31. Categories and 2-categories

Category **Cat** of (small) categories;
with $\mathbf{Cat}(D, C) = C^D =: [D, C]$ (functor category).

Cartesian closed monoidal $(\mathbf{Cat}, \times, \mathbf{1})$, **terminal category** $\mathbf{1} = \bullet\circlearrowright$ as the monoidal unit, and Currying $\mathbf{Cat}(A \times B, C) \cong \mathbf{Cat}(A, [B, C])$.

(Strict) **2-category:** (1-)category $\mathbf{C}$ enriched over **Cat**.
0-cells: objects $A, B, \ldots$ of $\mathbf{C}$.
1-cells: morphisms of $\mathbf{C}$, i.e., objects in the categories $\mathbf{C}(A, B)$ or elements of the sets $\mathbf{Cat}\big(\mathbf{1}, \mathbf{C}(A, B)\big)$.
2-cells: morphisms in the categories $\mathbf{C}(A, B)$.

Example: 2-category **Cat**. Categories as 0-cells. Functors as 1-cells.

Natural transformations $\tau\colon F \to G$ as 2-cells $\quad A \;\overset{F}{\underset{G}{\Downarrow \tau}}\; B$

Horizontal 2-cell composition: $\mathbf{Cat}(A, B) \times \mathbf{Cat}(B, C) \xrightarrow{\circ} \mathbf{Cat}(A, C)$;

$$A \;\overset{F}{\underset{G}{\Downarrow \tau}}\; B \;\overset{F'}{\underset{G'}{\Downarrow \tau'}}\; C \;\mapsto\; A \;\overset{F'\circ F}{\underset{G'\circ G}{\Downarrow \tau' \circ \tau}}\; C$$

with
$$\begin{array}{ccc} F' \circ Fa & \xrightarrow{\tau'_{Fa}} & G' \circ Fa \\ {\scriptstyle F'\tau_a}\downarrow & \searrow^{(\tau'\circ\tau)_a} & \downarrow{\scriptstyle G'\tau_a} \\ F' \circ Ga & \xrightarrow[\tau'_{Ga}]{} & G' \circ Ga \end{array}$$

Identity: $\mathbf{Cat}(\mathbf{1}, \mathbf{Cat}(C, C)) \ni j_C\colon \bullet\circlearrowright \;\mapsto\; C \;\overset{1_C}{\underset{1_C}{\Downarrow \mathrm{id}}}\; C$

Vertical 2-cell composition on $\mathbf{Cat}(A, B)$:
$$\begin{array}{ccc} Fa & \xrightarrow{(\tau\bullet\sigma)_a} & Ha \\ & {\scriptstyle \sigma_a}\searrow \quad \nearrow{\scriptstyle \tau_a} & \\ & Ga & \end{array}$$

Entropic or interchange law: $(\tau' \bullet \sigma') \circ (\tau \bullet \sigma) = (\tau' \circ \tau) \bullet (\sigma' \circ \sigma)$,
as bifunctorial horizontal composition respects vertical composition.

$(n+1)$**-category:** an n-category enriched over **Cat**.