28. Pointed sets, kernels, and cokernels

Pointed set X_e has chosen element e, so $e\colon \top \to X$ with image $\{e\}$.

Category of pointed sets is the slice category $(\top \downarrow \mathbf{Set})$.
Internal hom $[X_e, Y_d] = [X, Y]_{\{d\}}$ with constant $d\colon X \to Y; x \mapsto d$.

Currying: $(\top \downarrow \mathbf{Set})(X_e \wedge Y_d, Z_c) \cong (\top \downarrow \mathbf{Set})\big(X_e, [Y_d, Z_c]\big)$ with the **smash product** $X_e \wedge Y_d = \Big(\big((X \smallsetminus \{e\}) \times (Y \smallsetminus \{d\})\big) \cup \{(e,d)\}\Big)_{\{(e,d)\}}$.

Suppose category C has a zero object 0, e.g., $(\top \to \top)$ in $(\top \downarrow \mathbf{Set})$.

Zero morphism in $C(x,y)$ is the composite $(x \xrightarrow{0} y) = (x \to 0 \to y)$.

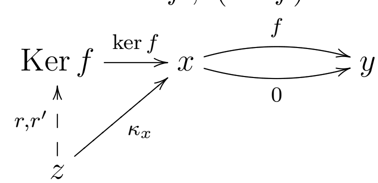
Kernel: $\operatorname{Ker} f \xrightarrow{\ker f} x$ is the equalizer of $x \underset{0}{\overset{f}{\rightrightarrows}} y$.

Cokernel: $y \xrightarrow{\operatorname{coker} f} \operatorname{Coker} f$ is the coequalizer of $x \underset{0}{\overset{f}{\rightrightarrows}} y$.

Lemma: $\operatorname{Ker} f \xrightarrow{\ker f} x$ is mono; and dually $y \xrightarrow{\operatorname{coker} f} \operatorname{Coker} f$ is epi.

Proof. $\forall\ z \xrightarrow{r,r'} \operatorname{Ker} f$, $(\ker f) \circ r = (\ker f) \circ r' =: \kappa_x$

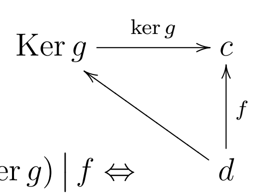
$\Rightarrow$



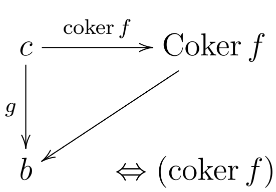
$\Rightarrow\ r = r'$. $\square$

Example: For $f \in (\top \downarrow \mathbf{Set})(X_d, Y_e)$, have $\operatorname{Ker} f = \big(f^{-1}\{e\}\big)_d \hookrightarrow X_d$.

Object c of C with zero, (co)kernels, preorders $\big(\partial_1^{-1}\{c\}, \big|\,\big)$ and $\big(\partial_0^{-1}\{c\}, \big|^{\mathsf{op}}\big)$.



adjunction



$(\ker g) \,\big|\, f \Leftrightarrow$ $\Leftrightarrow g \circ f = 0 \Leftrightarrow$ $\Leftrightarrow (\operatorname{coker} f) \,\big|\, g$

So: $\boxed{\ker g = \ker \operatorname{coker} \ker g}$ and $\boxed{\operatorname{coker} f = \operatorname{coker} \ker \operatorname{coker} f}$