colon G_m \times G_n \to G_{m+n}$

satisfying
$$\begin{array}{ccc} G_l \times G_m \times G_n & \xrightarrow{\rho_{l,m}\times 1} & G_{l+m} \times G_n \\ {\scriptstyle 1\times\rho_{m,n}}\downarrow & & \downarrow{\scriptstyle \rho_{l+m,n}} \\ G_l \times G_{m+n} & \xrightarrow[\rho_{l,m+n}]{} & G_{l+m+n} \end{array}$$
. Category $\mathbf{G}$ with $\mathbf{G}_0 = \mathbb{N}$

and $\mathbf{G}(m,n) = G_m$ for $m = n$ and $\emptyset$ for $m \neq n$.

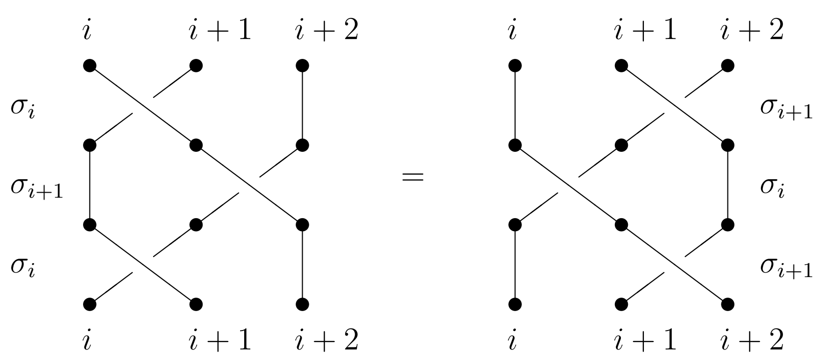
Monoidal category $(\mathbf{G}, +, 0)$ is $(\mathbb{N}, +, 0)$ at the object level,
with $+ = \bigcup \big(\rho_{m,n}\colon G_m \times G_n \to G_{m+n}\big)$ on morphisms.

Symmetric groups $S_n =$
$\langle \tau_1, \dots, \tau_{n-1} \mid \tau_i^2 = 1,\ \tau_i\tau_j = \tau_j\tau_i \text{ for } |i-j| > 1,\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}\rangle$.

General linear groups $\mathsf{GL}(K)$ with $\mathsf{GL}_n(K)$.

Braid groups
$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle$
give the **braid category B**.

Braid relation:



First string on top layer, second in middle, third on bottom layer.

- "Third Reidemeister move" in knot theory terms.
- "Yang-Baxter equation" in physics.