27. Copowers and free enriched categories

For a set S and an object b of a cocomplete category B, the colimit of the constant diagram $S \to \{b\}$ is the **copower** or **multiple**

$$S \cdot b = \sum_{s\in S} b, \text{ with insertions } \iota_s \colon b \to S \cdot b \text{ for } s \in S.$$

Example: For $X \in \mathbf{Set}$, have $\iota_s \colon X \to S \times X = S \cdot X; x \mapsto (s, x)$.

Example: For $V \in \mathcal{L}$, have $S \cdot V = \overbrace{V \oplus \ldots \oplus V}^{|S| \text{ copies}}$ for S finite.

For arbitrary S, have power $V^S = \mathbf{Set}(S, UV) \cong \mathcal{L}(FS, V) \in \mathcal{L}_0$,
and copower $S \cdot V = \big\{ f \colon S \to V \;\big|\; \infty > |\{s \in S \mid f(s) \neq 0\}| \big\}$,
a subobject of V^S, proper if S is infinite.

Category C, bicomplete closed symmetric monoidal base category $(\mathbf{B}, \otimes, I)$.

Free B-enriched category $\mathbf{B}C$ on C: left adjoint to impoverishment.

Object class $\mathbf{B}C_0 = C_0$.

For $x, y \in \mathbf{B}C_0 := C_0$, define $\mathbf{B}C(x, y) := C(x, y) \cdot I = \sum\limits_{f\in C(x,y)} I$.

For $x \in C_0$, define $j_x = \iota_{1_x} \colon I \to C(x, x) \cdot I$.

For $x, y, z \in C_0$, distributivity and unitality give $\mathbf{B}C(x, y) \otimes \mathbf{B}C(y, z) = \sum_{f\in C(x,y)} I \otimes \sum_{g\in C(y,z)} I = \sum_{f\in C(x,y)} \sum_{g\in C(y,z)} I \otimes I = \sum_{(f,g)\in C(x,y)\times C(y,z)} I$.

Then have composition

$$\begin{array}{ccc} \sum_{C(x,y)\times C(y,z)} I & \overset{\circ}{\dashrightarrow} & \sum_{C(x,z)} I \\ \uparrow{\scriptstyle \iota_{(f,g)}} & \nearrow{\scriptstyle \iota_{g\circ f}} & \\ I & & \end{array}$$

Example: For a category C, and Boolean algebra $\mathbf{2}$,
the free $\mathbf{2}$-enriched $\mathbf{2}C$ is the preorder
obtained by "forgetting arrow labels" of C.

Group rings: For linear $(\mathcal{L}, \otimes, K)$,
and a one-object group $G = G_1$ on $G_0 = \{*\}$,
the *group ring* over K is the one-object free $\mathcal{L}$-category $\mathcal{L}G$,
with morphism set $G \cdot K$.

Standard Hopf algebra notation: $\eta_G = j_* = \iota_{1_*} \colon K \to G \cdot K$.