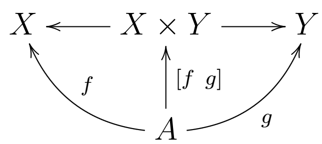
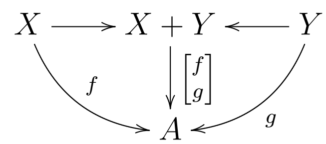
30. Enriching abelian categories

Abelian category $\mathbf{A}$ (Freyd's definition).

Matrices:



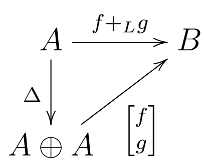


Exact: $0 \to X \xrightarrow{\iota_X} X+Y \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} Y \to 0,\ \ 0 \to X \xrightarrow{[1\ 0]} X \times Y \xrightarrow{\pi_Y} Y \to 0$

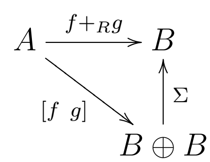
Theorem: $0 \to X+Y \xrightarrow{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}} X \times Y \to 0$ exact,
so $X+Y \cong X \times Y =: X \oplus Y$, **biproduct**.

Diagonal: $\Delta\colon X \xrightarrow{[1\ 1]} X \oplus X$. **Summation:** $\Sigma\colon X \oplus X \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} X$.

For $f, g \in \mathbf{A}(A,B)$, define



and



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Proposition: $0 +_L f = f = f +_L 0, \quad 0 +_R f = f = f +_R 0$.

Proposition: $(f +_L g) +_R (h +_L k) = (f +_R h) +_L (g +_R k)$.

Proof. Both sides are $A \xrightarrow{\Delta} A \oplus A \xrightarrow{\begin{bmatrix}f & h\\g & k\end{bmatrix}} B \oplus B \xrightarrow{\Sigma} B$. □

Theorem: $+_L = +_R$, commutative and associative.

Proof. Setting $g = h = 0$, have $f +_R h = f +_L h =: f + h$.
Setting $h = 0$, have $(f+g)+k = f+(g+k)$.
Setting $f = k = 0$, have $g + h = h + g$. □

Theorem: $\mathbf{A}(A,B)$ is an abelian group.

Proof. For $f\colon A \to B$, have $A \oplus A \xrightarrow{\begin{bmatrix}1 & f\\0 & 1\end{bmatrix}} B \oplus B$ monic and epic, so an isomorphism with inverse $B \oplus B \xrightarrow{\begin{bmatrix}1 & g\\0 & 1\end{bmatrix}} A \oplus A$, then $f + g = 0$. □