## 24. Heyting algebras and topologies

Preorder $(P, \leq)$ with all finite products, sufficiently including:

The empty product (terminal object) $\top$ with $\forall\ x \in P\ ,\ x \leq \top$;

The (comm., assoc.) **meet** or g.l.b with $\forall\ x, y \in P\ ,\ x \leftarrow x \cdot y \rightarrow y$.

For each fixed $a$ in $P$, functor $S(a)\colon (P, \leq) \rightarrow (P, \leq); x \mapsto (x \cdot a)$.

Suppose each $S(a)$ has a right adjoint $R(a)\colon z \mapsto (a \multimap z)$:

$$\forall\ x, y, z \in P\ ,\ \boxed{x \cdot y \leq z \ \Leftrightarrow\ x \leq y \multimap z} \qquad (*)$$

**Example:** Propositions, "and" is product; "deduce $q$ from $p$" is $p \rightarrow q$. Then $p \multimap q$ would be proposition "$p$ implies $q$".

**Bounded lattice:** poset, finite products, coproducts, $0 = \bot$, $1 = \top$.

**Complete lattice:** poset with all products and coproducts.

**Heyting algebra** is a bounded lattice with the adjunctions $(*)$.

**Prop:** Heyting algebras are distributive: $S(a)$ preserves coproducts.

**Prop:** Complete Heyting algebras are completely distributive.
By $(*)$, have $y \multimap z = \sum\{x \mid x \cdot y \leq z\}$.

**Example:** Boolean algebra with implication $p \multimap q = p \rightarrow q = (\neg p) \vee q$

**Negation** (pseudocomplement) $\neg x := x \multimap 0$ in any Heyting algebra.

**Example:** $\{0 \leq \frac{1}{2} \leq 1\}$, where $\frac{1}{2} \multimap 0 = \max\{x \mid x \cdot \frac{1}{2} \leq 0\} = 0$.
Then $\neg\neg\frac{1}{2} = \neg 0 = 1 \neq \frac{1}{2}$; "Law of the excluded middle" does not hold.

**Regular elements** $x = \neg\neg x$ in Heyting algebra form Boolean algebra.

**Topology:** In any topological space $(X, \mathcal{O})$, the subset $\mathcal{O}$ of $2^X$ comprising the open sets forms a complete Heyting algebra. Unions in $2^X$, but infinite intersections differ, take interior. Here $P \multimap Q = [(X \smallsetminus P) \cup Q]^\circ$

- **Indiscrete topology** $\mathcal{O} = \{\emptyset, X\}$
- **Discrete topology** $\mathcal{O} = 2^X$
- **Alexandrov topology** of poset $(P, \leq)$ is the set of all downsets.
- **Cofinite topology** of set $X$ has $\mathcal{O} = \{\emptyset\} \cup \{S \subseteq X \mid X \smallsetminus S \text{ finite}\}$
- For monoid $M$ and an $M$-set $X$, take $\mathcal{O}$ as the set of $M$-subsets. If $M$ is a group, get a Boolean algebra.