25. Currying

Heyting algebra: $\forall\, x, y, z \in P_0\,,\ \ P(x \cdot y, z) \cong P(x, y \multimap z)$
In particular, $\forall\, y, z \in P_0\,,\ \ P(y, z) \cong P(1, y \multimap z)$.

Currying: $\forall\, X, Y, Z \in \mathbf{Set}_0\,,\ \mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}\big(X, \mathbf{Set}(Y, Z)\big)$
In particular, $\forall\, Y, Z \in \mathbf{Set}_0\,,\ \mathbf{Set}(Y, Z) \cong \mathbf{Set}\big(\top, \mathbf{Set}(Y, Z)\big)$.

Tensor product: $\forall\, X, Y, Z \in \mathcal{L}_0\,,\ \mathcal{L}(X \otimes Y, Z) \cong \mathcal{L}\big(X, \mathcal{L}(Y, Z)\big)$
In particular, $\forall\, Y, Z \in \mathcal{L}_0\,,\ \mathcal{L}(Y, Z) \cong \mathcal{L}\big(K, \mathcal{L}(Y, Z)\big)$,
"linear spaces" as modules over commutative ring K., e.g., $\mathbb{Z}$ for $\mathbf{Ab}$.

Note $\mathcal{L}\big(X, \mathcal{L}(Y, Z)\big) \subseteq \mathbf{Set}\big(X, \mathbf{Set}(Y, Z)\big) \cong \mathbf{Set}(X \times Y, Z)$, so
$\mathcal{L}\big(X, \mathcal{L}(Y, Z)\big)$ tracks the **bilinear** maps $X \times Y \to Z$.

In all three cases, $\cong$ is a natural isomorphism of sets, so on the left hand side of the lower $\cong$ is a hom-set of the locally small category.

Strict symmetric monoidal category $(\mathbf{C}, \otimes, I)$: $X \otimes Y = Y \otimes X$,
$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$, and $I \otimes X = X = X \otimes I$.

E.g: Heyting algebra $(P, \cdot, 1)$, Cartesian $(\mathbf{Set}, \times, \top)$, linear $(\mathcal{L}, \otimes, K)$.

Closed monoidal category: Adjunction $\mathbf{C}(X \otimes Y, Z) \cong \mathbf{C}(X, [Y, Z])$
with **internal hom**-object $[Y, Z]$, set isom. $\mathbf{C}(Y, Z) \cong \mathbf{C}(I, [Y, Z])$.

Bifunctors: monoidal product $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$
and internal hom $[\_\_\,, \_\_]\colon \mathbf{C}^{\mathsf{op}} \times \mathbf{C} \to \mathbf{C}$.

Heyting algebras: monoid product $\to$ adjunction $\to$ internal hom.
Linear spaces: internal hom $\to$ adjunction $\to$ monoid product.

Note $\_\_ \otimes Y$ a left adjoint $\Rightarrow$ preserves coproducts $\Rightarrow$ distributivity:

$$(X + X') \otimes Y = (X \otimes Y) + (X' \otimes Y) \text{ or } \Big(\sum X_i\Big) \otimes Y = \sum (X_i \otimes Y)$$

Also $[Y, \_\_]$ a right adjoint $\Rightarrow$ preserves products $\Rightarrow$ "exponentiation":

$$[Y, Z_1 \times Z_2] = [Y, Z_1] \times [Y, Z_2] \text{ or } [Y, \textstyle\prod Z_i] = \prod [Y, Z_i]$$

Compare $\mathbf{C}(Y, \prod Z_i) \cong \prod \mathbf{C}(Y, Z_i)$

Arithmetic: $(l + l') \cdot m = l \cdot m + l' \cdot m$ and $(n_1 \cdot n_2)^m = n_1^m \cdot n_2^m$
in the skeleton $(\mathbb{N}, \cdot, 1)$ of $(\mathbf{FinSet}, \times, \top)$.