29. Factorization of morphisms, abelian categories

First Isom. Thm. for sets:
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle e}\searrow & \uparrow{\scriptstyle m} \\ & & f(X) \end{array} \qquad \begin{array}{l} \text{so } f = m \circ e\,, \\ \\ m \text{ mono}, e \text{ epi} \end{array}$$

Abelian category [Freyd]: (A0) $\mathbf{A}$ has a zero object;
(A1) For $A, B \in \mathbf{A}_0$, product $A \times B$ and coproduct $A + B$ exist;
(A2) For $A, B \in \mathbf{A}_0$ and $f \in \mathbf{A}(A, B)$,

have kernel $\boxed{\operatorname{Ker} f \xrightarrow{\ker f} A}$ and cokernel $\boxed{B \xrightarrow{\operatorname{coker} f} \operatorname{Coker} f}$

(A3) Every monomorphism is a kernel; every epimorphism is a cokernel.

Image: $[\operatorname{Im} f \xrightarrow{\operatorname{im} f} B] := [\operatorname{Ker}(\operatorname{coker} f) \xrightarrow{\ker \operatorname{coker} f} B]$,
a monomorphism, smallest subobject of B that divides f.

Coimage: $[A \xrightarrow{\operatorname{coim} f} \operatorname{Coim} f] := [A \xrightarrow{\operatorname{coker} \ker f} \operatorname{Coker}(\ker f)]$,
an epimorphism, smallest quotient of A that divides f.

Factorization $[A \xrightarrow{f} B] = [A \xrightarrow{q} \operatorname{Im} f \xrightarrow{\operatorname{im} f} B]$ with q an epimorphism.
Indeed, $\operatorname{coker} q \neq 0$ would mean f divided by a smaller subobject of B.

Theorem: $f \colon A \to B$ mono and epi $\Rightarrow$ f is an isomorphism.

Proof. Have $B \xrightarrow{\operatorname{coker} f} 0$ since f epi, and $1_B = \ker \operatorname{coker} f$.
Since f is mono, $A \xrightarrow{f} B$ is also a kernel of $\operatorname{coker} f$ [and so $A \cong B$].
Thus f has a section: $f \circ s = 1_B$. Dually, it has a retraction: $r \circ f = 1_A$.
Since $r = r \circ 1_B = r \circ f \circ s = 1_A \circ s = s$, have f invertible. $\square$

Corollary: $\operatorname{Im} f \cong \operatorname{Coim} f$, and $\boxed{f = \operatorname{im} f \circ \operatorname{coim} f}$

See `https://math.stackexchange.com/questions/3268091/coimage-and-image-in-abelian-categories`