26. Enriched categories

Bicomplete category: All limits and colimits.

Base category: bicomplete symmetric monoidal category $(\mathbf{B}, \otimes, I)$, e.g., $(\mathbf{Set}, \times, \top)$, $(\mathcal{L}, \otimes, K)$, poset $\Big(([0,\infty], \geq\,), +, 0\Big)$ with $x + \infty = \infty$.

B-enriched category: quiver C with $\forall\, x, y \in C_0\,,\; C(x,y) \in \mathbf{B}_0$ and:

- **composition:** $\forall\, x, y, z \in C_0\,, \circ \in \mathbf{B}\big(C(x,y) \otimes C(y,z), C(x,z)\big)$
- **identities:** $\forall\, x \in C_0\,, j_x \in \mathbf{B}(I, C(x,x))$ with commuting:

$$\begin{array}{ccc} C(w,x) \otimes C(x,y) \otimes C(y,z) & \xrightarrow{\circ\otimes 1} & C(w,y) \otimes C(y,z) \\ {\scriptstyle 1\otimes\circ}\downarrow & & \downarrow{\scriptstyle \circ} \\ C(w,x) \otimes C(x,z) & \xrightarrow[\circ]{} & C(w,z) \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} C(x,y) & \xrightarrow{j_x \otimes 1} & C(x,x) \otimes C(x,y) \\ {\scriptstyle 1\otimes j_y}\downarrow & \searrow{=} & \downarrow{\scriptstyle \circ} \\ C(x,y) \otimes C(y,y) & \xrightarrow[\circ]{} & C(x,y) \end{array} \qquad \begin{array}{l} [\text{recall } B = I \otimes B,\ \text{etc.}] \\ \\ \ldots \text{ for } w, x, y, z \in C_0\,. \end{array}$$

Locally small category is enriched over $(\mathbf{Set}, \times, \top)$.

Pre-additive category is enriched over $(\mathbf{Ab}, \otimes, \mathbb{Z})$.

Linear category $\mathcal{L}$ is enriched over $(\mathcal{L}, \otimes, K)$.

Closed monoidal category is enriched over itself.

Preorder is enriched over the Boolean algebra $\mathbf{2} = \big(\{\bot < \top\}, \wedge, \top\big)$.

Directed metric spaces are enriched over $([0,\infty], +, 0)$.
Thus $d(x,y) \in [0,\infty]$ for $x, y \in C_0$,
and composition means $d(x.y) + d(y,z) \geq d(x,z)$.

If the symmetric monoidal $(\mathbf{B}, \otimes, I)$ is closed, can "impoverish" the enriched category C to $C_\circ$ with $C_\circ(x,y) = \mathbf{B}(I, C(x,y))$ for $x, y \in C_0$.